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Interactive Learning and Behavioral Sunspots

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Abstract

This paper shows how to extend the adaptive learning approach to a truly behavioral uncertainty problem. It investigates the simplest case of a first order self-referential model where two agents have a non-negligible impact on aggregate expectations. Unlike the standard setting, agents are not boundedly rational in that they acknowledge their own influence on output and they are perfectly informed about the exogenous determinants of the economy. Nevertheless no common knowledge is assumed; agents are epistemically isolated and they can only have noisy perceptions of the other agent's simultaneous expectation. In order to have consistent forecasts, each agent has to learn how the other agent's expectations will affect the economy. I prove there exist two types of learnable equilibria: (a) a unique rational expectation equilibrium (REE) and (b) at least one behavioral sunspot equilibrium (BSE). The latter may arise because the learning dynamics can generate the self-fulfilling reciprocal belief that the other agent is irrationally exuberant. Finally, numerical simulations illustrate how an unpredictable switch from the REE to a BSE may occur endogenously when both coexist and are learnable.

KEYWORDS: excess volatility, behavioral uncertainty, intrinsic heterogeneity, statistical learning, structural change.

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1 Introduction

1.1 The main idea

This paper aims to contribute to the discussion on emergence of epistemic foundations of rational expectations equilibria[1]. Rational expectations (RE) is the cornerstone of modern macroeconomic theory. It implies that agents do not commit systematic errors in forecasting the determinants of the economic system. In particular, since economic outputs generally respond to agents' beliefs, RE implies behavioral consistency, that is, each agent has not biased expectations about others'expectations. Agents holding RE fulfil a Nash equilibrium, called REE.

Besides the controversial nature of this stringent hypothesis, nobody can doubt that a great part of everyday business dealing is supported by agents' consistent expectations. This is more manifest for expectations about others' behavior. In real economic system agents operate being generally confident in a particular way of acting of others, normally labeled "rational" or "conventional". Communication between me and you through this paper is an immediate example. I have written expecting you understand my language and you are reading because you expect I have chosen an understandable language. Hopefully, we did not commit systematic errors, so that, your understanding of this paper is a sort of REE.

But where rational expectations about others' behavior came from? Generally economists justify behavioral consistency entailed by REE assuming common knowledge. In other words, no agent doubts about what the others know, so that, each agents trivially holds unbiased expectations about what the others are doing. Nevertheless, if an economist faces the problem of endogenous emergence of behavioral consistency using the concept of common knowledge she gets into troubles. Firstly, modelling a dynamic convergence in the infinite dimension space of higher order beliefs is a titanic effort. More essentially, the concept of common knowledge is not strictly required for emergence of REE. A simple example can illustrate last point.

We can model a game of chess assuming players hold common knowledge of the rules of the game. All the same, we cannot use the concept of common knowledge if one of the players is an automaton. What does it mean that a computer know that a player know that it knows? Knowledge seems to be a truly human faculty that hardly we can attach to a machine. But games played by human beings and automatons are indistinguishable. Therefore, the concept of common knowledge entails an epistemic sharing of knowledge that it is not necessary for players to play a game. What an automaton actually does is to react to a move on the basis of a bunch of already played games stored in its memory. In other words, it doesn't share knowledge with the opponent, but rather it expects by analogy a certain statistically relevant behavior. This case-based mechanism gives us a new perspective on how to model emergence of behavioral consistency among agents.

[1] From here onward REE will label both "rational expectations equilibria" and "rational expectations equilibrium".

The main idea of this paper is to investigate whether and how epistemically isolated agent can solve behavioral uncertainty in real time using the same tools scientists have to investigate reality. Agents, in guise of statisticians observing an exogenous process, form expectation about others' behavior on the basis of behavioral regularities observed and tested in time. In a situation of behavioral consistency a profile of agents' behaviors is self-sustained in that nobody has incentive to modify her behavior on the basis of the story of others' behavior she observed.

The gain for this change of perspective is that as long as behavioral consistency is generated endogenously, stochastic departures from an equilibrium can end up in 'jumps' to new equilibria where behavioral consistency requirement is equally satisfied. This happens because the system is not bounded by an a-priori epistemic condition. The paper presents a very simple formal exercise showing the potentiality of this idea. The modelled dynamics can contribute to the explanation of empirically observed structural breaks in time series focusing on the effects of agents trying to solve behavioral uncertainty in real time.

In sum, everyday social interactions are driven by expectations about others' behavior that are built and tested in time. If we want to model how economic agents think about the future we must have a theory on how they use information about the past, including information about others' behavior. Here I want to propose a competitive idea on how agents are able to achieve (or to fail) behavioral consistency. The move from the concept of common knowledge to a more case-based vision of expectations formation is the key for the interpretation of the results in this paper.

1.2 The paper

The paper investigates a simple first order self referential model with a unique rational expectations equilibrium. The task is to spell out under what conditions rational agents can solve behavioral uncertainty in real time. To make the exercise strictly focused on the aim I assume agents are fully informed about the model underlying the economy, but fully ignorant about what the others know. In other words, they perfectly know all the exogenous determinants of the economy, but they are epistemically isolated. This is the first essential hypothesis in the paper.

Nevertheless, macroeconomists normally assume agents to be infinitesimally small so that, in settings where endogenous uncertainty is present, their best response (expectation) functions are approximated by neglecting the influence of each individual in the model. This is true as long as each agent has the same impact on the aggregate expectation. Therefore, to meaningfully argue about the importance to form expectations about others' expectations, we have to assume that in the economy there exist at least two agents whose expectations have a not negligible impact on the aggregate. This is the second essential hypothesis. It deserves a careful justification that promptly follows.

Even if the economy is populated by a continuum of infinitesimal agents, we can think expectations are polarized by few institutional forecasters. They are the only ones endowed with the resources to gather and analyse efficiently available information. Institutional forecasters act as focal point for the most part of the public in that the latter just imitate the formers' expectations. A short list of institutional forecasters would include central bank, fiscal authorities, rating agencies, market leaders etc. As a result, in self-referential models, interactions between institutional forecasters matter as long as every institutional forecaster has a non-negligible impact on the aggregate expectations.

The behavioral uncertainty problem entailed by the two assumptions above is stated as follow. The unique rational expectations equilibrium of the model occurs if and only if all institutional forecasters forecast the fundamental price. Each institutional forecaster has the power to displace the actual course of the economy away from the fundamental one provided they have non-negligible group of followers. Hence, to have consistent expectations, each institutional forecaster has to care about others' possibly "non rational" simultaneous expectations. But, since they are epistemically isolated, they cannot have instantaneous perfect knowledge of others' simultaneous expectations. Because of the simultaneous and endogenous nature of the coordination problem, they only can have noisy signals of others' simultaneous expectations . For that behavioural uncertainty is in place.

As argued in the section above, rationality of others is something that can be in principle assessed in real time testing the persistence of a certain behavior. The idea is that each institutional forecaster can rationally exploit available information to extract the signal of others' rationality running recursive regression on available data[2]. Their information set is composed by all information necessary to compute the fundamental price, that is, the one forecasted in case all institutional forecasters hold rational expectations, and noisy observations of others' simultaneous expectations. Observational errors are drawn from a very generic distribution whose only restriction is to have bounded second moment. Agents cannot avoid observational errors, but they can test the hypothesis that observed displacements of the actual price from the fundamental one are independent from observed displacements of others expectations from rational expectation. If this is the case, each institutional forecaster would not care any longer about others' expectations and they would forecast just the fundamental price fulfilling the rational expectations equilibrium.

Orthodoxly the independence of the equilibrium from higher order beliefs is obtained assuming agents hold common knowledge others' expectations are the same as their own. Here agents have to extract this signal in real time using statistical tools. The specific research questions this paper wants to answer are the following ones. Are heterogeneous adaptive learners able to fill a gap of behavioural consistency under a statistical learning dynamics? And then, is rational expectations equilibrium still unique and learnable?

[2]Notice that OLS regression is the solution of the minimization problem of the forecast errors variance, that is it is consistent with the assumption rational exploitation of available information.

The paper basically shows three results in the simple case of two institutional forecasters. *i*) The unique rational expectations equilibrium arises from a disequilibrium prior in a large region of the coefficients space without relying on any common knowledge property. It results from interacting learning dynamics in which every agent tests the rationality of others in real time. Moreover, the rational expectations equilibrium attainability is proved to be particularly robust to the correlation and systematic bias in observational errors. *ii*) Nevertheless *even if* institutional forecasters are perfectly informed about the economic fundamentals and they acknowledge their own influence on the economy, one learnable equilibrium different from the rational expectations equilibrium may arise (I name such equilibria behavioural sunspot equilibria). This happens because in some cases agents' observational errors can generate the self-sustaining beliefs of others' irrational exuberance in the economy. These equilibria exhibit excess volatility around the fundamental value and possibly a drift away from rational expectations equilibrium depending on observational errors. In contrast to other sunspot equilibria and consistently with the general approach, behavioural sunspot equilibria don't imply any common knowledge property, rather they arise from endogenous coordination among non-cooperative agents. *iii*) Moreover the existence of behavioural sunspot equilibria is not alternative to rational expectations equilibria, but they coexist in a large region of the parameter space. In such a region, real time learning dynamics selects among them and endogenous and unpredictable switches from one equilibrium to the other can generally arise. I show with numerical simulations how a structural switch from the rational expectations equilibrium to behavioural sunspot equilibrium may occur endogenously.

2 Methodology

This section aims to clarify the task of assessing foundations of REE by learning. It gives connection of the present work with the two main learning approaches in macroeconomics, namely adaptive learning and eductive learning. The main methodological achievement of this paper is to establish a bridge between the two, making adaptive learning techniques to work in a typical eductive learning problem.

2.1 Why introducing learning?

A Nash equilibrium is a strategic profile consistent with each agent being rational and knowing the game and the strategic choices of the others. A REE is a Nash equilibrium of a game in which expectations are actions and the payoffs inversely proportional to the forecast error variance. In particular, assuming rational expectations implies that every agent in the model acts optimally in that she maximizes her utility function given *consistent* forecasts on the output of the economy. Biased expectations on the processes involved in the stochastic maximization result in a loss of efficiency for agents. In other words, in a stochastic economy agents violating rational expectations hypothesis would carry out a sub-optimal action.

Every stochastic model can be typically reduced or approximated by a linear system in which some endogenous variables depend on both behavioral variables (agents' expectations) and exogenous variables. To have consistent forecasts on the output of the economy agents have to know the model and to hold consistent beliefs on distributions of *both* exogenous and behavioral variables. The following proposition states conditions for the emergence of a REE.

Proposition 1 *In the context of self-referential models the emergence of a REE occurs with the simultaneous satisfaction of:*

a) procedural rationality *each agent maintains an optimizing behavior given her beliefs don't contradict available information*[3]*;*

b) exogenous consistency *each agent knows (or she has correct beliefs about) the model and the exogenous processes involved in the economy;*

c) behavioral consistency *each agent knows (or she has correct beliefs about) others' expectations.*

The expression *"or she has correct beliefs about"* in the proposition is to stress that what actually matters for the emergence of the equilibrium is the realization of a certain strategy profile. Strictly speaking the concept of knowledge is not necessary. The proposition will be shown formally working in the model presented later.

Assessing the emergence of REE means to clarify how the three conditions above are satisfied. All the same, more then one expectations profile can equally satisfies conditions above. Therefore, a primary concern for economist is to answer the question: how do agents coordinate among possible different REE? Or, restating the same from a normative point of view, which REE should the economist select to calibrate a policy intervention? Multiplicity is still a big issue for macroeconomics and it arises within RE paradigm as consequences of its acceptance.

From a more microeconomic point of view, one can ask under what conditions, if any, the great amount of knowledge assumed by RE paradigm would be maintained by rational agents. According to common sense, rationality should be intended as a procedure, rather than an outcome. It is not enough to identify such equilibria for which agents' expectations are correct, as rational expectations paradigm imposes. A more rigorous methodology would aim to clarify under what *rational* procedure, if any, agents would be able to correct their beliefs given an initial disequilibrium condition. The relevant question is when would procedural rationality be sufficient to achieve exogenous consistency or (and) behavioral consistency? What conditions justify epistemic foundations of rational expectations equilibria?

Learning give us the opportunity to take into account both these points. The general idea is the following. Agents start from a disequilibrium initial condition for which they hold possibly

[3]Procedural rationality is a strength of Bayesian rationality, the kind of rationality implied by a temporary equilibrium, that is, a general economic equilibrium in which agents' beliefs are simply given.

wrong beliefs about exogenous or behavioral variables distributions. That is, they suffer a lack of exogenous and/or behavioral consistency. But, since they are endowed with procedural rationality, they *rationally* exploit all information they have to fill their knowledge gap. In self-referential models where expectations influence the output, agents can recover new informative data about the actual working of the economy just observing their forecast error. Poorly speaking learning produces a dynamics of agents' beliefs in real (or notional) time possibly converging to a rest point, namely to a rational expectations equilibrium.

