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The weight of argument and non-additive measures: a note

Abstract

This note argues that a representation of the epistemic state of the individual through a non-additive measure provides a novel account of Keynes's view of probability theory proposed in his *Treatise on Probability*. The paper shows, first, that Keynes's "non-numerical probabilities" can be interpreted in terms of decisional weights and distorsions of the probability priors. Second, that the degree of non-additivity of the probability measure can account for the confidence in the assessment without any reference to a second order probability. And, third, that the criterion for decision making under uncertainty derived in the non-additive literature incorporates a measure of the degree of confidence in the probability assessment. The paper emphasises the Keynesian derivation of Ellsberg's analysis: the parallel between Keynes and Ellsberg is deemed to be significant since Ellsberg's insights represent the main starting point of the modern developments of decision theory under uncertainty and ambiguity.

KEYWORDS: uncertainty, probabilities, Keynes

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1. Introduction

Thought experiments with urns containing balls of different colours constitute the factual basis of both Ellsberg's rejection of Savage's subjective expected utility models and the modern developments of decision theory under uncertainty and ambiguity (Camerer and Weber 1992). But the analysis of decision making in uncertain contexts conducted in terms of choices among urns containing coloured balls in known or unknown proportions dates back at least to Knight and Keynes. In his *Treatise on Probability*, in particular, Keynes (1921) exemplified the relevance of the notion of weight of argument by means of the black and white urn example. Ellsberg (1962, pp.11-12) recognised this notion as akin to his own notion of ambiguity, denoting the degree of confidence in a probability assessment.

For long the weight of argument has been measured as a kind of second order probability (Marshack 1975). The notion of epistemic reliability of the available information introduced by Gardenfors and Sahlin (1982) revived interest in a more elaborated interpretation of weight which aimed to incorporate a "Keynesian" measure of confidence in the probability assessment in a generalised Bayesian setting. This is one in which the existence of a single additive probability function representing the "credal state" of the individual is rejected in favour of a set of probability functions (Levi 1974). More recently even this second interpretation has been disputed by Keynesian scholars who started investigating the relationship between the notion of uncertainty used by Keynes in the *General Theory* and his earlier work on probability (Lawson 1985, and Carabelli 1988). The consensus reached by these so-called Keynesian fundamentalists implied a rejection of the standard interpretations of Keynes's *Treatise* on the grounds of Keynes's insistence that the epistemic state of the individual may imply probability assessments that are "non-numerical," thus either limiting probability assessments to qualitative comparisons or even excluding the comparability between probability assessments (Carabelli 1995). From this viewpoint, most of the emphasis Keynes put on uncertainty issues in his economics works is better explained with reference to the impossibility of applying probability calculus, a point stressed in the *Treatise* (Hillard 1992, and Dow 1995).

This paper argues that a representation of the epistemic state of the individual through a non-additive measure provides a novel representation of Keynes's approach to probability theory that can stand the critical remarks of the above mentioned Keynesian scholars. The first part of the paper (sections 2-4) discusses Keynes's place in the history of modern decision theory by focusing on two theses: first, the role that Keynes's treatment of probability had in the revival of the epistemic notion of probability – a notion suggested by Jacob Bernoulli's work on "the art of conjecturing" – and, second, the application of probability to conduct as put forward by Keynes in the *Treatise*, with specific regard to the causes of Keynes's rejection of expected utility maximization. In agreement with some Keynesian scholars (Runde 2001 and Gerrard 2003) our analysis suggests that many arguments of the modern critics of subjective expected utility have a Keynesian flavour. The second part of the paper (section 5) makes the relationship between Keynes's peculiar approach and modern decision theory explicit and points out a number of aspects which have been overlooked even by the more accurate Keynesian scholars. It is argued, first, that Keynes's understanding of "non-numerical" probabilities hints at the notion of decision weights introduced by Edwards (1954) and Kahneman and Tversky (1979); Keynes's view is consistent with the interpretation of these weights as probability measures that do not satisfy the property of additivity (Tverski and Kahneman 1992, and Tverski and Wakker 1995). Second, it is shown that the degree of non-additivity of the probability measure (Schmeidler 1989) can account for confidence in a probability assessment without any reference to a second order probability, as suggested by Keynes's the weight of argument. Third, it is maintained that the criteria for decision making under uncertainty put forward in the non-additive literature (like the one discussed in Eichberger and Kelsey 1999) incorporate a measure of the degree of confidence in the probability assessment, as suggested by Keynes in a usually ignored section of the *Treatise*. With regard to all these three issues the paper emphasises the Keynesian derivation of Ellsberg's (1961) analysis: the parallel between Keynes and Ellsberg is significant because Ellsberg's insights constitute the starting point of the modern developments of decision theory under uncertainty and ambiguity discussed in the paper.[1]

To be sure, the scattered elements of Keynes's philosophy of probability dealt with in the first part of the paper emphasize, in the main, those aspects which are at the basis of the new Keynesian fundamentalism. However our understanding of Keynes's rejection of well-defined probability functions, and maximisation as a guide to human conduct, leads to a theoretical option that is different from the one favoured by most Keynesian interpreters. There are increasingly different viewpoints in recent studies, but the main thread of Keynesian fundamentalism can be summarised as follows: "For Keynes uncertainty is an

[1] Ellsberg did not quote Keynes in his 1961 article introducing the paradox, but recognised most similarities with Keynes's *Treatise* in his 1962 doctoral thesis, which remained unpublished until 2001 (see Feduzi 2007).

absence of probabilistic reasoning" (Hillard 1992, p. 69).[2] And even when it is recognised that Keynes's scepticism towards rational action was not bereft of constructive analysis, the emphasis remains on the impossibility of providing a general theory of behaviour under uncertainty with probabilistic content (Dequech 2000, and Davidson 2003).

Our interpretation, on the contrary, does not imply the rejection of probability theory *tout-court*. We take stock of the claim that standard probability calculus is inappropriate to analyse genuine uncertainty, a point forcefully made by George Shackle (1949 and 1961), the founding father of Keynesian fundamentalism; but we also discuss a reformulation of what probability theory can encompass, with specific regard to non-additive probability measures. It is our contention that this approach can provide a sounder theoretical background for some recent Keynesian appraisals, even coming from the Keynesian fundamentalist camp, that have acknowledged that there is a link between certain new developments in modern decision theory and Keynes's probability theory (Fontana and Gerrard 2004).

2. An antecedent: Jacob Bernoulli's epistemic probabilities

To put it schematically, there are two main traditions in the history of probability theory. On the one hand, there is the conventional view intending probability as aleatory probability. On the other hand, since Jacob Bernoulli there has developed a current referring to epistemic probability as well. Aleatory probabilities are numbers assigned to the possible outcomes of a chance event: the aleatory probability of a certain outcome measures the propensity of the outcome to occur. Epistemic probabilities are numbers that represent the degree to which an individual is certain of something or to which she believes it.[3]

Keynes's discussion of probability is full of references to the early developers of the notion of probability. Keynes mainly intended to put emphasis on the differences between the logical approach he was introducing and the frequency approach, dominant in the early 1920s. However he also wanted to make clear that the founders of probability reasoning discussed not only the notion of aleatory probability but also that of epistemic probability, and that his own proposal belonged to this second, possibly less developed tradition. In this respect, Jacob Bernoulli constitutes the main reference.

2 Lawson (1985) provided the first elaborate account of this viewpoint.
3 See in particular Hacking (1975, Ch. 6), and Shaffer (1978, pp. 312-313).

Most of Bernoulli's *Ars Conjectandi* is about games of chance, not unlike the works by Port Royal Logicians like Pascal and Fermat. But the notion of probability he presented in his opus was essentially epistemic. He defined probability as a part of certainty and considered it as a measure of human knowledge. In Bernoulli's words, probability "is the degree of certainty and differs from the latter as a part from the whole." Probabilities are calculated from arguments: "probabilities are estimated both by the number and the weight of the arguments which somehow prove or indicate that a certain thing is, was, or will be. As to the weight, I understand it to be the force of the proof" (Bernoulli 1713, p. 211, 214 as translated in Sheynin 2005, p. 8, 10). To Bernoulli arguments are contingent, and contingency is related to human knowledge.

It is with explicit reference to this approach that, in the *Treatise on Probability*, Keynes interpreted probability as different from chance or frequency. Keynes argued that "the identification of probability with statistical frequency is a very grave departure from the established use of words; for it clearly excludes a great number of judgments which are generally believed to deal with probability" (1921, p. 103). The emphasis on the epistemic interpretation of probability signals a renewed interest in an approach that, in the following years, was to be endorsed by many other authors and become the dominant interpretation. Indeed Keynes (1921, p. 317) stressed that

> "a careful examination of all cases in which various writers claim to detect the presence of 'objective chance' confirms the view that 'subjective chance,' which is concerned with knowledge and ignorance, is fundamental, and that so-called 'objective chance,' however important it may turn out to be from the practical or scientific point of view, is really a special kind of 'subjective chance' and a derivative type of the latter."

A similar point was made a few years later by Frank Ramsey (1926) and Bruno de Finetti (1937) in order to pursue a purely subjective theory of probability, subsequently encapsulated in Savage's (1954) axiomatic system. While neither de Finetti nor Ramsey followed Keynes in the contention that probability is a purely logical relation, the critical remarks they raised against the frequency approach are basically analogous to Keynes's (Kyburg and Smokler 1964, pp. 5-6).

Before moving on, it is also worth noting that Bernoulli (1713, p. 218 as translated in Sheynin 2005, p. 13) distinguished between pure and mixed arguments:

> "I call an argument pure if in some cases it proves a thing in such a manner that on other occasions it does not prove anything positively. A mixed argument, however, is such that in certain cases it thus proves a thing that on other occasions it proves the contrary in the same manner."

As commented on by Shafer (1978, p. 329), this distinction is remarkable since in the definition of pure arguments lies the possibility of non-additivity: as a matter of fact "the evidence of a mixed argument points to both the questions it addresses, whereas the evidence of a pure argument points only to the positive side." Given that our reconstruction stresses the role of non-additive probabilities in the current developments of decision theory, it is significant to point out that at the origin of epistemic probability the additivity of the probability function was not taken for granted.[4]

3. Keynes's logical probability and "non-numerical" probabilities

In Keynes's view (1921, p. 4) probability expresses a logical relation between two propositions, namely a conclusion and the evidence for it:

> "the terms *certain* and *probable* describe the various degrees of rational belief about a proposition which different amount of knowledge authorise us to entertain"

Keynes aimed to extend Russell's deductive logic to include probability relations. He argued that probability is a logical relation that is akin to the logical consequence between propositions, although one weaker than it (Braithwaite 1973). Keynes started from the logical relationship between a proposition *a* and a proposition *h*, and attributed certain belief (probability 1) to *a/h* if *a* is a logical consequence of *h*, and certain disbelief (probability 0) if *non-a* is a logical consequence of *h*. If neither of these two is the case, that is, "in cases where it is not possible to argue demonstratively from one [proposition] to the other" (Keynes 1921, p. 9), the individual who knows *h* is assumed to have a degree of partial belief in *a*, namely a belief that is intermediate between certain belief and certain disbelief. The subject matter of the theory of probability is defined thus: "Between two sets of propositions, …, there exists a relation, in virtue of which, if we know the first, we can attach to the latter some degree of rational belief" (Keynes 1921, pp. 6-7).[5]

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[4] Hacking (1975, p. 144) noted: "until 1713 it was not in the least determined that the addition law for probability would be accepted. Bernoulli was the last master to contemplate non-additive probabilities."