This methodology allow to model explicitly an endogenous acquisition of knowledge required to sustain a certain REE and, at the same time, to produce a natural path-dependent selection among possibly multiple equilibria. Learning approaches to rational expectations solve both multiplicity and epistemic foundation problem linked to economic equilibria. Of course, learning dynamics can fail to achieve consistency, so that, according to learning approaches a REE may be not a relevant one even if unique.

2.2 Eductive and Adaptive Learning

In this section we will briefly review the two main learning approaches to rational expectations equilibria, namely eductive and adaptive learning. The former applies to a behavioral uncertainty problem in that maintain exogenous consistency and relaxes endogenous consistency, whereas the latter bypasses behavioral consistency and focuses on learning to achieve exogenous consistency. This paper wants to investigate a typical eductive learning problem with adaptive learning techniques.

Eductive learning (Guesnerie 1992, 2005) takes a game-theoretic point of view. Agents are assumed to maintain procedural rationality and fully exogenous consistency, whereas they suffer a lack of behavioral consistency. In particular agents hold common knowledge of rationality and of the game, that nevertheless it is not enough to establish behavioral consistency. The relevant question is under what conditions this would be enough to induce behavioral consistency and hence to justify a REE. More specifically, eductive learning investigates under what conditions a particular rational expectations equilibrium corresponds to the game theoretic solution concept of rationalizable set (Pearce 1984 and Bernheim 1984). The rationalizable set is built by iterative deletion of strictly dominated strategies triggered by the only common knowledge of rationality and of the game. This mechanism entails a dynamics that, in the context of self referential models, is interpreted as an updating of beliefs in notional time. In case of multiplicity great emphasis is placed on common priors in that different initial beliefs can trigger convergence to different rational expectations equilibria.

In the context of self-referential models adaptive learning literature assumes agents acting like econometricians. Specifically, a representative agent run a recursive regression of temporary equilibrium data on a structural equation representing a perceived law of motion (PLM). In the basic framework the PLM has a form consistent with the rational expectations equilibrium that is independent from agents' beliefs. In other words, agents neglect their own influence in the economy even if this independence can be achieved only asymptotically. Referring to such kind of misspecification agents are presented in literature as bounded rational in that they miss the self referential nature of the economy. Holding estimated parameters and observing new realizations of observable exogenous variables the representative agent form a representative expectation (equal to the aggregate expectations) that in turn generate new temporary equilibrium data. Updated estimates of coefficients is obtained according to new data, and the story repeats. Anyway, because the specific misspecification mentioned above, convergence of estimates to the REE ones don't hold in general. Evans and Honkapohja (2001) shows extensively variants and analytical results of this class of problems.

Table 1 summarizes methodological differences between two approaches and the point of view taken in this paper. Eductive Approach assumes full procedural rationality and exogenous consistency, whereas it assume only common knowledge of rationality (that doesn't imply behavioral consistency). Eductive Learning investigates under what conditions this would be enough for agents to hold behavioral consistency. More specifically when the recursive succession of higher order beliefs spanned in notional time converges to REE. Adaptive learning assumes a kind of bounded rationality in that agents employ statistically optimal technique to forecast the course of the economy but they don't recognize the self-referential nature of the economy. For that, the problem of behavioral consistency is not considered. Nevertheless behavioral consistency is eventually fulfilled in the long run as every agent successfully learns to hold the same rational expectation. Normally agents are assumed to know and perfectly observe the relevant variables affecting the equilibrium, but they ignore the impact of such variables. The general aim with adaptive learning is to spell out conditions to fully achieve exogenous consistency, specifically, whether or not a statistical procedure of beliefs updating in real time converges to correct parameters estimates.

	Eductive	Adaptive	Interactive
Procedural Rationality	Full	Bounded	Full
Exogenous Consistency	Full	Lack	Full
Behavioral Consistency	Lack	Ignored	Lack
Learning Dynamics	Notional time	Real time	Real time

Table 1

A general shortcoming of bounded rationality hypothesis is that they results often as ad-hoc hypothesis. In adaptive learning approach it is not clear why agent should hold that particular form of cognitive limitedness. The issue is more relevant in light of the fact that the findings depend crucially on this hypothesis. In the following model this weakness is overcome by assuming explicitly agents acknowledge the self-referential nature of the economy. Interactions among rational agents arises properly because they recognize their own (not negligible) influence on the output dynamics. This paper built up a more general perspective on the issue of learning interactions[4] extending the field of application of adaptive learning to behavioral uncertainty problems. It bridges the two approaches making adaptive approach working in a typical eductive approach problem where interactions are relevant. I will assume full procedural rationality and exogenous consistency to show how statistical learning can overcome a lack of behavioral consistency in real time. Last column of table 1 will have a formal specification in reference to the presentation of the analytical model in section 3.

3 The Model

3.1 Basic Framework

A simple self-referential model. Consider an ocean of infinitesimal agents belonging to $A = (0, 1) \subset \mathbb{R}$. They populate a self-referential economy in which price at time $t \in \mathbb{N}$, namely $p_t \in \mathbb{R}^+$ depends on a constant $\alpha \in \mathbb{R}$ and on aggregate expectation $E_{t-1}p_t$ of p_t at time $t-1$. This economy is represented by

$$p_t = M(E_{t-1}p_t) = \alpha + \beta E_{t-1}p_t + \eta_t \tag{1}$$

where $M(E_{t-1}p_t) : \mathbb{R}^+ \to \mathbb{R}^+$ is a model function whose argument is the aggregate expectation, $\beta \in \mathbb{R}$ measures the impact of aggregate expectation on p_t and η_t is a centred i.i.d. shock normally distributed with variance δ_η. Formally, the aggregate expectation is expressed by

$$E_{t-1}p_t = \int_A [p_t]_a^{t-1} \, da \tag{2}$$

where $[\cdot]_a^{t-1}$ is a function belonging to a family of generic beliefs operator $\mathcal{F}$, whose image is agent a's belief at time $t-1$ of $(\cdot)$, with $a \in A$. The rational expectation equilibrium for the model at hand is

$$p^{RE} = \overline{p} + \eta_t \tag{3}$$

where the rational expectation is $E_{t-1}^{RE}p_t = \overline{p} = \alpha\,(1-\beta)^{-1}$. All agents are endowed with the following utility function

$$U_a = \mathbf{E}\left(p_t - [p_t]_a^{t-1}\right)^{-2} \tag{4}$$

[4] Recent literature faces the issue of structural heterogeneity in Adaptive Learning models (Berardi (2007), Branch and Evans (2006), Giannitsarou (2003), Evans, Honkapohja and Marimon (2001), Guse (2005), Honkapohja, Mitra (2006)) but the problem of interaction among learners is not considered. To the best of my knowledge Granato, Guse and Wong (2007) is the only work investigating the case of a minimal social interaction between two types of agents using adaptive learning. The latter uses a Stackelberg framework in which a group of leading firms has a well specified perceived law of motion and a group of followers just imitate the leaders' forecasts. Again there leaders don't care about heterogeneity present in the economy, but followers exploit it. In the model of Granato, Guse and Wong (2007) there is no learning about behavioral variables, but just one-way imitation. In sum, in all these works each agent is not aware of the existence of heterogeneity in the model.

where $[p_t]_a^{t-1}$ is the belief of agent a at time $t-1$ of p_t, and $\mathbf{E}[\cdot]$ is the mathematical expectation operator. Agents' welfare depends on their ability to correctly forecast on p_t.

This concludes the description of a very simple model often employed as benchmark in economic dynamics literature. For instance, it is consistent with the model proposed by Muth (1961) in his seminal paper on rational expectations or Lucas's framework in his 1973 famous paper. This will guarantee transparent comparability of the proposed approach with existing literature. Two substantial hypothesis will be assumed in the context of the above model, namely expectations are polarized and agents are epistemically isolated.

Polarized expectations. Standard perspective in macroeconomics is that all agents are equal and have equally negligible effect on aggregate variable. Nevertheless, we can image more realistically that individuals expectations are not independent. It should be not difficult to persuade the reader that the most part of agents doesn't actually have a theory on how economy works. On the other hand, only few agents in the economy are able to produce statistically correct expectations exploiting available information. A brief list of such professional forecasters would include market leaders, monetary and fiscal authorities, rating agencies and financial institutes. The latter are the only ones among agents holding procedural rationality and they act as focal points in front of the public.

Let' define formally this form of structural heterogeneity assuming institutional forecasters beliefs are formed according to

$$E_{t-1}^i p_t \equiv \mathbf{E}[p_t|\Omega_{t-1}^i], \quad \forall i \in A^* \tag{5}$$

with $A^* \subset A$ being the set of institutional forecasters in the economy where $E_{t-1}^i \in \mathcal{F}$. Professional forecasters maintain mathematical expectation of the generic process x_t conditioned to available information up to time $t-1$. Assumption (5) is a formal specification of procedural rationality. The set A^* has zero measure with respect the sigma-algebra generated by A. It is natural to think professional forecasters are very few because information processing presents strong scale economy effects. Differently, the public have the following expectation function specification

$$[p_t]_z^{t-1} = \widetilde{E}_{t-1}^z p_t = E_{t-1}^{i_z} p_t, \quad \forall z \in A/A^* \;\text{ with } i_z \in A^* \tag{6}$$

where $\widetilde{E}_{t-1}^z(\cdot) \in \mathcal{F}$ is nothing else then an imitation correspondence and i_z is the professional forecaster imitated by agent z.

If this working hypothesis is reasonable, agents expectations are polarized around few institutional forecasters' forecasts. Given (6) then the aggregate expectation in the economy is given by

$$E_{t-1}p_t = \sum_{i \in A^*} \lambda_i E_{t-1}^i p_t \ , \quad \sum_{i \in A^*} \lambda_i = 1 \tag{7}$$

where $\lambda_i \in (0, 1)$ represents the size of the public relying on agent i's expectation. The extent of agent i's basin of audience, represented by λ_i, measures the average impact of agent i's expectation on the aggregate expectations[5]. Equation (7) invalidates the negligibility of agents individual impact in the economy as assumed in general equilibrium perspective. In particular the impact of each institutional forecaster in the economy is proportional to its basin of audience. The arising of the unique REE depends uniquely on the game played by institutional forecasters.

Epistemic isolation hypothesis. Since the aim of the present work is to understand how and whether institutional forecasters can solve behavioral uncertainty in real time, we will assume they have perfect knowledge of the actual mechanism underlying the economy $M(\cdot)$ and they perfectly observe the relevant exogenous variables[6]. In this sense agents hold exogenous consistency as stated in last column of table 1. This is a standard hypothesis in eductive learning literature. Moreover we assume the set A^* and functional form of (7) to be known by agents in the economy. That is, institutional forecasters expectations are public information and their role as focal point is commonly acknowledged. What institutional forecasters don't know are behavioral data, namely how agents form their expectations and what are the arguments of their utility function. In this sense agents are epistemically isolated. They suffer a complete lack of behavioural consistency as noted in table 1.

Since institutional forecasters are perfectly informed about the structural form of the model and fundamentals, but not about other agents beliefs, they hold the following expectation function:

$$E^j_{t-1}p_t = (1-\beta)\,\overline{p} + \beta E^j_{t-1}p_t E_{t-1}p_t \tag{8a}$$

where

$$E^j_{t-1}E_{t-1}p_t = \sum_{i\in A^*} [\lambda_i]^{t-1}_j \, E^j_{t-1}E^i_{t-1}p_t \ , \quad \sum_{i\in A^*} [\lambda_i]^{t-1}_i = 1 \tag{9}$$

for each $j \in A^*$. The use of the generic beliefs operator $[\cdot]^{t-1}_j$ for λ_i is to emphasize that any optimal property is required on beliefs on audience shares for following arguments. Institutional forecasters are aware that the actual dynamics of the economy will be stuck around the REE if and only if all institutional forecasters have rational expectations (neglecting very unlikely events). Notice that differently from standard adaptive learning approach, we are explicitly assuming institutional forecasters acknowledge the self-referential nature of the economy. In this sense they are not bounded as usually assumed in adaptive learning literature. The first box of table 1 stresses this feature.

Institutional forecasters know fundamentals, but, as any agent in the economy, they are not interested in the fundamental *in se*. Their unique concern is to have consistent forecast of the actual course of the economy. To this aim institutional forecasters have to take in to account their own power of displacing the actual course far away from the rational expectations

[5]The ones filling uncomfortable with mere imitation implied by (6) can imagine more sophisticated expectation-takers agents weighting all expectations-makers' expectations. From this perspective λ_i would be interpreted as the public average consideration of agent i in expectations formation. This would not change the substance and mathematics of following arguments.