[5] The probability argument can be represented as $p(a/h)$, where p represents the probability of a certain proposition a, given the available evidence h.

In his endeavour Keynes (1921, p. 12) distinguished between direct and indirect knowledge, with the latter as "the part [of our rational belief] which we know by argument." In other words,

> "our knowledge of propositions seems to be obtained in two ways: directly by contemplating the objects of acquaintance; and indirectly, *by argument*, through perceiving the probability-relation of the proposition, about which we seek knowledge, to other propositions."

It seems that Keynes was following Bernoulli's path because he identified probability with indirect knowledge and argued (1921, pp. 15-16) that indirect knowledge is based on argument and involves a degree of probability lower than certainty:

> "knowledge … of a secondary proposition involving a degree of probability lower than certainty, together with the knowledge of the premises of the secondary proposition, leads only to *a rational belief of the appropriate degree* in the primary proposition. The knowledge present in this latter case I have called knowledge *about* the primary proposition or conclusion of the argument, as distinct from knowledge *of* it."

The *Treatise*, therefore, should be viewed as an attempt to develop a logic meant to derive knowledge from probability arguments. Indeed, probability is "that part of logic which deals with arguments which are rational but not conclusive" (Keynes 1921, p. 241). Jointly with the works of Rudolf Carnap and Harold Jeffrey, the *Treatise* is considered the cornerstone of the so-called logical approach to probability theory. The peculiarity of this approach is that the probability of a conclusion given certain evidence corresponds to the degree of belief that is rational to hold, and this probability is "objective," since it corresponds to the logical deduction that can be drawn from the evidence. Depending on the knowledge to which it is related, Keynes (1921, p. 4) argued, probability may appear subjective, but

> " a proposition is not probable because we think it so. When once the facts are given which determine our knowledge, what is probable or improbable in these circumstances is fixed objectively, and is independent of our opinion. The theory of probability is logical, therefore, because is concerned with the degree of belief which is *rational* to entertain in given conditions, and not merely with the actual beliefs of particular individuals."

In Keynes's view, this logical relation of probability can be perceived by individuals.

This is a crucial point insofar as it leads to split the epistemic probability camp into two sub-groups: Keynes's logical tradition and the subjectivists. The goal of the logical approach is to identify principles of inductive rationality that are so powerful that different individuals sharing the same evidence are constrained by these principles to agree on definite probability judgements. This perspective encountered strong resistance after the emergence of the subjectivist-personalist approach.[6] Ramsey's (1926, p. 63) objection to Keynes's view of probability relations is well known: "I do not perceive them, and … I shrewdly suspect that others do not perceive them either, because they are able to come to so very little agreement as to which of them relates any two given propositions."[7] Further, George Shackle's peculiar theory of decision although mainly inspired by Keynes and in spite of its critical stance towards the subjectivist position, did not endorse Keynes's viewpoint that the logic of probability is independent of evidence.[8]

After clarifying the meaning of probability and its relationship with the theory of knowledge, Keynes developed his own formal logic of probability in Part II of the *Treatise*. These formal developments do not survive the test of a consistent axiomatic system usually adopted by the later subjectivist mainstream. But the rationale of Keynes's attempt is of great interest nonetheless: his criticism of frequency probability challenges any theory of probability based on a unique additive distribution (Runde 2001). Moreover Keynes's critical remarks underpin various scholars' contention that probabilities form only a partial order (Kyburg 1995). In particular, Keynes rejected the idea that probabilities can always be given a representation by real numbers, an idea that is implied by the definition of frequency probability, and to which subjectivists are committed through the Dutch book argument. Indeed, before tackling the formal logic of his own system of probability, Keynes was keen to restrict the range of applicability of probability theory, with specific regard to the two related issues of measurability and comparability.

Keynes (1921, p. 21) argued against the generally accepted opinion that "a numerical comparison between the degrees of any pair of probabilities is not only conceivable but it is actually within our power." Critical of the frequentist viewpoint that the numerical character of probability is necessarily involved in the definition of probability as the ratio between "favourable cases" and the "total number of cases," he provided various instances from "the experience of the practical man" that in many cases "no rational basis have been discovered for numerical comparison" (Keynes 1921, p. 23-32). As a result, he maintained that

> "there are some pairs of probabilities between the members of which *no* comparison of magnitude is possible; that we can say, nevertheless, of some pairs of relations of

[6] Even though systematised and re-elaborated by Carnap (1950), the logical approach became a minority view in the following developments of the theory of probability (Fine 1983).

[7] On whether Keynes's obituary of Frank Ramsey represents a turn toward a subjectivist position see Raffaelli (2006) and the literature referred to there.

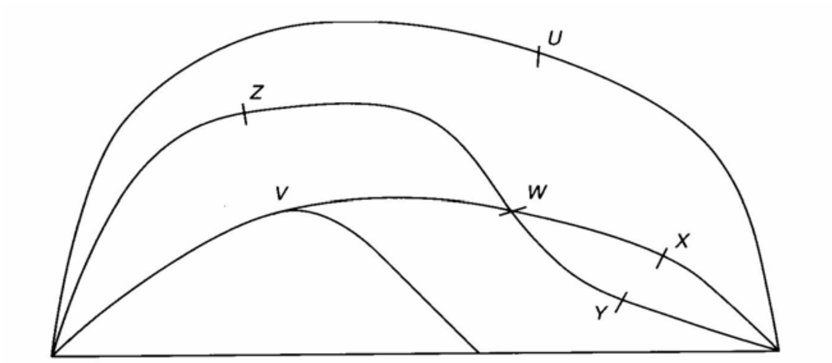
[8] See Shackle (1979). On the contrary, an upshot of Keynes's approach is that to say that "a probability is unknown ought to mean that it is unknown to us through our lack of skill in arguining from given evidence," since "evidence justifies a certain degree of knowledge" (Keynes 1921, p. 34).

probability that the one is greater and the other less, although it is not possible to measure the difference between them; and that in a very special case, …, a meaning can be given to a *numerical* comparison of magnitude”[9]

Keynes conceded that, being probability a concept intermediate between impossibility and certainty, when one argues that one probability is “greater” than another, “this precisely means that the degree of our rational belief in the first case lies *between* certainty and the degree of the rational belief in the second case” (Keynes 1921, p. 37). But he also stressed that the probabilities of two quite different arguments can be impossible to compare. Probabilities can be compared if they belong to the same “ordered series,” that is, if they “belong to a single set of magnitude measurable in term of a common unit.” In contrast, probabilities are impossible to compare if they belong to two different arguments and one of them is not (weakly) included in the other. As Keynes (1921, p. 38) put it:

> “Some probabilities are not comparable in respect to more or less, because there exists more than one path, so to speak, between proof and disproof, between certainty and impossibility; and neither of two probabilities, which lie on independent paths, bears to the other and to certainty the relation of ‘between’ which is necessary for quantitative comparison.”

Keynes’s discussion about measurability and comparability in chapter 3 of the *Treatise* is illustrated in a diagram (Keynes 1921, p. 42) which is reproduced below (from Feduzi 2007).



On the horizontal axis this diagram features the scale of probability ranging from 0 to 1: every point on this axis is a numerical representation of the logical degree of belief in a proposition.

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9 Keynes distinguished between four alternatives (1921, p. 33): “Either in some cases there is no probability at all; or probabilities do not all belong to a single set of magnitudes measurable in terms of a common unit; or these measures always exist, but in many cases are, and must remain, unknown; or probabilities do belong to such a set and their measures are capable of being determined by us, although we are not always able so to determine them in practice.” It is to be noted that a similar but simpler taxonomy is proposed by Einhorn and Hogarth (1986) in order to make room for Ellsberg’s notion of ambiguity.

In the plane created by the diagram there are some different paths, all starting with 0 and ending with 1, which do not lie on the straight line between the extremes. Each point on every non-linear path represents what Keynes (1921, p. 42) calls a “non-numerical probability” or a “numerically undetermined probability.” As reported in the previous quotes, Keynes suggested that probabilities lying on the same path can be compared among themselves, but that comparability is limited to points on the same path (or among paths that have points in common); and “the legitimacy of such comparisons must be a matter for special enquiry in each case” (Keynes 1921, p. 40).

At first it might appear that Keynes did not attempt to provide a mathematical structure for these probability values and left the interpretation of the vertical dimension undetermined. However, in the second part of the *Treatise* Keynes tried to give a meaning to a numerical measure of a relation of probability through “numerical approximation, that is to say, the relating of probabilities, which are not themselves numerical, to probabilities, which are numerical.” He (1921, p. 176) argued as follows:

> “Many probabilities, which are incapable of numerical measurement, can be placed nevertheless *between* numerical limits. And by taking particular non-numerical probabilities as standards a great number of numerical comparison or appropriate measurements become possible.”

It can be argued thus that Keynes’s interrelated notions of unmeasurable and incomparable probabilities can be given operational content through the notion of interval of probabilities. Keynes’s attempt to develop what he called “a systematic method of approximation” was later taken up by Koopman (1941), who provided an axiomatisation of Keynes’s ideas by introducing upper and lower probabilities, thus paving the way for the modern treatment of “imprecise probabilities” (Walley 1991).[10]

Moreover, as will be detailed in the section after next, from the viewpoint of modern decision theory the Keynesian paths can be given a straightforward interpretation. These paths closely resemble what nowadays would be identified with distorted probabilities, that is, contractions or expansions of prior linear probabilities. The theme of distorted probabilities was introduced by Ellsberg (1961) and Fellner (1961) in the *Quarterly Journal of Economics* symposium on the violations of Savage’s axioms of subjective probabilities, the symposium made famous by the introduction of the Ellsberg Paradox. On that occasion, after a brief round of discussions Ellsberg’s results were put aside, “simply because researchers at the time

[10] Remarkably in Keynesian fundamentalists’ texts there are only scattered footnote references on these developments (for instance see Runde 1995, p. 198 fn.)

were helpless to address them" (Machina 2001, xxxix).[11] As is well known, the theme of distorted probabilities has begun receiving increasing attention since Kahneman and Tversky's (1979) discussion of decision weights, while the later Rank Dependent Expected Utility (Quiggin 1982, and Yaari 1987) and Choquet Expected Utility (Schmeidler 1989) were mostly motivated by the aim to provide a theory of decisions capable of accommodating individuals' perception of probabilities through probability weighting functions.

4. The weight of argument and the application of probability to conduct

The previous section presented Keynes's rationale for criticising the use of frequency probability. Keynes's insistence on the limited applicability of numerical probabilities was obscured by the need for consistency of the subjectivist mainstream, before re-emerging in the works of Keynesian scholars emphasising the role of uncertainty in economic discourse. But Keynes's critical remarks did not concern only the use of probability distributions. Apart from the degree of probability, there is a second relevant aspect in the measurement of an argument in Keynes's view: its weight. Indeed, the measurement of probabilities should encompass both the magnitude of the probability of an argument and the degree of confidence in the argument. Keynes (1921, p. 77) maintained that the weight of an argument is correlated to, but independent of, the size of probability: probability and weight are "independent properties." On the one hand,

> "the magnitude of probability of an argument depends upon a balance between what may be termed the favourable and the unfavourable evidence; a new piece of evidence which leaves the balance unchanged, also leaves the probability of the argument unchanged."

On the other hand, arguments can be compared according to the weight, that is,

> "a balance, not between the favourable and the unfavourable evidence, but between the *absolute* amounts of relevant knowledge and of relevant ignorance respectively."