[6]The model includes the only constant as exogenous determinant. This is in sake of simplicity, but nothing changes in the result assuming a generic stochastic exogenous process in place of the only constant.

equilibrium. In particular, given procedural rationality and exogenous consistency they form expectations according to

$$E^j_{t-1}p_t = [M']^{t-1}_j(E^j_{t-1}E^{-j}_{t-1}p_t) = \frac{1-\beta}{1-\beta\,[\lambda_j]^{t-1}_j}\overline{p} + \frac{\beta}{1-\beta\,[\lambda_j]_j}\sum_{i\in A^*/\{j\}}[\lambda_i]^{t-1}_j\,E^j_{t-1}E^i_{t-1}p_t \quad (10)$$

for each $j \in A^*$ where $E^j_{t-1}E^{-j}_{t-1}p_t$ is a vector of agent j's expectations on $E^i_{t-1}p_t$ where $i \in A^*/\{j\}$ and $[M']^{t-1}_j(E^{-j}_{t-1}p_t)$ a function from $E^j_{t-1}E^{-j}_{t-1}p_t$ to $E^j_{t-1}p_t$. Function $[M']^{t-1}_j(E^j_{t-1}E^{-j}_{t-1}p_t)$ is a best response function conditioned to a given (possibly wrong) others' expectations profile. Notice that (10) depends strictly on the fact that agents hold procedural rationality and exogenous consistency and not at all on any common knowledge property. In other words, this framework properly embodies the meaning of Bayesian rationality.

Knowledge of simultaneous expectations. Given hypothesis above, what really matters for solving agents maximization problem is knowledge of others' simultaneous expectations, that is, the satisfaction of behavioral consistency. Let's devote this paragraph to shed light on this point.

In sake of clarity and to make following arguments analytically handy, let's assume form here onward that there are only two professional forecasters in the economy, namely $A^* = \{1, 2\}$.. Both hold a mirror-like expectation function

$$E^j_{t-1}p_t = \frac{1-\beta}{1-\beta\,[\lambda_j]^{t-1}_j}\overline{p} + \frac{\beta(1-[\lambda_j]_j)}{1-\beta\,[\lambda_j]_j}E^j_{t-1}E^i_{t-1}p_t \quad (11)$$

$\forall\ (i, j) \in A^{*2}$ and $i \neq j$. Equation (11) is just the two agent specification of (10). Now, rewrite agents' perceived law of motion (PLM) as

$$E^j_{t-1}p_t = \left(1-[\theta_j]^{t-1}_j\right)\overline{p} + [\theta_j]^{t-1}_j\,E^j_{t-1}E^i_{t-1}p_t \quad (12a)$$

$\forall\ (i, j) \in N^2$ where $\theta_j \equiv \beta\,(1-\lambda_j)\backslash(1-\beta\lambda_j)$.

Behavioral consistency holds whenever

$$E^j_{t-1}p_t \in \Omega^i_{t-1} \quad (13)$$

$\forall\ (i, j) \in N^2$ and $i \neq 2$, that means agent i knows agent j's simultaneous price expectation. Notice that $[p_t]^{t-1}_j$ is a $t-1$ datum and $E^j_{t-1}x_t \in \Omega^j_{t-1}$ is trivially always true. To see how (13) is really what agents need to optimally solve their problem let's assume no behavioral uncertainty, namely $\Omega^i_{t-1} = \Omega^j_{t-1}$. Iterated expectations law implies

$$E^j_{t-1}E^i_{t-1}p_t = \mathbf{E}[\mathbf{E}[p_t|\Omega^i_{t-1}]|\Omega^j_{t-1}] = E^i_{t-1}p_t \quad (14)$$

so that, solving the linear system (12) we have

$$E^1_{t-1}p_t = E^2_{t-1}p_t = \overline{p}$$

independently from $[\theta_1]_1^{t-1}$ and $[\theta_2]_2^{t-1}$. In other words, no matter if agents' beliefs on public audience shares are correct or not, knowledge of simultaneous expectations and of the model implies that the aggregate expectation is a rational expectation. Rearranging PLMs (20) as

$$E^1_{t-1}(p_t - \overline{p}) = [\theta_1]_1^{t-1}\left(E^2_{t-1}p_t - \overline{p}\right) \tag{15a}$$
$$E^2_{t-1}(p_t - \overline{p}) = [\theta_2]_2^{t-1}\left(E^1_{t-1}p_t - \overline{p}\right) \tag{15b}$$

it is now clear from (15) that $[\theta_i]_i^{t-1}$ doesn't matter if and only if $E^j_{t-1}p_t = \overline{p}$. If this is not the case, the extent of displacements of actual dynamics from the steady state do will depends on $[\theta_i]_i^{t-1}$. The two agents case can be easily extended to a numerable set of agents without changing conclusions. This is not surprising since if agents are procedural rational, they hold exogenous consistency and behavioral consistency then they maintain rational expectations.

3.2 Dealing with behavioral Uncertainty

Condition (13) is a very strong requirement. Knowledge of simultaneous expectation (actions) is the implicit hypothesis underlying a "Robinson Crusoe" economy populated by a single representative agent, in which every coordination failure is avoided by externalities internalization. To introduce behavioral uncertainty in the model is to specify an epistemic system in which (13) is violated. Breaking (13) generates naturally agents' heterogeneity in the economic system. This kind of heterogeneity doesn't require neither different forecasting functions (structural heterogeneity) nor different priors, but it arises from the only fact that agents are separated individuals so that they cannot have instantaneous knowledge of simultaneous actions. We will refer to this kind of heterogeneity as intrinsic heterogeneity.

Intrinsic heterogeneity. Instead of considering agent instantaneous perfect knowledge of others' simultaneous expectations, as stated in (14), let's assume more realistically that they commit observational errors in detecting others' simultaneous expectations. It is possible that professional forecasters share a lot of public information and they know each other so well that they are able to predict how the other will react to a certain shock. Nevertheless agents can apply margins of subjective interpretation to public information. The simultaneous nature of their expectations makes them to generally fail to have instantaneous perfect information of what the other is expecting at the same time. Suppose two individuals namely i and j. At time $t-1$ it is asked agent i to write down: i) her own expectation on price at time t, labeled $E^i_{t-1}p_t$ and ii) her own expectations on agent j's analogous expectation denoted by $E^i_{t-1}E^j_{t-1}p_t$. The same is asked to agent j. Observational error of agent i (j) on agent j (i)'s expectation is measured by $E^i_{t-1}E^j_{t-1}p_t - E^j_{t-1}p_t$ ($E^j_{t-1}E^i_{t-1}p_t - E^i_{t-1}p_t$). We are assuming this error to be a stochastic component drawn from a generic distribution.

Formally intrinsic heterogeneity is stated by

$$\left(E^i_{t-1}E^j_{t-1}p_t - E^j_{t-1}p_t\right) \equiv v_i \sim \Upsilon\left(\mu_i, \delta_i\right) \tag{16}$$

with the following specifications

$$\begin{aligned} v_i &= \mu_i + \sqrt{\varrho_i \overline{p}}\epsilon_{i,t} && \forall i \in A^* && \text{(17a)} \\ v_z &= \mu_z + \sqrt{\varrho_z \overline{p}}\left(\gamma_z(E^{i_z}_{t-1}p_t - \overline{p}) + (1-\gamma_z)\,\epsilon_{z,t}\right) && \forall z \in A/A^*,\ i \in A^* && \text{(17b)} \end{aligned}$$

where $\Upsilon(m,\delta)$ is a generic distribution function with finite mean m and finite variance δ, $\varrho_a \in (0,1)$ is a scale factor, μ_a denotes a drift in the agent a's perception of an other agent's simultaneous expectation.

In particular we assume $\epsilon_{i,t} \sim \Upsilon\left(0,1\right)$ and $\epsilon_{z,t} \sim \Upsilon\left(0, \left(1-\gamma_z^2 \mathbf{E}(E^{i_z}_{t-1}p_t - \overline{p})^2\right)/\left(1-\gamma_z\right)^2\right)$ so that $\mathbf{E}(v_a^2) = \varepsilon_a = \left(1+\mu_a^2\right)\ \forall a \in A$.

The specific forms of observational errors as maintained by (17) don't add nothing of substantial in the general framework. The factor $\sqrt{\varrho_a \overline{p}}$ just states the variance of expectations errors increases linearly with the level of the fundamental at a rate of ϱ_a.Then γ_z controls for the covariance between the observational error made by the public, and the distance of the imitated expectation from the fundamental. Notice the latter is a proxy for the amount of behavioural uncertainty in the economy. In other word, this specification takes account of the idea that the public receives a biased information whose distortion is possibly further amplified during the transmission. Finally assume $\mu_a, \gamma_a, \varrho_a$ are drawn from a normal distribution whose mean is respectively μ, γ, ϱ and variance δ_μ, δ_γ, δ_ϱ. Notice that

$$\int_{a\in A} v_{a,t}\ da = \mu + \sqrt{\varrho\overline{p}}\gamma\left(\sum_{i\in A^*}\lambda_i(E^i_{t-1}p_t - \overline{p})\right), \tag{18}$$

so that the aggregate expectation of $E_{t-1}p_t$, is now equal to

$$E_{t-1}p_t = \overline{p} + \mu + \left(1 + \sqrt{\varrho\overline{p}}\gamma\right)\sum_{i\in A^*}\lambda_i\left(E^i_{t-1}p_t - \overline{p}\right). \tag{19}$$

Coming back to our two professional forecasters example, condition (16) adds a very simple but meaningful change in individual PLMs. They become

$$\begin{aligned} E^1_{t-1}\left(p_t - \overline{p}\right) &= \left[\theta_1\right]_1^{t-1}\left(E^2_{t-1}p_t + v_{1,t-1} - \overline{p}\right) && \text{(20a)} \\ E^2_{t-1}\left(p_t - \overline{p}\right) &= \left[\theta_2\right]_2^{t-1}\left(E^1_{t-1}p_t + v_{2,t-1} - \overline{p}\right) && \text{(20b)} \end{aligned}$$

The observational error is independent from the noisy observation, from the true value, across agents[7] and with the exogenous noise of the model. This specification is very general and can

[7] Notice that since the error is correlated neither with $E^{*i}_{t-1}E^{*j}_{t-1}p_t$ nor with $E^{*j}_{t-1}p_t$, v_i is a measurement error that doesn't affect the consistency and variance of the coefficient estimates.

account for very different specification of the basic problem. We don't enquiry how happened agents suffer a measurement error or rather they have biased information about others' expectations. We don't assess how agents can correct systematic errors (if any). Assumption (16) just gives a general specification of failure of (13).

Now we can formally state the information set held by institutional forecaster i at time t, namely $\Omega_{i,t} \equiv (M(\cdot),\ \{p_\tau\}_{\tau=0}^{t-1}\ ,\ \{E^j_{\tau-1}p_t + v_{i,\tau-1}\}_{\tau=1}^{t-1})$ composed by information about the exogenous determinants of the economy, the story of observed prices and the story of observed other's expectation up to time $t-1$. They can forecast either the rational expectation $\overline{p}$ or a conditioned expectation $[M']_i^{t-1}(E^j_{\tau-1}p_t + v_{i,t-1})$.

What institutional forecasters have to learn? If at least one institutional forecaster has not rational expectations, agents will observe actual deviations from the rational expectation $\overline{p}$ that are not due to the exogenous unpredictable noise of the model η_t. Hence, they can test if such deviations are correlated with the noisy observation they have of others simultaneous expectations. Noisy observations defined by (16) are the best proxy for agents' actual behavior. Remember that agent i cannot disentangle $E^j_{t-1}p_t$ form his perception of agent j's simultaneous expectation $E^j_{t-1}p_t + v_i$ since $E^{*i}_{t-1}p_t \notin \Omega^j_{t-1}$, that is, intrinsic heterogeneity is in play. It's important to clarify that agents cannot learn to avoid observational errors defined by (**??**). Those errors have more an ontological root rather then an epistemic one. That is, heterogeneity of agents is defined through (16).

Call $b = [\theta_1]_1^{t-1}$ and $c = [\theta_2]_2^{t-1}$. Agents 1 and 2 can estimate respectively b and c regressing data according to the following structural equations derived from 20

$$E^1_{t-1}\left(p_t - \overline{p}\right) \quad = \quad b\left(E^2_{t-1}p_t + v_{1,t-1} - \overline{p}\right) \tag{21a}$$

$$E^2_{t-1}\left(p_t - \overline{p}\right) \quad = \quad c\left(E^1_{t-1}p_t + v_{2,t-1} - \overline{p}\right) \tag{21b}$$

If $(b,c) = (0,0)$ dependence from noisy observation of simultaneous expectations is rejected agents will forecast the fundamental value, so that aggregate expectation will be a rational expectations. In other words, if all institutional forecasters are rational they would not need to condition their expectations on noisy observations of the other one's simultaneous expectations. But if this is not the case considering noisy observations do improve the accuracy of forecasts. Learning is valuable exactly because this form of behavioral uncertainty is introduced.