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[11] It is likely that the fact that Ellsberg never became an academic – because of his involvement first with Rand Corporation as strategic analyst and then with the U. S. Government as special assistant at the Defence Department during the Vietnam War – contributed to the decline of interest in his paradox. This neglect has been unfortunate, however, because Ellsberg's dissertation submitted to the Economics Department of Harvard University in 1962 is a major achievement. Ellsberg's thesis, published as late as 2001, provides such a thorough discussion of decision-criteria for solving the paradox that, had it been published in the 1960s, would have surely contributed to the progress of research discussing the behaviour brought into light by the paradox.

Keynes's aim was to emphasise that in most (economic) decisions the uncertainty surrounding the individual cannot be represented only through probability, since there exists another dimension in the epistemic state of the individual, that is, the confidence in the probability assessment itself. Keynes (1921, p. 77) clarified the relationship between probability and weight of an argument as follows:

> "the magnitude of the probability of the argument may either decrease or increase, according as the new knowledge strengthens the unfavourable or favourable evidence; but something seems to have increased in either case … an accession of new evidence increases the weight of an argument. New evidence will sometimes decrease the probability of an argument, but it will always increase its weight."[12]

As a result, Keynes (1921, p. 83) contended that the weight of argument should have a role in the application of probability theory to human conduct. However, being dubious about the "practical significance" of the theory of evidential weight, he also argued that the conclusion

> "that the 'weight' and the 'probability' of an argument are independent properties, may possibly introduce a difficulty into the discussion of the application of probability to practice."

As will be pointed out below, the weight is one of the issues Keynes emphasised to contrast the use of mathematical expectation in the "application of probability to conduct."

Keynes's definition of weight was not accurate enough to be interpreted unambiguously. As reported in the above statement regarding the balance between amounts of "relevant knowledge" and "relevant ignorance," he hinted at a way to calculate the weight, but the point was not taken further. And when he used the weight in discussing decision criteria for choice, Keynes (1921, p. 345) admitted that he considered the significance of the weight "highly perplexing;" additionally, he gave a slightly different definition of it by arguing that the weight of an argument is "the degree of completeness of the information upon which a probability is based."

There is a substantial literature about the Keynesian notion of weight of argument. Possibly because of Keynes's inability to adhere to a precise definition, the weight has been

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[12] Keynes (1921, p. 82) exemplified the way the standing of a probability assessment depends on the information on which the assessment is based introducing the example of balls drawn from a urn. He admitted that, by appeal to the principle of indifference, the probability of drawing a white ball from an urn known to contain black and white balls in equal proportion is equal to the probability of drawing a white ball from an urn containing an unknown proportion of white and black balls. But he argued that, in the first case, a greater weight supports the argument in favour of the conclusion that the probability is ½. It is regrettable that Ellsberg did not comment on Keynes's dealing of the urn example in his doctoral thesis recognising the ample similarities of his approach with Keynes's (on this point see Feduzi 2007).

given different readings. For long, the leading interpretation was to relate the weight of argument to the notion of second order probability distribution. The point of this interpretation is that, granted that the weight of argument is only a probability distribution over the probability distribution on the set of events, it is always possible to compare two arguments simply by applying the rule of reduction of compound lotteries (Borch 1968, and Marschak 1975). And, indeed, the distinction between a probability assessment and the degree of confidence in it has no room in a strict Bayesian set-up, in which the behavioural distinction between risk and uncertainty is meaningless. But, as evidenced by a number of Keynesian scholars in the late 1980s (for a brief survey see Gerrard 1992), this interpretation betrays Keynes's idea of probability in a crucial sense: although the weight of argument expresses a degree of confidence in a probability assessment, it is quite a different thing from a probability of a second order. Apart from Keynes's explicit claim that probability and weight are independent properties, the latter seems to refer to something different from the absolute amount of knowledge on which the probability assessment is based, as even the admittedly scant references to it in his major economic works suggest.[13] Runde (1990) was probably the first commentator to call attention to the fact that there are different definitions of weight in the *Treatise*. Runde emphasised the importance of the definition of evidential weight as the degree of completeness of information on which a probability assessment is based, rather than the mere absolute amount of evidence implicit in the second order probability interpretation. In Keynes's approach, that is, new evidence can increase the relevant ignorance more than the relevant knowledge, thus decreasing the weight of argument.[14]

Ellsberg's (1961, p. 657) claim that the nature of the individual's information concerning the likelihood of events is a relevant dimension of the decision problem contributed much to the renewed interest in the topic of the confidence in probability assessments. Ellsberg proposed to call this dimension the ambiguity of information, "a quality

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[13] In the *General Theory* Keynes hinted at the weight of argument in discussing confidence in a crucial chapter on expectations (Keynes 1936, p. 148 fn.). That the notion of liquidity preference can be better understood in terms of the two dimensions of probability and weight has been shown by Runde (1994). It can be also argued that Keynes's emphasis on radical uncertainty in his 1937 summary of the *General Theory*, where he claims that with regard to certain matters "we simply do not know" (Keynes 1937, p. 214), can be expressed by a weight of argument of zero degree. See also Anand (1991).

[14] Runde (1990) has proposed the following notation for the different notions of weight (V) of a certain proposition (a) given the available evidence (h), in terms of knowledge (K) and ignorance (I): absolute amount of relevant knowledge: $V(a/h)=K$; balance of absolute amounts of relevant knowledge and ignorance: $V(a/h)=K/I$; and degree of completeness of information: $V(a/h)=K/(K+I)$.

depending on the amount, type, reliability and 'unanimity' of information," expressing the individual's "degree of confidence in an estimate of relative likelihoods." Moreover, Ellsberg (1961, p. 661) insisted that ambiguity can be brought about by "highly conflicting" information, and noted (1962, p. 11) that Keynes had made the same point in arguing that, in going out for a walk, it is an "arbitrary matter to decide for or against the umbrella … if the barometer is high, but the clouds are black" (Keynes 1921, p. 32).

Levi (1974) and Gardenfors and Sahlin (1982) contrasted the interpretation of the weight in terms of a probability of second order with the notions of credal states and epistemic reliability of the probability assessment; they motivated on these grounds the use of decision criteria alternative to the maximisation of expected utility. More recently, Schmeidler (1989, p. 571) justified the introduction of non-additive probabilities on the grounds that the probability attached to an uncertain event "does not reflect the heuristic amount of information that led to the assignment of that probability." As we shall detail in the next section, a new representation of weight has been provided within the non-additive probability approach (Dow and Werlang 1992).

The question of what decision criterion is suggested by taking into account the weight of argument leads to Keynes's third criticism of probability theory, one that is of major interest from a modern viewpoint. Keynes devoted Chapter 26 of his *Treatise* to the application of probability to conduct. His problem was the interpretation of "goodness" of choice when "it is not rational for us to believe that the probable is true." With reference to the selection of an appropriate rule of choice, Keynes (1921, p. 343) wrote that "normal ethical theory at the present day makes two assumptions: first, that degrees of goodness are numerically measurable and arithmetically additive, and second, that degrees of probability also are numerically measurable." As a result, ethical theory decides among alternative acts on the basis of their mathematical expectations, which Keynes presented as "a technical expression originally derived from the scientific study of gambling and games of chance, and stands for the product of the possible gain with the probability of attaining it."[15]

Keynes (1921, p. 344) disagreed with a generalized application of mathematical expectation for three reasons. First, because the assumption that "quantities of goodness are duly subject to the law of arithmetic, appears to me to be open to a certain amount of doubt;" second, because to assume that

[15] Keynes discussion is based on Moore's analysis of which are the appropriate behavioural rules to be used in ethics. As in Moore, in Keynes there is not a proper analysis expected utility values, but only of expected values. On the relationship between Moore and Keynes see Bateman (1991).

> "degrees of probability are wholly subject to the law of arithmetic, runs directly counter to the view which has been advocated in part I… [that] mathematical expectations, of goods or advantage, are not always numerically measurable, and hence even if a meaning can be given to the sum of a series of non-numerical mathematical expectations, not every pair of such sums are numerically comparable in respect of more and less,"

and, third, because it

> "ignores what I have termed the weights of arguments, namely the amount of evidence upon which each probability is founded."

To sum up, "it is not always possible by a mere process of arithmetic to determine which of the alternative ought be chosen" (Keynes 1921, p. 344-345).

Keynes (1921, p. 348) contended that an alternative to the notion of mathematical expectations does not lie "in the discovery of some more complicated function of the probability wherewith to compound the proposed good;" this entails that the search for an alternative criterion for choice was not among his aims. However, even in this case, he made an effort at constructive analysis, one that is usually disregarded by Keynesian scholars. Keynes argued that probability and weight should be compounded into a coefficient to be used to shape a normative theory of decision making. As suggested by Brady (1993), Keynes's coefficient relates to probability in much the same way as the kind of decision weights whose contour is coherent with Ellsberg's results does.[16]

Even with respect to this third issue, there is a relevant connection with Ellsberg's analysis. As is well known, Ellsberg rejected the rule of mathematical expectation applied to subjective probabilities because the decision-makers he analysed, in declaring to be unrepentant violators of the axioms of the theory, showed to take a measure of the their degree of ignorance into account. In his analysis of alternative criteria for decision making, Ellsberg applied an evaluation method that explicitly includes the individual's attitude towards the ambiguity of the context of choice. Thus, the rationale underlying this last aspect of Keynes's criticism of aleatory probability – as much as the one underlying the two previous issues – indicates that Ellsberg's rejection of mathematical expectation has a definite Keynesian origin.

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[16] Keynes introduced the following coefficient: $c = 2pw/(1+q)(1+w)$, where p is the probability of an event, $q = 1-p$ the probability of its complement, and w is the weight. On the role of Keynes's coefficient c of probability and weight, see also Feduzi 2007.

5. Modern decision theory and distorted probabilities

This section reformulates the three critiques Keynes raised against frequency probability from the viewpoint of modern decision theory. Each critique is re-interpreted through Schmeidler's (1989) non-additive probability approach. Our reading is not an attempt at rational reconstruction; our aim is to show that Keynes's points have counterparts in modern decision theory. It is taken for granted that subjective expected utility fails to account for Keynes's insights, as stressed by Keynesian fundamentalists. But modern developments explaining behavioural patterns such as the ones related to the Ellsberg paradox and prospect theory represent a significant shift in current decision theory that goes in a Keynesian direction, a fact which has not received enough attention in the appraisal of Keynesian scholars.

As seen in the previous section, the first point Keynes raised against frequency probability was that of "non-numerical" probabilities. Although the theme was to disappear from mainstream decision theory after the development of the subjective approach of Ramsey, de Finetti, and Savage, it influenced the works of a few mathematicians and statisticians like Bernard Koopman, Irvin Good and Cedric Smith, who were critical of a strict interpretation of the Bayesian doctrine. As already indicated, in dealing with vagueness and numerical measurement of probabilities Keynes introduced the notion of interval probabilities as a possible representation of uncertain contexts. Koopman (1940) followed Keynes in an attempt to provide an axiomatic treatment of intervals of probability. The notion of upper and lower probabilities was taken up by Good (1950), who argued that the necessity for intervals arises from the fact that individuals' initial judgements are inherently imperfect, and was developed by Smith (1961). Ellsberg (1961) referred to Keynes, Koopman and Good as instances of the existence of a subjectivist tradition emphasising vagueness and indeterminacy in probability judgements and contending that a precise treatment of them can be given. Ellsberg (1962, p. 121) also noted that interval probabilities can explain his paradox and that "lower probabilities" do not conform to the additive property proper of a probability measure.[17]

[17] This lively, though minor, tradition of thought opposing the restrictive interpretation of subjectivism proposed by Savage has been continued by statisticians like Dempster (1967), Shafer (1976), and Walley (1991) and by followers of the fuzzy-measure approach of Zadeh (1978), like Dubois and Prade (1988). There is no space here to deal with this issue in detail (on which see Basili and Zappia 2007), but it is worth noting that both belief functions and fuzzy measures can be interpreted as instances of non-additive measures (Gilboa 2004).