To see in detail the mechanism let's write down the problem agent 1 wants to solve[8]. Let's abstract for a moment from v_z setting $v_z = 0\ \forall z \in A/A^*$. Firstly consider the case agent 2 maintains rational expectations. Agent 1's maximization problem is

$$\min_b U_1 = \mathbf{E}\left(p_t - E^1_{t-1}p_t\right)^2, \tag{22}$$

[8]Agent 2's problem is mirror-like.

that, given (1)-(21a) and $E^2_{t-1}p_t = \overline{p}$, becomes

$$\min_b U_1 = \delta_\eta + (\beta(1-\lambda_2)-1)^2 b^2 \varepsilon_1^2 \tag{23}$$

whose solution is $\widehat{b}_{RE} = 0$. Differently, assume agent 2 deviates form RE, formally $E^2_{t-1}p_t = \overline{p} + \kappa_t$, where $\kappa_t \sim \Upsilon(\mu_\kappa, \delta_\kappa)$ and κ_t is independent form b, then solution to (22) is given by

$$\widehat{b} = \frac{\beta\lambda_2 (\delta_\kappa + \mathbf{E}(v_{1,t-1}\ \kappa_t))}{(1-\beta(1-\lambda_2))(\delta_\kappa + 2\mathbf{E}(v_{1,t-1}\ \kappa_t) + \varepsilon_1)} \tag{24}$$

where if $\delta_\kappa \neq 0$ or $\mathbf{E}(v_{1,t-1}\ \kappa_t) \neq 0$ then $\widehat{b} \neq 0$. Notice that if $\varepsilon_1 = 0$ but $\delta_\kappa \neq 0$ then $\widehat{b} = \theta_1$; finally, if $\varepsilon_1 \neq 0$ but $\delta_\kappa = 0$ then $\widehat{b} = 0$. Therefore, using (**??**) to regress available data, agents can estimates respectively b and c - i.e. correlations between actual displacements from the fundamental value and noisy observation of other one's simultaneous expectations. Those parameters measures also the degree of optimal reaction to others' deviation form rationality. If $(b, c) = (0, 0)$ institutional forecasters act as if they held knowledge of simultaneous expectations.

Behavioral Sunspot Equilibria. Looking at equations (21) it is clear that both expectations differ from the fundamental value for a stochastic component. In case of $\mu_i = 0$ this stochastic component has the form of a martingale. Rational expectation plus a matingale is the typical form of a sunspot solution. We know that statistical learning is able to discard sunspot solution as long as common factor representation is not used[9]. This comes form the fact that RLS-algorithm solves recursively problem (22) exactly as we just did analytically.

The problem for agents is that until the process of learning about others' rationality is not successful completed (that is $(b, c) = (0, 0)$), even if $\mu = 0$ (that is observational errors are not systematic) they hold non rational expectation of the form

$$E^1_{t-1}p_t = \overline{p} + \kappa_{1,t}(b, c, v_{1,t-1}, v_{2,t-1}) \tag{25a}$$
$$E^2_{t-1}p_t = \overline{p} + \kappa_{2,t}(b, c, v_{1,t-1}, v_{2,t-1}) \tag{25b}$$

where $\kappa_{i,t}(0, 0, v_{1,t-1}, v_{2,t-1}) = 0$. The system entails an interaction between agents for which REE ($(b, c) = (0, 0)$) is an equilibrium. Nevertheless other equilibria could exist such that

$$\widehat{b} = \frac{\beta\lambda_2 \left(\delta_{\widehat{\kappa}_{2,t}} + \mathbf{E}(v_{1,t-1}\widehat{\kappa}_{2,t})\right)}{(1-\beta(1-\lambda_2))\left(\delta_{\widehat{\kappa}_{2,t}} + 2\mathbf{E}(v_{1,t-1}\widehat{\kappa}_{2,t}) + \varepsilon_1\right)} \tag{26a}$$

$$\widehat{c} = \frac{\beta(1-\lambda_2)\left(\delta_{\widehat{\kappa}_{1,t}} + \mathbf{E}(v_{2,t-1}\widehat{\kappa}_{1,t})\right)}{(1-\beta\lambda_2)\left(\delta_{\widehat{\kappa}_{1,t}} + 2\mathbf{E}(v_{2,t-1}\widehat{\kappa}_{1,t}) + \varepsilon_2\right)} \tag{26b}$$

where $\widehat{\kappa}_{i,t} \equiv \kappa_{i,t}(\widehat{b}, \widehat{c}, v_{1,t-1}, v_{2,t-1})$ and $(\widehat{b}, \widehat{c}) \neq (0, 0)$.

Such equilibria are defined behavioral sunspots equilibria (BSEs). The word sunspot refers to the presence of a extra stochastic component in agents' expectation function, namely the

[9] For a comprehensive understanding of this mechanism see Evans, McGough (2007).

observational errors. Both in BSEs and classical sunspot equilibria there are additional variables uncorrelated with any exogenous variable from the reduced form model, but they differ for the kind of belief required to self-sustain them. In classical sunspot equilibria is required agents to hold common knowledge that a certain observable extra variable affects the equilibrium. Differently, BSEs require all agents must believe[10] that displacements of the actual price from the fundamental one depends on others' deviation from rational expectations. Next section will explore the robustness of REE learneability and the possibility of adaptive learners being stuck in a BSE.

4 Equilibria and Learning Analysis

4.1 The Dynamic System

From (21) it is simple to check PLMs can be expressed as linear function of the fundamental price and observational errors. Formally we have

$$E^1_{t-1}p_t = \overline{p} + \frac{b_{t-1}c_{t-1}}{1-b_{t-1}c_{t-1}}v_{2,t-1} + \frac{b_{t-1}}{1-b_{t-1}c_{t-1}}v_{1,t-1} \tag{27a}$$

$$E^2_{t-1}p_t = \overline{p} + \frac{b_{t-1}c_{t-1}}{1-b_{t-1}c_{t-1}}v_{1,t-1} + \frac{c_{t-1}}{1-b_{t-1}c_{t-1}}v_{2,t-1} \tag{27b}$$

provided $bc \neq 1$. Processes (27) cannot be inferred by agents since they cannot disentangle observational errors. According to (1) and (19), specifications (27) makes the actual law of motion to follow the following process

$$p_t = \overline{p} + \beta^* \left(\frac{b_{t-1}\left(\lambda + (1-\lambda)c_{t-1}\right)}{1-b_{t-1}c_{t-1}}v_{1,t-1} + \frac{c_{t-1}(\lambda b_{t-1} + (1-\lambda))}{1-b_{t-1}c_{t-1}}v_{2,t-1} \right) + \beta\mu + \eta_t \tag{28}$$

where $\beta^* \equiv \beta(1+\overline{p}\sqrt{\varrho}\gamma)$. Notice that $p_t = \overline{p}$ if i) $v_{1,t-1} = 0$ and $v_{2,t-1} = 0$, or ii) $b^1_{t-1} = 0$ and $c_{t-1} = 0$, or iii) $\beta = 0$.

4.2 Equilibria

First order conditions of the minimization problem (22) for both agents are given by

$$\mathbf{E}[\phi^1_{t-1}\left(p_t - \overline{p} - T_b\phi^1_{t-1}\right)] = 0 \tag{29}$$

$$\mathbf{E}[\phi^2_{t-1}\left(p_t - \overline{p} - T_c\phi^2_{t-1}\right)] = 0 \tag{30}$$

with $\phi^i_{t-1} = E^j_{t-1} + v_i - \overline{p}$, where $T.$ map is a function giving the coefficients of the linear forecast of p_t with the smaller mean square error variance conditioned on the available information set. For

[10] Notice that it is not required that they believe that others believe.

a technical reference on projections and convergence properties of adaptive learning algorithms used in what follows, see Marcet and Sargent (1989).

Proposition 2 *T-map takes the form*

$$T_b = \beta^* \left(\frac{b\left(\lambda + (1-\lambda)c\right)\left(\varepsilon_1 + c\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right) + c(\lambda b + (1-\lambda))\left(c\varepsilon_2 + \sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)}{\varepsilon_1 + \varepsilon_2 c^2 + 2\overline{p}\sqrt{\varepsilon_1\varepsilon_2}\rho_v c} \right) \tag{31}$$

$$T_c = \beta^* \left(\frac{b\left(\lambda + (1-\lambda)c\right)\left(b\varepsilon_1 + \sqrt{\varepsilon_1\varepsilon_2}\rho_v\right) + c(\lambda b + (1-\lambda))\left(\varepsilon_2 + b\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)}{\varepsilon_2 + \varepsilon_1 b^2 + 2\overline{p}\sqrt{\varepsilon_1\varepsilon_2}\rho_v b} \right) \tag{32}$$

Proof. See appendix A1. ■

Notice T-map depends on error variances ratio $\varepsilon_1/\varepsilon_2$ and not at all on the extent of them. Moreover if $\varepsilon_1 = \varepsilon_2$ errors variances simply disappear from equations. Fix points of the T-map give the optimal (b, c) such that professional forecasters don't commit systematic error given available information[11].

Equilibria of the learning process are rest points of T-maps ($T_b = b, T_c = c$). They are given by the system:

$$b = \frac{\beta^* c(1-\lambda)(c\varepsilon_2 + \sqrt{\varepsilon_1\varepsilon_2}\rho_v)}{((1-\beta^*\lambda)\varepsilon_2 - \beta^*(1-\lambda)\sqrt{\varepsilon_1\varepsilon_2}\rho_v)c^2 + (2(1-\beta^*\lambda)\sqrt{\varepsilon_1\varepsilon_2}\rho_v - \beta^*(1-\lambda)\varepsilon_1)c + (1-\beta^*\lambda)\varepsilon_1} \tag{33a}$$

$$c = \frac{\beta^* b\lambda(b\varepsilon_1 + \sqrt{\varepsilon_1\varepsilon_2}\rho_v)}{((1-\beta^*(1-\lambda))\varepsilon_1 - \beta^*\lambda\sqrt{\varepsilon_1\varepsilon_2}\rho_v)b^2 + (2(1-\beta^*(1-\lambda))\sqrt{\varepsilon_1\varepsilon_2}\rho_v - \beta^*\lambda\varepsilon_2)b + (1-\beta^*(1-\lambda))\varepsilon_2} \tag{33b}$$

assuming $bc \neq 1$. It is easily proved by substitution that the fundamental rational expectation solution $(0, 0)$ it is always a rest point of the T-map.

Other non fundamental solutions of the system depend on the parameters λ, $\varepsilon_1, \varepsilon_2$. Unfortunately is not possible to provide a closed form solution for all values. To assess the existence of BSEs we focus on the simple symmetric agents case $\lambda = 1/2$, $\varepsilon_1 = \varepsilon_2 = \varepsilon$. In such case we restrict the domain of possible solutions to the one dimensional space where $b = c$. This is evident from inspection of equations (33). The system (33) is now expressed by

$$c\left(c^2\left((1-\beta^*/2) - (\beta^*/2)\,\rho_v\right) - \left(\beta^* - (2-\beta^*)\,\rho_v\right)c + (1-\beta^*/2) - (\beta^*/2)\,\rho_v\right) = 0 \tag{34}$$

[11] Notice that agents could commit systematic errors because they cannot disentangle an eventual drift in perception of other agent's simultaneous expectations. In such cases institutional forecasters optimally exploit biased information, they commit sistematic error not because they are bounded rational but because they have distorted information. No assumption in this model implies that professional forecasters have a drift on others' simultaneous expectations. The framework is shaped in such a way to be as general as possible. I don't want to discuss the possibility agents have distorted information, but I just assume agents use optimally the information they have.

whose solutions other than $(0,0)$ are (b_+, c_+), (b_-, c_-) where

$$c_+ = b_+ = \frac{\beta^* - (2-\beta^*)\rho_v + 2\sqrt{(\beta^*-1)(1-\rho_v^2)}}{2-\beta^*-\beta^*\rho_v} \tag{35}$$

$$c_- = b_- = \frac{\beta^* - (2-\beta^*)\rho_v - 2\sqrt{(\beta^*-1)(1-\rho_v^2)}}{2-\beta^*-\beta^*\rho_v} \tag{36}$$

that is, at least one non fundamental solution arises for $\beta^* \geq 1$. For a continuity argument we can state the following proposition.

Proposition 3 *The fundamental solution is always a solution of the system. It is not the unique one in at least a not trivial open region in the coefficient space $(\beta^*, \lambda, \varepsilon_1, \varepsilon_2, \rho_v)$.*

BSEs take shape as a coordination failure. After a certain threshold, for agent j it is a best action to condition her expectations to noisy observation of agent i's expectation, given agent i is doing the mirror-like action. More specifically, an agent conditioning her own expectation on noisy signal of the other agent's expectation results to be as if it was "irrational exuberant"; in turn, an agent testing irrational exuberance of the other agent has incentive to condition her own expectation to the noisy observation of the other agent's expectations. This way observational errors variance transmits to the actual price course through aggregate expectation. This mechanism could be an explanation on how excess variance phenomena can be generated by behavioral uncertainty only. The exercise at hand shows that this issue is relevant even if agents know everything about the exogenous determinants of the economy.

Remark 4 *Notice that BSE are kind of limited-informed rational expectations (Sargent 1991) in that agents cannot observe separately all the stochastic components of the actual law of motion. However, differently from the standard setting proposed by Sargent (1991), here the subjective information set $\Omega_{i,t}$ does not result in a subset of the objective one $\Omega_t = (M(\cdot), \{p_\tau\}_{\tau=0}^{t-1}, \{v_{1,\tau-1}\}_{\tau=1}^{t-1}, \{v_{2,\tau-1}\}_{\tau=1}^{t-1})$. Rather the former is a coarse set in that observational errors appear as correlated by non avoidable non linear constraints resulting from the system (21). This feature is at the basis of non linearity of the T map and the emergence of BSEs.*

Behavioral sunspot solutions ($b \neq 0$, $c \neq 0$) have the feature to include behavioral variables, namely observational errors, among the ones affecting actual price. Behavioural sunspot equilibria would present, as any sunspot solution, a variance higher than the fundamental solution, but they can eventually present also a drift. The extent of the drift is given by

$$\beta^* \left(\frac{b\,(\lambda + (1-\lambda)c)}{1-bc}\mu_1 + \frac{c(\lambda b + (1-\lambda))}{1-bc}\mu_2 \right) + \beta\mu \tag{37}$$

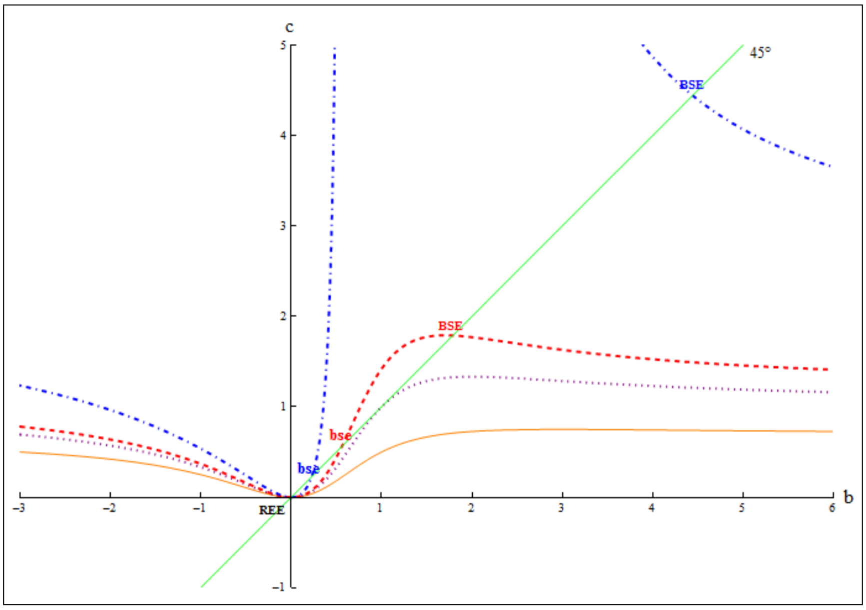

Figure 1: T-map for the case of two simmetrical agents ($\lambda = 0.5$ and $\varepsilon_1 = \varepsilon_2$) and no correlation between observational errors $\rho_v = 0$. Curves are obtained for different values of β^*, respectively: 0.8 normal, 1 dotted, 1.08 dashed , 1.4 dotted-dashed. "BSE" denotes the high behavioral Sunspot Equilibrium and "bse" the low one.

whereas the excess variance is measured by

$$\beta(1 + 2\overline{p}\sqrt{\varrho}\gamma)^2 \left(\frac{b^2 \left(\lambda + (1-\lambda)c\right)^2}{(1-bc)^2} \overline{p}\varrho_1 + \frac{c^2(\lambda b + (1-\lambda))^2}{(1-bc)^2} \overline{p}\varrho_2 \right) +$$
$$+ \beta(1 + 2\overline{p}\sqrt{\varrho}\gamma)^2 \left(2\frac{bc(\lambda b + (1-\lambda)) \left(\lambda + (1-\lambda)c\right)}{(1-bc)^2} \left(\sqrt{\varepsilon_1\varepsilon_2}\rho_v - \mu_2\mu_1\right) \right)$$

and it is increasing in $\beta, b, c, \overline{p}\varrho_1$ and $\overline{p}\varrho_2$ and ρ_v.

In figure 1 it is plotted the right hand side of equation (33a) with $\lambda = 0.5$ and $\varepsilon_1 = \varepsilon_2$ and $\rho_v = 0$ for different values of β^*, namely $0.8, 1, 1.08, 1.4$, in the space (b, c). The intersection between this curve and the bisector identifies the rest point of T-map in case of two symmetrical professional forecaster having equal sized basin of audience.

As proved analytically equilibria different from REE ($b = 0, c = 0$) arise for $\beta^* \geq 1$. As β^* increases two BSEs originate, a "low" one ($b_- = c_- < 1$) and an "high" one ($b_+ = c_+ > 1$) where

$$c_+ = b_+ = \frac{\beta^* + 2\sqrt{(\beta^* - 1)}}{2 - \beta^*} \tag{38a}$$

$$c_- = b_- = \frac{\beta^* - 2\sqrt{(\beta^* - 1)}}{2 - \beta^*} \tag{38b}$$

so that, for $\beta^* = 1$, $b_- = c_+ = 1$. The reader can easily prove[12] $\mathrm{d}c_+ \backslash \mathrm{d}\beta^* > 0$ and $\mathrm{d}c_+ \backslash \mathrm{d}\beta^* < 0$. Since $\lim_{\beta^* \to 2_-} b_+ = +\infty$, and $\lim_{\beta^* \to 2_-} b_- = 0$, the low (high) BSE is monotonically decreasing (increasing) in β^* and it collapses to REE (goes to infinitum) as β^* approaches 2.

4.3 Learneability

The concept of learneability refers to the nature, stable or unstable, of the learning dynamics around the equilibria computed above. In particular, to check learneability we must to investigate the Jacobian of the T-map. If the matrix of all partial derivative of T map for the solution at hand has all eigenvalues in the unit root, we can say the solution to be stable under learning. The Jacobian for T-map takes the form

$$JT(b,c) = \begin{pmatrix} \frac{dT_b(b,c)}{db} & \frac{dT_b(b,c)}{dc} \\ \frac{dT_c(b,c)}{db} & \frac{dT_c(b,c)}{dc} \end{pmatrix}$$

where

$$\frac{\mathrm{d}T_b(b,c)}{\mathrm{d}b} = \beta^* \left(\frac{(\lambda + (1-\lambda)c)\left(\varepsilon_1 + c\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right) + c\lambda\left(c\varepsilon_2 + \sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)}{\varepsilon_1 + \varepsilon_2 c^2 + 2c\sqrt{\varepsilon_1\varepsilon_2}\rho_v} \right) \tag{39a}$$

$$\begin{aligned} \frac{\mathrm{d}T_b(b,c)}{\mathrm{d}c} = {} & \beta^* \frac{b\left(\varepsilon_1 + c\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)(1-\lambda) + b\sqrt{\varepsilon_1\varepsilon_2}\rho_v\left(c(1-\lambda)+\lambda\right) + \left(2c\varepsilon_2 + \sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)(1-\lambda+\lambda b)}{\varepsilon_1 + \varepsilon_2 c^2 + 2c\sqrt{\varepsilon_1\varepsilon_2}\rho_v} - \\ & - \beta^* \frac{\left(2c\varepsilon_2 + 2\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)\left(b\left(\varepsilon_1 + c\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)\left(c(1-\lambda)+\lambda\right) + c\left(c\varepsilon_2 + \sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)(1-\lambda+\lambda b)\right)}{\left(\varepsilon_1 + \varepsilon_2 c^2 + 2c\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)^2} \end{aligned} \tag{39b}$$

$$\begin{aligned} \frac{\mathrm{d}T_c(b,c)}{\mathrm{d}b} = {} & \beta^* \frac{c\left(\varepsilon_2 + b\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)\lambda + c\sqrt{\varepsilon_1\varepsilon_2}\rho_v\left(b\lambda + 1 - \lambda\right) + \left(2b\varepsilon_1 + \sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)\left(\lambda + (1-\lambda)c\right)}{\varepsilon_2 + \varepsilon_1 b^2 + 2b\sqrt{\varepsilon_1\varepsilon_2}\rho_v} - \\ & - \beta^* \frac{\left(2b\varepsilon_1 + 2\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)\left(c\left(\varepsilon_2 + b\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)\left(b\lambda + 1 - \lambda\right) + b\left(b\varepsilon_1 + \sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)\left(\lambda + (1-\lambda)c\right)\right)}{\left(\varepsilon_2 + \varepsilon_1 b^2 + 2b\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)^2} - \end{aligned} \tag{39c}$$

$$\frac{\mathrm{d}T_c(b,c)}{\mathrm{d}c} = \beta^* \left(\frac{(1-\lambda+\lambda b)\left(\varepsilon_2 + b\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right) + b(1-\lambda)\left(b\varepsilon_1 + \sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)}{\varepsilon_2 + \varepsilon_1 b^2 + 2b\sqrt{\varepsilon_1\varepsilon_2}\rho_v} \right) \tag{39d}$$

Proposition 5 *Let's define*

$$K_{(\bar{b},\bar{c})} \equiv JT(b,c)|_{(\bar{b},\bar{c})} - \mathbf{I}_n \tag{39}$$

where $\mathbf{I}_n$ is a $n \times n$ identity matrix. If $K_{(\bar{b},\bar{c})}$ has negative eigenvalues then learneability of $(\bar{b},\bar{c})$ holds.

[12] Exploit the fact that $\beta - 2\sqrt{(\beta-1)} > 0$ is alwais true.

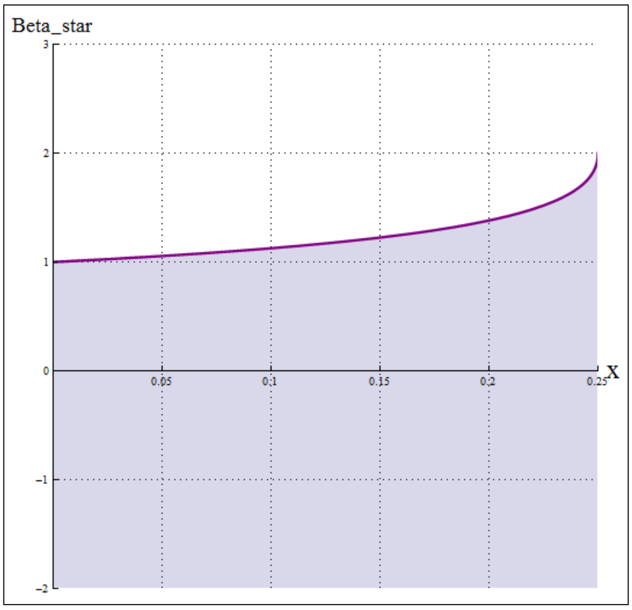

Figure 2: REE Learneability. The shadow area denotes the learneability region in the coefficient space $(\beta^*, \rho_v, \lambda)$ for any pair $(\varepsilon_1, \varepsilon_2)$ where $x \equiv \lambda(1-\lambda)(1-\rho_v^2)$.

Fundamental rational expectations equilibrium matrix K is given by

$$K_{(0,0)} = \begin{pmatrix} \beta^*\lambda - 1 & \beta^*(1-\lambda)\frac{\sqrt{\varepsilon_1\varepsilon_2}\rho_v}{\varepsilon_1} \\ \beta^*\lambda\frac{\sqrt{\varepsilon_1\varepsilon_2}\rho_v}{\varepsilon_2} & \beta^*(1-\lambda) - 1 \end{pmatrix}. \tag{41}$$

We can easily prove the following

Proposition 6 *The beliefs dynamics converges asymptotically to the REE solution* $(b, c) = (0, 0)$ *whenever*

$$\rho_v < 1 \quad and \quad \beta^* \leq \frac{1 - \sqrt{1 - 4\lambda(1-\lambda)(1-\rho_v^2)}}{2\lambda(1-\lambda)(1-\rho_v^2)} \tag{42}$$

where $\rho_v < 1$.

Proof. See appendix A2. ■

The region under the bold curve in figure 2 represents the learneability region for the fundamental rational expectations equilibrium. Notice that For $\mu_i = 0$, we have ρ_v is the correlation ratio between observational errors of the institutional forecasters. Moreover for $\rho_v = 0$ (that implies $\mu_i = 0$) condition (42) collapses to

$$Max[\beta^*\lambda, \beta^*(1-\lambda)] < 1. \tag{43}$$

These results give us conditions under which behavioral consistency can be filled for the model at hand. Notice again that (42) does not depend on second moment absolute values of observational errors distributions but only on the ratio between them. Remember that such observational errors were defined in a very general way. In particular REE learneability holds in a region bigger than $\beta < 1$ even if observational errors are correlated or they have a systematic bias. Nevertheless learneability of REE is never global.