It is worth stressing that both the issue of alternative representations of individuals' beliefs and the related one of alternative criteria for choice were paramount in decision theory at the time of the making of Savage's subjective interpretation of probabilities. From a theoretical point of view, Shackle (1949) strongly objected to the use of probability functions on the grounds that the list of possible states of the world conditioning crucial entrepreneurial choices cannot be assumed as given. Decision theorists and statisticians like Wald, Hurwicz, Hodges, and Lehmann discussed criteria for decisions under complete or partial ignorance alternative to expected utility maximisation (Luce and Raiffa 1957). Moreover, since the inception of experimental economics it was evident that expected utility theory was descriptively inadequate (Preston and Baratta 1948): in particular, Edwards (1954) argued that experimental evidence challenged the hypothesis that decision-makers act as if they were endowed with an additive probability distribution.

Edwards's experimental evidence constitutes the main starting point of Kahneman and Tversky's work on the use of decision weights, instead of probability functions, to represent the way decision-makers feel about probabilities within the framework of a descriptive model of decision making under risk, that is, even probabilities are assumed to be objective. After Kahneman and Tversky (1979) it has become usual to use weighted functions to represent how decision-makers over-weigh low probabilities and under-weight high probabilities, a pattern of behaviour observed under both risk and uncertainty (Wu and Gonzales 1999). Tversky and Kahneman (1992) later rationalised decision making under uncertainty through a cumulative version of prospect theory, by combining the empirical realism of their original prospect theory with the theoretical tractability of non-additive measures. Tversky and Wakker (1995) eventually showed that cumulative prospect theory can be axiomatised by means of a non-additive measure and hence turned into a non-additive model of decision-making under uncertainty.[18]

As documented in section 3, Keynes's "non-numerical" probabilities can be interpreted as instances of decision weights that do not meet the standard rule of probabilities. Each distorted path in Keynes's diagram can be represented through a non-additive measure, and the degree of non-additivity can be viewed as the degree of distortion of the linear probability prior. In order to make this point clearer we briefly introduce the notion of capacity and its significance in modern decision theory. Let $\Omega=\{w_1,...,w_n\}$ be a finite non empty set of states of the world and let $S=2^{\Omega}$ be the set of all events. A function $\mu: S \rightarrow R_+$ is a

[18] If one focuses only on gains, cumulative prospect theory coincides with Schmeidler's (1989) non-additive probability theory (see Wakker 2006).

non-necessarily-additive probability measure, or capacity, if $\mu(\emptyset)=0$ and $\mu(\Omega)=1$, and if for all $s_1,s_2 \in S$ such that $s_1 \supset s_2$, $\mu(s_1) \geq \mu(s_2)$. A capacity is convex if for all $s_1,s_2 \in S$, $\mu(s_1 \cup s_2) \geq \mu(s_1) + \mu(s_2) - \mu(s_1 \cap s_2)$. A convex capacity is also called super-additive. The convexity of a capacity is the property suggested by the Ellsberg paradox, since the belief in the (unambiguous) event of drawing a black or a white ball strictly exceeds the sum of the beliefs in the (ambiguous) events that a black ball is drawn or a white ball is drawn.[19]

Since the capacity is a non-additive measure, the integration of a real-valued function with respect to a capacity is impossible in the Lebesgue sense. In a seminal paper, Schmeidler (1989) provided an axiomatic structure to show that the proper integral for a capacity is the Choquet integral. Schmeidler's approach maintains the distinction between preferences and beliefs that is proper to Savage's settings: preferences are represented by a Choquet integral of utility with respect to a capacity, that is, a weighted average of utilities multiplied by non-additive probabilistic weights. In view of the fact that this procedure generalises the subjective expected utility model to account for ambiguous events, the non-additive approach is usually called Choquet Expected Utility.[20]

Eichberger and Kelsey (1999) showed that a specific class of non-additive probabilities can be parametrised in such a way that additive probabilities correspond to a particular parameter value, and that the parameters have a behavioural interpretation. They introduced the notion of simple capacity: a capacity μ is said to be simple if there exists an additive probability distribution p on S and a real number $\rho \in [0,1]$ such that $\mu(s) = \rho\, p(s)$, where ρ can be interpreted as the individual's confidence in his estimate of the "true" probability distribution p. Eichberger and Kelsey followed Ellsberg in interpreting the degree of distortion of the additive prior as the degree of confidence in a probabilistic assessment. As a result, simple capacities are distortions of additive probability distributions and can provide the formal support for the interpretation of Keynes's non linear paths in the diagram of the

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19 That Ellsberg noted this point from the outset is made apparent by his doctoral thesis (1962, pp. 121). An event is said to be unambiguous if it can be represented through an additive probability distribution, that is, if $p(A)+p(A^c)=1$. An ambiguous event is characterised by $p(A)+p(A^c)\neq 1$. Sarin and Wakker (1992) refer to Keynes in order to characterise the ambiguity in the probability of an event as the one that is caused by lack of available information relative to the amount of conceivable information.