Now let's analyse learneability of BSEs. Substituting for $\lambda = 1/2$, $\varepsilon_1 = \varepsilon_2$, matrix $K_{(\bar{b},\bar{c})}$ becomes

$$K_{(\bar{b},\bar{c})} = \begin{pmatrix} \frac{((\beta^*/2)(1+\rho_v)-1)\bar{b}^2+((\beta^*/2)(1+2\rho_v)-2\rho_v))\bar{b}+(\beta^*/2)-1}{1+\bar{b}^2+2\bar{b}\rho_v} & (\beta^*/2)\frac{(2\rho_v^2-1)\bar{b}^3+3\rho_v\bar{b}^2+3\bar{b}+\rho_v}{\left(1+\bar{b}^2+2\bar{b}\rho_v\right)^2} \\ (\beta^*/2)\frac{(2\rho_v^2-1)\bar{b}^3+3\rho_v\bar{b}^2+3\bar{b}+\rho_v}{\left(1+\bar{b}^2+2\bar{b}\rho_v\right)^2} & \frac{((\beta^*/2)(1+\rho_v)-1)\bar{b}^2+((\beta^*/2)(1+2\rho_v)-2\rho_v))\bar{b}+(\beta^*/2)-1}{1+\bar{b}^2+2\bar{b}\rho_v} \end{pmatrix} \tag{44}$$

where $\bar{b} = \bar{c}$. A certain equilibrium $\left(\bar{b}, \bar{c}\right)$ is learnable if and only if the matrix $K_{(\bar{b},\bar{c})}$ has all negative eigenvalues.

Proposition 7 *For $\lambda = 1/2$, $\varepsilon_1 = \varepsilon_2$ and $\rho_v = 0$, whenever solution (b_+, c_+) exists it is the unique learnable BSE.*

Proof. See appendix A3. ■

According to a continuity argument we can extend the statement on learneability of BSE in the model at hand for a non trivial open region of the parameters space $(\beta^*, \lambda, \varepsilon_1, \varepsilon_2, \rho_v)$. From numerical analysis in figure 3 we can visualize learneability region of both BSEs in case of symmetrical agents for different values of ρ_v. The analytical finding in correspondence of $\rho_v = 0$ are confirmed. Nevertheless a positive correlation can make the lower BSE to be learnable. With perfect positive correlation the proposition above is overturned. Finally a negative correlation can generate a learnable high BSE even for β^* bigger than 2. Finally for perfect negative correlation no BSE is learnable.

4.4 Real Time Simulations with Constant Stochastic Gradient

This section is devoted to the real time simulation of the dynamic system at hand. The aim is to show convergence to learnable equilibria and how, in case of multiplicity, unpredictable shifts from the REE to BSE can occur endogenously.

First of all let's give intuition of how differently sized basins of attraction are determined in case of multiple learnable equilibria. Figure 1 give us the opportunity to visualize the extent of basins of attraction restricted to the space $b = c$ with $\rho_v = 0$. In two cases ($\beta^* = 1.08$ and $\beta^* = 1.4$) two learnable equilibria exist - the rational expectation equilibrium and the high BSE. In both cases the low BSE (labeled "bse") is the unstable equilibrium under learning dynamics.

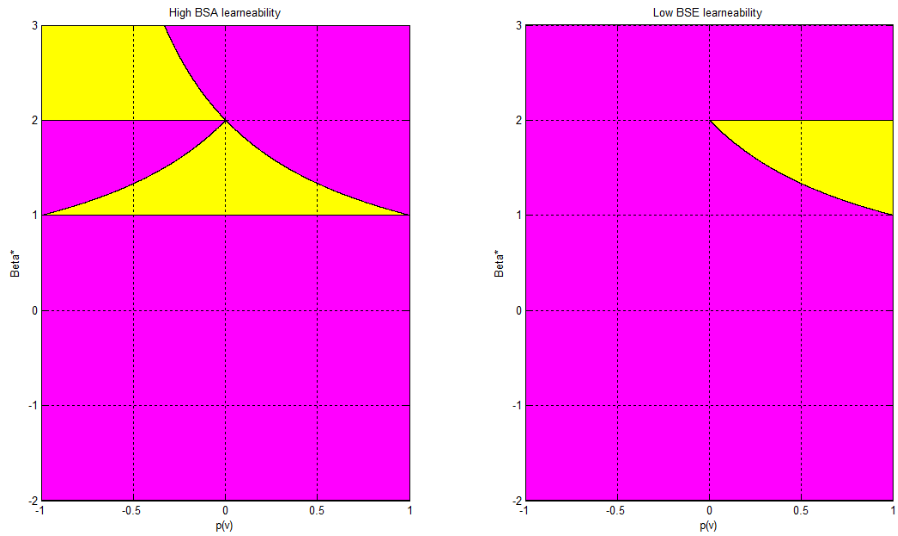

Figure 3: BSEs learneability. Clear area denotes learneability region in the coefficient space (β^*, ρ_v) in case of simmetric agents $(\varepsilon_1 = \varepsilon_2,\ \lambda = 0.5)$.

Therefore the latter identifies the threshold between the basins of attractions of the rational expectation equilibrium and the high BSE along the bisector. Notice that only initially, for β^* very close to unity, the rational expectations equilibrium is the dominant one in that it has a bigger basin of attraction.

To simulate the dynamic system I chose to use the Constant Stochastic Gradient algorithm (CSG)

$$\begin{aligned} b_t &= b_{t-1} + \gamma[E^2_{t-1} - \overline{p} + v_{1,t-1}](p_{t-1} - E^1_{t-1}) \\ c_t &= c_{t-1} + \gamma[E^1_{t-1} - \overline{p} + v_{2,t-1}](p_{t-1} - E^2_{t-1}) \end{aligned}$$

where, differently from the recursive version of OLS, the estimated correlation matrix is settled equal to the identity matrix and the gain is fix. CSG converges to an ergodic distribution centred on the rest point of the T-map in the same cases recursive OLS would punctually converge. CSG exhibits permanent learning since a bigger weight is given to more recent data. This feature is common to all constant gain algorithm. This makes these class of algorithm to be particularly suitable for learning structural changes. Recursive OLS on the contrary converges at the cost of a huge stickiness of the dynamics for t big enough. Moreover CSG algorithm are also derived as optimal solution to a forecast errors variance minimization problem, given agents are "sensitive" to risk in a particular form. For details see Evans, Honkapohja and Williams (2005). For our purposes CSG has the advantages of showing convergence to equilibria and, at the same time, testing the possibility of endogenous and unpredictable shifts from the rational expectations equilibrium to a BSE.

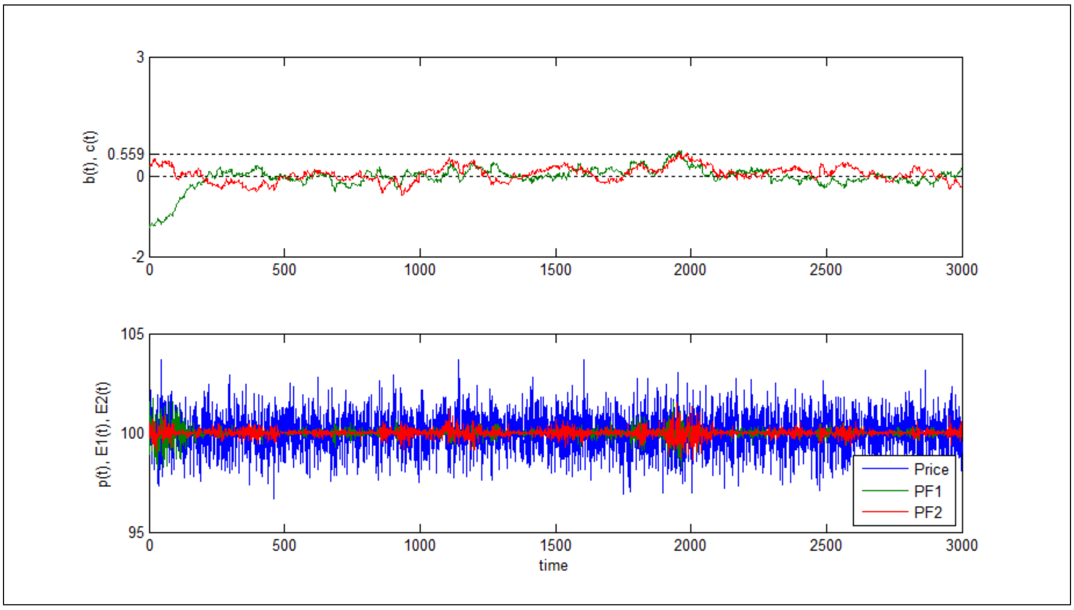

Figure 4: Benchmark case: REE equilibrium. Parameters: $\beta = 0.98$, $\alpha = 2$, $\varrho = \varrho_i = 0.1, \varepsilon_i = \varepsilon$, $\lambda = 0.5$, $\gamma = 0$, $\mu_i = 0$, $\rho_v = 0$, $gain = 0.025$, Equilibria: Learneable REE $(0, 0)$.

All simulations considers two symmetrical agents ($\varepsilon_i = \varepsilon$, $\lambda_i = \lambda$) so that analytical results can be contrasted with experiments. The following parameters are set equal in all simulations: $\beta = 0.98$, $\alpha = 2$, $\varrho = \varrho_i = 0.1$ (so that $\overline{p} = 100$, and $\sqrt{\varrho \overline{p}} = 1$) All the figures are generated with the same series of errors and with the same initial conditions, but in figure 7 where some ad hoc initial conditions are necessary in order to trigger local convergence. The exogenous shocks are all Gaussian white noises with unitary variance. In all figures the following conventions hold. In the upper box is showed the dynamics of coefficients where green, red are colors respectively denoting agent 1 and 2 corresponding coefficients dynamics (respectively b_t and c_t). The lower box shows the corresponding dynamics of both p_t and agents' expectations. The latter are plotted with the same color correspondences as in the upper box.

Figure 4 shows the benchmark case. Agents coefficients b and c converge in distribution to REE ($b = 0, c = 0$) after less then 500 repetitions. This implies a very low volatility of agents expectations around the fundamental value as it is proved by the very thin oscillation around zero of the red and green paths visible in the lower box. Even smaller expectations oscillations can be generated with gains smaller then 0.025. This plot shows how equilibrium can arise in time without assuming any common knowledge property. Moreover if we assume $\gamma = 0$ this equilibrium is unique in the feasible region ($\beta < 1$) of the model and absolutely robust with respect a wide range of specification of behavioral uncertainty (see (16)).

Figure 5 shows how the dynamics changes if public observational error has small covariance with the distance between the fundamental price and the aggregate ($\gamma = 0.1$). Exactly the same dynamics than in figure 4 is exhibited up to about 1800 repetitions. After that, the stochastic nature of the process, to which the CSG algorithm is sensible, displaces the coeficients above

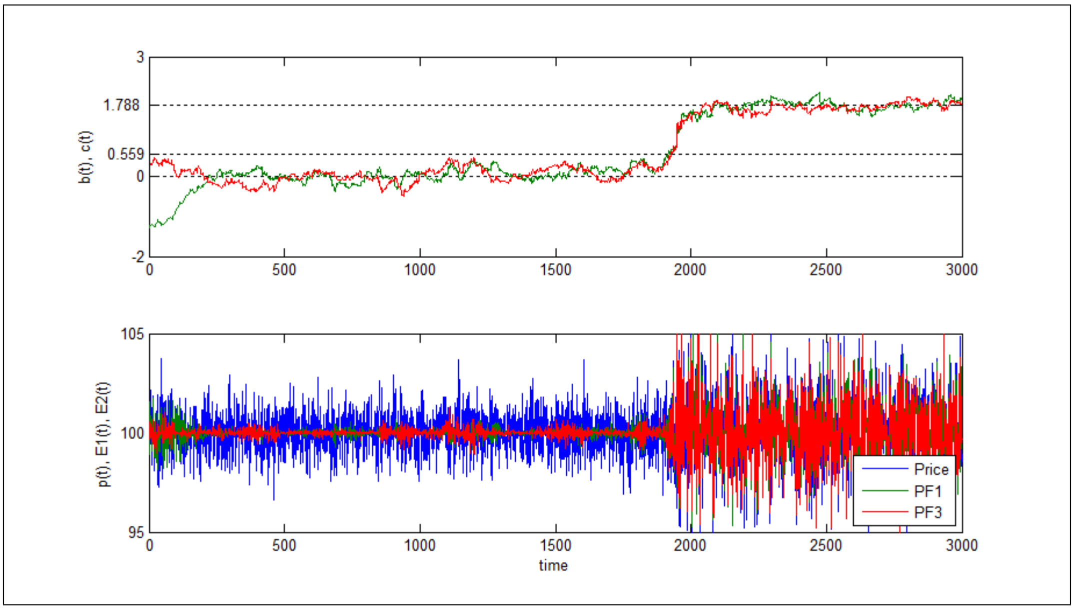

Figure 5: Unpredictable and endogenous emergence of BSE.Parameters: $\beta = 0.98$, $\alpha = 2$, $\varrho = \varrho_i = 0.1, \varepsilon_i = \varepsilon$, $\lambda = 0.5$, $\gamma = 0.1$, $\mu_i = 0$, $\rho_v = 0$, $gain = 0.025$, Equilibria: Learneable REE $(0, 0)$, unlearneable low BSE $(0.559, 0.559)$, learneable high BSE $(1.788, 1.788)$.

the low BSE value 0.559 (in the basin of attraction of high BSE, 1.788). This trigger a jump that make coefficients to asymptotically converge to the high BSE. Correspondingly the process p_t almost double in variance (REE variance is about 1) and agents' expectations exhibit a huge increase in volatility that now matches the one of p_t.

Figure 6 confirms learneability of REE and high BSE in case $\rho_v = 0.5$ with the same γ value as before. As for figure 5 an increase in price and expectations' variances is triggered in correspondence of the jump of the coefficients. Figure 7 shows that with an higher γ and ρ_v the dynamics changes drastically. REE and BSE are no longer learnable whereas low BSE it is. This confirms our finding from the numerical analysis in figure 3.

Finally, figure 8 plots the dynamics in case of drift in observational errors ($\mu_i = -0.8$). The coefficients path is similar to the one of figure 5. REE is still learnable even if observational error and and endogenous jump to high BSE occurs around time 1800. The corresponding p_t and agents' expectations processes exhibits usual increase in variance, but now a drift in level occurs as well once BSE convergence occurs. This simulation confirms the quite controintuitive result that systematic observational errors don't affect convergence to REE. Nevertheless such sistematic errors are latent and they come out to affect the actual price once BSE is attained.

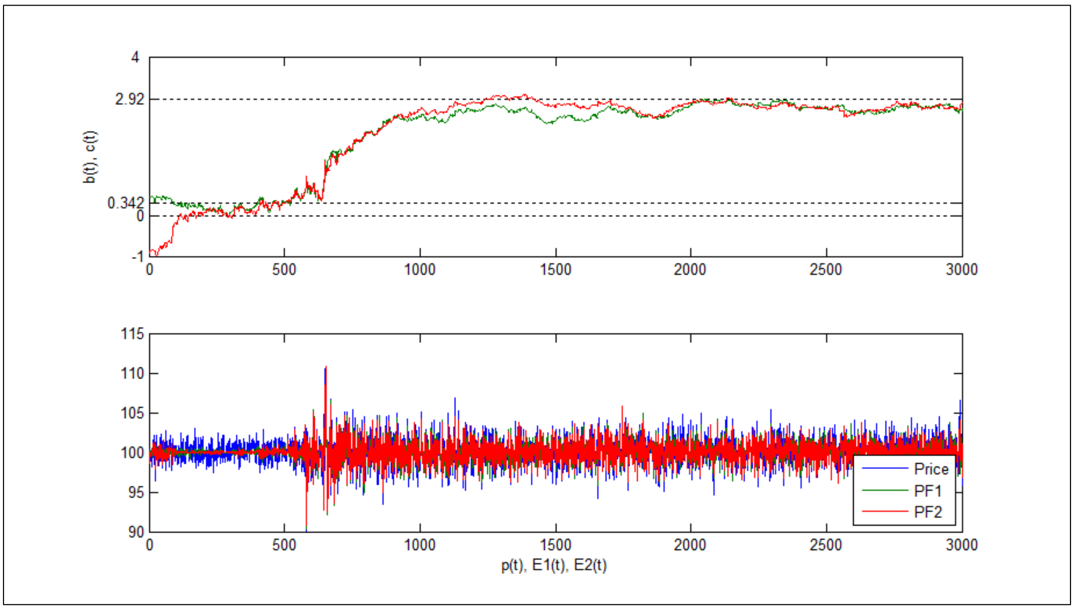

Figure 6: Correlation between observational errors. Parameters: $\beta = 0.98$, $\alpha = 2$, $\varrho = \varrho_i = 0.1, \varepsilon_i = \varepsilon$, $\lambda = 0.5$, $\gamma = 0.1$, $\mu_i = 0$, $\rho_v = 0.5$, $gain = 0.025$, Equilibria: learneable REE $(0, 0)$, unlearneable low BSE $(0.342, 0.342)$, learneable high BSE $(2.92, 2.92)$.

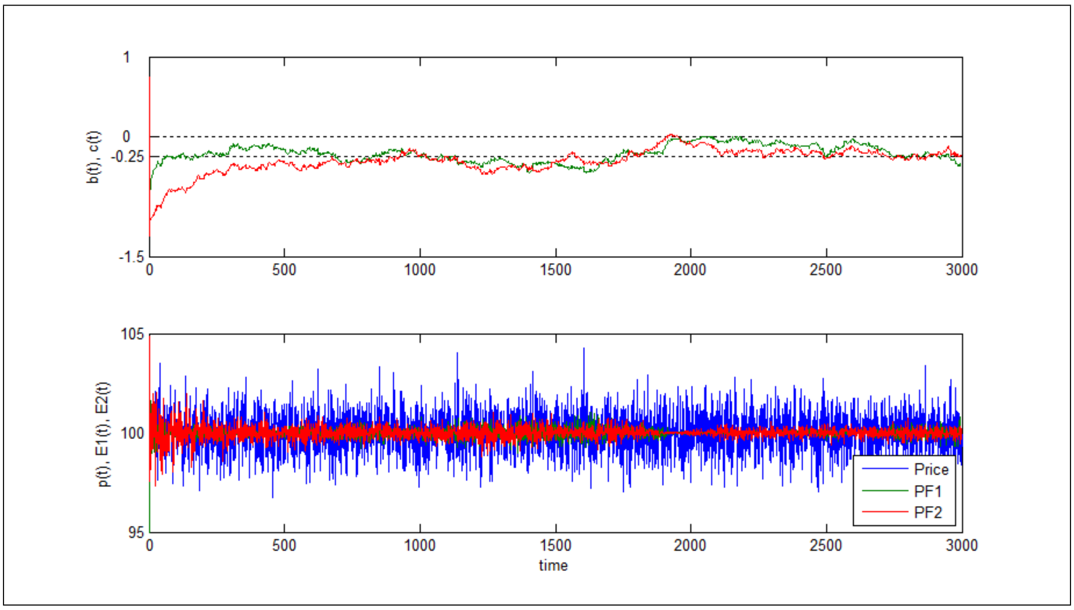

Figure 7: Not learneable REE.Parameters: $\beta = 0.98$, $\alpha = 2$, $\varrho = \varrho_i = 0.1, \varepsilon_i = \varepsilon$, $\lambda = 0.5$, $\gamma = 0.42$, $\mu_i = 0$, $\rho_v = 0.75$, $gain = 0.01$, Equilibria: unlearneable REE $(0, 0)$, learneable low BSE $(-0.25, -0.25)$, unlearneable high BSE $(3.97, 3.97)$.

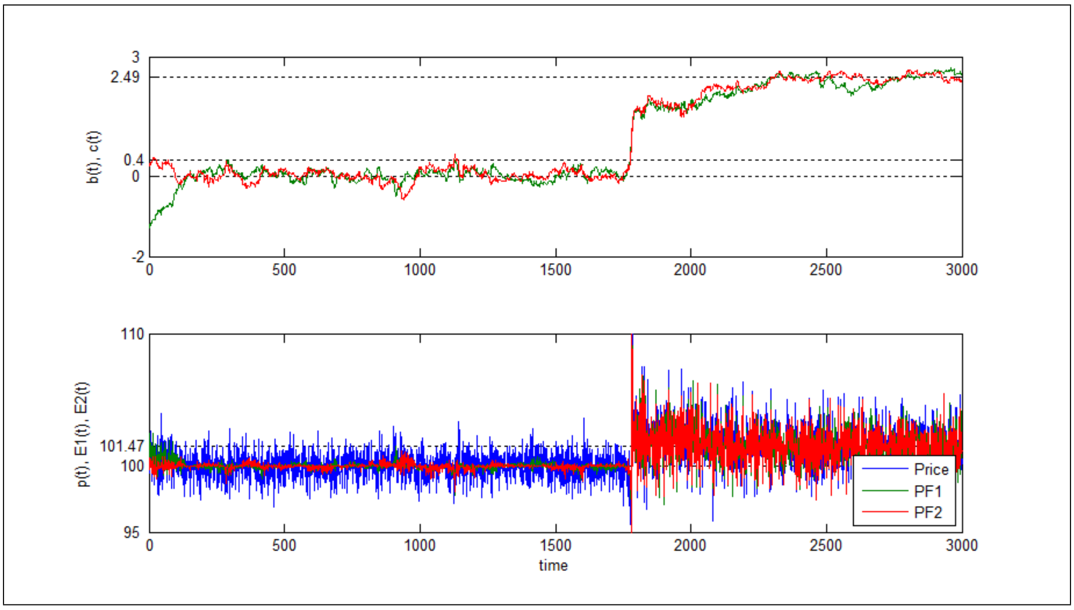

Figure 8: Drift in observational errors. Not learneable REE.Parameters: $\beta = 0.98$, $\alpha = 2$, $\varrho = \varrho_i = 0.1, \varepsilon_i = \varepsilon$, $\lambda = 0.5$, $\gamma = 0.1$, $\mu_i = -0.8$, $\rho_v = 0.39$, $gain = 0.025$, Equilibria: learneable REE $(0, 0)$, unlearneable low BSE $(0.4, 0.4)$, learneable high BSE $(2.49, 2.49)$.

5 Conclusions

This paper builds on the idea that, even if the model is populated by a continuum of infinitesimal agents, expectations are polarized by few institutional forecasters. Institutional forecasters act as focal point for the most part of the public, in that the latter just imitate the formers' expectations. A short list of institutional forecasters would include central bank, fiscal authorities, rating agencies, market leaders etc. As a result, in self-referential models interactions between institutional forecasters matter as long as every institutional forecaster has a non-negligible impact on the aggregate expectations. Therefore, to have consistent expectations, every institutional forecaster has to care about others possibly "non rational" simultaneous expectations. But since they are epistemically isolated, they cannot have point-wise knowledge of others' simultaneous expectations.

Rationality of others is something that can be in principle tested in real time observing their actions. I showed how it is possible to extend adaptive learning to face the behavioural uncertainty problem stated above in real time as eductive learning does in notional time. The idea is that every institutional forecaster can rationally exploit available information to extract the signal of others' rationality, running recursive regression on available data. To isolate the behavioural uncertainty problem, I assume agents hold perfect knowledge of the model, but they don't know what the others believe. They pay an observational error in detecting the simultaneous expectation of the other agent.

Institutional forecasters cannot correct their observational error, but they can test the hypothesis that displacements of the actual price from the fundamental one are independent from observed displacements of others expectations from the fundamental. If this is the case, each institutional forecaster would not care any longer about others' expectations and they would forecast just the fundamental price. Specifically, institutional forecasters want to learn about others' rationality, because I assume that they fully recognize the self-referential nature of the economy. Therefore, institutional forecasters are not bounded rational as is generally assumed in the standard approach to adaptive learning. This is a central innovation in my methodology.

The questions this paper has answered are: are heterogeneous adaptive learners able to fill a gap of behavioural consistency? under adaptive real time learning dynamics, is rational expectations equilibrium still unique and learnable?

I have proved how the unique fundamental rational expectations equilibrium can arise from a disequilibrium prior. This is done without relying on any common knowledge property. It results from interacting learning dynamics in which every agent tests the rationality of others in real time. Moreover, the rational expectations equilibrium attainability is proved to be particularly robust to the correlation and systematic bias in observational errors.

Nevertheless *even if* institutional forecasters are perfectly informed about the economic fundamentals and they acknowledge their own influence on the economy, one learnable equilibrium different from the rational expectations equilibrium may arise (I name such equilibria behavioural sunspot equilibria). This happens because in some cases the agent's observational errors can generate the self-sustaining and reciprocal belief of the other one being irrational exuberant, producing excess volatility. These equilibria exhibit larger variance around the steady state and drift away from rational expectations equilibrium depending on observational errors. In contrast to other sunspot equilibria and consistently with the general approach, behavioural sunspot equilibria don't imply any common knowledge property, rather they arise from endogenous coordination among non-cooperative agents. Moreover the existence of behavioural sunspot equilibria is not alternative to rational expectations equilibria, but they coexist in a large region of the parameter space. In such a region, real time learning dynamics selects among them and endogenous and unpredictable switches from one equilibrium to the other can generally arise. I show numerically how a structural switch from the rational expectations equilibrium to behavioural sunspot equilibrium may occur endogenously.

Appendix

A1.

In order to recover the T-map of the system the following projection conditions must be satisfied:

$$\begin{aligned} E[\phi_{t-1}^{1}\left(p_t-\overline{p}-T_b\phi_{t-1}\right)] &= 0,\ \left(\phi_{t-1}^{1}\right)'=[E_{t-1}^{2}+v_{1,t-1}-\overline{p}] \\ E[\phi_{t-1}^{2}\left(p_t-\overline{p}-T_c\phi_{t-1}\right)] &= 0,\ \left(\phi_{t-1}^{2}\right)'=[E_{t-1}^{1}+v_{2,t-1}-\overline{p}] \end{aligned}$$

Spelling out above conditions, for agent 1's T-map (agent 2 is mirror like),we have

$$\beta^*\left(\frac{b\left(\lambda+(1-\lambda)c\right)}{\left(1-bc\right)^2}\varepsilon_1+\frac{c^2(\lambda b+(1-\lambda))}{\left(1-bc\right)^2}\varepsilon_2+\frac{c(\lambda b+(1-\lambda))+cb\left(\lambda+(1-\lambda)c\right)}{(1-bc)^2}\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)+$$
$$-T_b(\frac{1}{\left(1-bc\right)^2}\varepsilon_1+\frac{c^2}{\left(1-bc\right)^2}\varepsilon_2+2\frac{c}{\left(1-bc\right)^2}\sqrt{\varepsilon_1\varepsilon_2}\rho_v)=0$$

$$\beta^*\left(\frac{b\left(\lambda+(1-\lambda)c\right)}{\left(1-bc\right)^2}\left(\varepsilon_1+c\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)+\frac{c(\lambda b+(1-\lambda))}{\left(1-bc\right)^2}\left(c\varepsilon_2+\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)\right)+$$
$$-T_b\left(\frac{1}{\left(1-bc\right)^2}\varepsilon_1+\frac{c}{\left(1-bc\right)^2}\left(c\varepsilon_2+2\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)\right)=0$$

finally the projected T-map for b and c is given by

$$T_b=\beta^*\left(\frac{b\left(\lambda+(1-\lambda)c\right)\left(\varepsilon_1+c\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)+c(\lambda b+(1-\lambda))\left(c\varepsilon_2+\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)}{\varepsilon_1+c^2\varepsilon_2+2c\sqrt{\varepsilon_1\varepsilon_2}\rho_v}\right)$$

$$T_c=\beta^*\left(\frac{b\left(\lambda+(1-\lambda)c\right)\left(b\varepsilon_1+\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)+c(\lambda b+(1-\lambda))\left(\varepsilon_2+b\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)}{\varepsilon_2+\varepsilon_1b^2+2b\sqrt{\varepsilon_1\varepsilon_2}\rho_v}\right)$$

A2.

To find conditions for which (39) has negative eigenvalues it is enough to evaluate signs of trace and determinant. In particular both eigenvalues are negative if and only if track is negative (it is the sum of eigenvalues) and determinant is positive (it is the product of eigenvalues). Track of (41) in $(0,0)$, namely $\mathbf{T}(K_{(0,0)})$ is given by

$$\mathbf{T}(K_{(0,0)})=\beta^*\lambda-1+\beta^*(1-\lambda)-1=\beta^*-2$$

so that $\mathbf{T}(K_{(0,0)})<0$ implies $\beta^*<2$. Determinant $\mathbf{\Delta}(K_{(0,0)})$ takes the form

$$\mathbf{\Delta}(K_{(0,0)})=(1-\rho_v^2)\lambda(1-\lambda)\beta^{*2}-\beta^*+1$$

The determinant $\mathbf{\Delta}(K_{(0,0)})$ is greater than zero if and only if

$$0 \leq \rho_v^2 \leq 1 \text{ and } \beta^* \leq \frac{1-\sqrt{1-4\lambda(1-\lambda)(1-k)}}{2\lambda(1-\lambda)(1-k)}, \quad \beta^* \geq \frac{1+\sqrt{1-4\lambda(1-\lambda)(1-k)}}{2\lambda(1-\lambda)(1-k)} \tag{45}$$

$$\rho_v^2 > 1 \text{ and } \frac{1-\sqrt{1-4\lambda(1-\lambda)(1-k)}}{2\lambda(1-\lambda)(1-k)} \leq \beta^* \leq \frac{1+\sqrt{1-4\lambda(1-\lambda)(1-k)}}{2\lambda(1-\lambda)(1-k)} \tag{46}$$

Now, notice that $\rho_v \in [-1,1]$ as direct consequence of the Cauchy–Schwarz inequality. Finally

$$\frac{1-\sqrt{1-4\lambda(1-\lambda)(1-k)}}{2\lambda(1-\lambda)(1-k)} \leq 2$$

implies

$$(1-4\lambda(1-\lambda)(1-k))^2 < 1-4\lambda(1-\lambda)(1-k)$$

that is always true since $0 < 4\lambda(1-\lambda)(1-k) < 1$. This states (42) to be the only relevant condition.

A3.

To discuss the sign of eigenvalues of a 2×2 matrix it is enough to study both trace and determinant sign. In particular the matrix at hand has all negative eigenvalues if and only if it has negative trace (the sum of eigenvalues is negative) and positive determinant (the product of eigenvalues is positive). We will discuss them separately in the case $\rho_v = 0$.

Trace The two entries of the main diagonal are equal, hence if the first entry is negative, the trace is negative too. The relevant inequality is

$$((\beta^*/2)-1)\,\bar{b}^2 + (\beta^*/2)\,\bar{b} + (\beta^*/2) - 1 < 0 \tag{47}$$

whose determinant is

$$\Delta = -3\,(\beta^*/2)^2 + 8\,(\beta^*/2) - 4$$

that is greater than zero only if $4/3 < \beta^* < 4$. Recalling that for existence of BSE it must be $\beta^* \geq 1$, and since for $\beta^* > 2$ (47) is a convex curve, its solution is

$$\nexists\, b = c \text{ with } \beta^* > 4 \tag{48a}$$

$$s_- \leq b = c \leq s_+ \text{ with } 2 \leq \beta^* \leq 4 \tag{48b}$$

$$b = c \leq s_-, \quad b = c \geq s_+ \text{ with } \frac{4}{3} \leq \beta^* < 2 \tag{48c}$$

$$\forall b = c \text{ with } 1 \leq \beta^* < \frac{4}{3} \tag{48d}$$

where

$$
\begin{aligned}
s_- &= \frac{\beta^* - 2\sqrt{-3\left(\beta^*/2\right)^2 + 8\left(\beta^*/2\right) - 4}}{2-\beta^*} \\
s_+ &= \frac{\beta^* + 2\sqrt{-3\left(\beta^*/2\right)^2 + 8\left(\beta^*/2\right) - 4}}{2-\beta^*}
\end{aligned}
$$

The satisfaction of the conditions above for (38) can be checked easily from the analysis of the following inequality

$$
-\left(3/4\right)\beta^{*2} + 4\beta^* - 4 < \beta^* - 1
$$

that always holds since $-3((\beta^*/2)-1)^2 < 0$. We can conclude that both (b_+, c_+) and (b_-, c_-) satisfy condition (48c)-(48d), so that (47) holds in the range

$$
\beta^* \in (-\infty, 2) \tag{49}
$$

Now, remember that both (b_+, c_+) and (b_-, c_-) exist only if $\beta^* \geq 1$. In this range $b_+ = c_+ > 1$ and $0 < b_- = c_- < 1$. Notice that the following inequality always holds

$$
\beta^* - 2\sqrt{-3\left(\beta^*/2\right)^2 + 8\left(\beta^*/2\right) - 4} < 0
$$

so that $b = c \leq s_-$ is not a feasible solution of (47). We concludes solution (b_+, c_+) satisfies condition (47) in the range

$$
\beta^* \in [1, 2) \tag{50}
$$

whereas solution (b_-, c_-) does in the range

$$
\beta^* \in [1, 4/3) \tag{51}
$$

We will use condition (50)-(51) to restrict the analysis of the determinant.

Determinant The determinant of $K_{(\bar{b},\bar{c})}$ is equal to $A^2 - B^2$ where

$$
A = \frac{\left(\left(\beta^*/2\right) - 1\right)\bar{b}^2 + \left(\beta^*/2\right)\bar{b} + \left(\beta^*/2\right) - 1}{1 + \bar{b}^2}
$$

$$
B = \frac{-\left(\beta^*/2\right)\bar{b}^3 + 3\left(\beta^*/2\right)\bar{b}}{\left(1 + \bar{b}^2\right)^2}
$$

the relevant condition is

$$
A^2 - B^2 > 0 \tag{52}
$$

that is true if and only if

$$
|A| > |B| \tag{53}
$$

now learneability conditions requires A to be negative, so that we must distinguish only two cases with B negative or positive. Let's analyse second entry sign.

The relevant inequality is

$$-(\beta^*/2)\,\overline{b}^3 + 3(\beta^*/2)\,\overline{b} < 0 \tag{54}$$

so that, being $(\beta^*/2)$ positive in the relevant range, (54) is negative if and only if

$$\overline{b} > \sqrt{3}$$

since $\overline{b}$ must be always positive. Now we can distinguish two cases.

First case: if B is negative than (53) is expressed by

$$A < B \tag{55}$$

Recall in such a case $b \geq \sqrt{3}$, and let's go through explicit calculation of

$$\left(((\beta^*/2)-1)\,\overline{b}^2 + (\beta^*/2)\,\overline{b} + (\beta^*/2) - 1\right)\left(1+\overline{b}^2\right) < -(\beta^*/2)\,\overline{b}^3 + 3(\beta^*/2)\,\overline{b}$$

We obtain

$$\begin{aligned} ((\beta^*/2)-1)\,\overline{b}^4 + \beta^*\overline{b}^3 + 2\,((\beta^*/2)-1)\,b^2 - \beta^*\overline{b} + (\beta^*/2) - 1 &< 0 \qquad (56)\\ \left(b^4 + \frac{2\beta^*}{2-\beta^*}b^3 + 2b^2 - \frac{2\beta^*}{2-\beta^*}\overline{b} + 1\right)((\beta^*/2)-1)^{-1} &< 0 \qquad (57)\end{aligned}$$

where $((\beta^*/2)-1)^{-1}$ is always negative in the relevant range so that we obtain

$$b^4 + \frac{2\beta^*}{2-\beta^*}b^3 + 2b^2 - \frac{2\beta^*}{2-\beta^*}\overline{b} + 1 > 0$$

now, from the observation that $b^3 > b$ since $b > \sqrt{3}$, and $2-\beta^*$ is positive in the relevant range we can maintain

$$b^4 + \frac{2\beta^*}{2-\beta^*}b^3 + 2b^2 - \frac{2\beta^*}{2-\beta^*}\overline{b} + 1 > b^4 + 2b^2 + 1 > 0$$

so that for transitivity (55) is proved.

Second case: if B is positive than (53) is expressed by

$$A < -B \tag{58}$$

Recall in such a case $b < \sqrt{3}$, and let's go through explicit calculation of

$$\left(((\beta^*/2)-1)\,\overline{b}^2 + (\beta^*/2)\,\overline{b} + (\beta^*/2) - 1\right)\left(1+\overline{b}^2\right) < +(\beta^*/2)\,\overline{b}^3 - 3(\beta^*/2)\,\overline{b}$$

We obtain

$$((\beta^*/2)-1)\,\bar{b}^2+(\beta^*/2)\,\bar{b}+(\beta^*/2)-1+((\beta^*/2)-1)\,\bar{b}^4+(\beta^*/2)\,\bar{b}^3+((\beta^*/2)-1)\,b^2<$$
$$<+(\beta^*/2)\,\bar{b}^3-3\,(\beta^*/2)\,\bar{b}$$

$$((\beta^*/2)-1)\,\bar{b}^4+2\,((\beta^*/2)-1)\,b^2+2\beta^*\bar{b}+(\beta^*/2)-1<0 \tag{59}$$
$$\left(\bar{b}^4+2b^2+\frac{4\beta^*}{\beta^*-2}\bar{b}+1\right)((\beta^*/2)-1)^{-1}<0 \tag{60}$$

where the following inequality is equivalent in the relevant range $\beta^* \in (1,2)$

$$f(b)=\bar{b}^4+2b^2+\frac{4\beta^*}{\beta^*-2}\bar{b}+1>0 \tag{61}$$

Notice that for $b=1$, that is for $\beta^*=1$, $f(b)$ is equal to zero. Moreover notice that derivative of $f(b)$ is

$$f\,'(b)=4\left(\bar{b}^3+b+\frac{\beta^*}{\beta^*-2}\right)$$

that is positive for $\beta^*=1$ (implying $b=1$) and always monotonically increasing in b in the relevant range, since

$$f\,''(b)=4\left(\bar{b}^2+1\right)$$

so we can conclude that $b_+ \in (1,\sqrt{3}]$ satisfies (61) whereas $b_- \in (0,1]$ does not.

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