20 Formally the Choquet integral of an act f with respect to the capacity μ is

$$\int f d\mu = \int_0^{\infty} \mu(\{w | f(w) \geq t\})dt + \int_{-\infty}^{0} [\mu(\{w | f(w) \geq t\}) - \mu(\Omega)]dt$$

For a non-technical introduction see Gilboa (2004). For an overview of economic applications of Schmeidler's model see Mukerji and Tallon (2004).

Treatise. Moreover, the Choquet integral of distorted probability provides a solution for the calculus of mathematical expectation with respect to Keynesian non-numerical probabilities.

Let us now turn to the second issue, the interpretation of the weight of argument. As noted before, Keynes suggested that the weight represents the degree of completeness of information on which the decision maker has to act. To represent the intuition behind Keynes's weight of evidence, Gardenfors and Sahlin (1982) introduced the notion of epistemic reliability. Epistemic reliability refers to a situation in which an individual believes that more than one subjective probability distribution over the set of states is possible, and that she is able to select an interval of plausible priors among all the possible priors, though she does not have a second-order probability over the set of priors. As discussed later, the rejection of the second-order probability explanation is made apparent in Gardenfors and Salin by the indication that only a maximin-type behaviour is compatible with ambiguity. Basically, the idea of representing the epistemic state of the individual by means of multiple priors was Ellsberg's proposal to accommodate the behaviour of unrepentant violators of Savage's axioms. Ellsberg (1962) had noted that the degree of confidence in a probabilistic assessment could be represented either through a distortion of linear probabilities or through a set of "reasonably acceptable" probability functions.[21] Indeed, Einhorn and Hogarth (1986, p. 229) defined the notion of ambiguity as "the uncertainty associated with specifying which of a set of appropriate distributions is appropriate in a given situation," that is, "as an intermediate state between ignorance (no distributions are rule out) and risk (all distributions but one are ruled out)."

In the language of modern decision theory, given an event A, the relevant ignorance can be defined as $\psi = 1 - (\mu(A) + \mu(A^C))$, that is, as the difference between complete knowledge, normalised to the unity, and the probability of occurrence of the event plus the probability of its complement (negation of the event). The weight of argument, as a result, can be represented by $\omega = 1 - \psi$. If μ is an additive probability measure, the relevant ignorance is zero and the weight of argument is 1. But if μ is non-additive – to be precise, if it is a convex capacity – the relevant ignorance is different from zero and the weight of argument belongs to the interval between zero and one. As relevant ignorance increases relative to complete

[21] Ellsberg (1962, p. 182) summarises the rationale under the use of multiple probabilities as follows: "under most circumstances of decision-making you can *eliminate*, from the set Y of all possible probability distributions over the relevant states of the world, certain distributions as *unacceptable* for representing your opinion … But … there may remain a sizeable subset Y° of distributions … that still seem 'reasonable acceptable' … that do not contradict your ('vague') opinions [and that] may yet be large, particularly when relevant information is perceived as scanty, unreliable, contradictory, *ambiguous*."

knowledge the weight decreases. In the case of a simple capacity, in as much as the degree of distortion with respect to the "subjectively true" additive probability increases, the weight decreases, signalling a decreasing confidence in the initial prior. This has an obvious counterpart in the representation of the appropriate decision rule under uncertainty, as will be shown below.

The notion of relative ignorance referred to was introduced by Dow and Verlang (1992) in order to characterise uncertainty aversion with non-additive measures, drawing on Schmeidler's (1989) hint that the convexity of the capacity μ indicates the decision-maker's confidence in the probability assessment. Kelsey (1994) discussed a representation of the weight of evidence in terms of a range of possible probabilities, and related it to Gilboa and Schmeidler's (1989) maximin expected utility (on which see below). Vercelli (1999) put emphasis on the link between the Keynesian weight and non-additive probability theory.

According to our interpretation of Keynes's thought, the significance of the weight of argument emerges only when the decision-maker is not endowed with a unique additive probability measure and, in which case she cannot behave as an expected utility maximiser. From the viewpoint of modern decision theory, Keynes's unwillingness to compare magnitude and weight of different arguments can be ascribed to the lack of a coherent theory by which the decision-maker's beliefs could be transformed into distorted probabilities. Moreover, as implicitly assumed by Keynes, the necessity to introduce a weighted probability is related to the degree of ambiguity the decision-maker takes into account in a given situation. Ellsberg's rationalisation of the behaviour observed in urn problems followed the same line of reasoning, a fact that shows that Ellsberg's analysis is squarely in the Keynesian tradition. Therefore it is not surprising that, in his treatment of alternative decision criteria, Ellsberg points to a solution that includes Keynes's notion of weight.

The third Keynesian argument against frequency probability, his objection to mathematical expectation, can now be discussed in terms of the decision criteria emerging from the non-additive representation of beliefs. In the light of the interpretation of the relationship between weight of argument and additivity just presented, it follows that, since the mathematical expectation neglects the weight of argument, this is an adequate criterion for choice only if the probabilities are additive, that is, only if the weight of argument is one. In all other cases it cannot be applied. Given the mentioned difficulties to derive indications for practical conduct from the theory of probability, Keynes did not investigate alternative criteria for choice in detail. But Keynes's coefficient c serves to emphasise that when the weight is relevant, that is, when it is not 1, a new measure should be elaborated. Keynes (1921, p. 348-

349) was sceptical about the capability of a more complicated measure to provide a solution: however, the interpretation of Keynes's "non numerical" probabilities as distorted probabilities opens up the possibility to use the criteria currently put forward in modern decision theory as a solution. In the remaining of this section, we discuss one of the solutions available in modern decision theory, a solution inspired by the work of Ellsberg.

As is now made clearer by the recent publication of his 1962 doctoral thesis, even as regards decision rules Ellsberg follows in Keynes's footsteps.[22] In his own proposal of a criterion for choice Ellsberg (1961, pp. 664-665) introduced a parameter representing the relative ignorance of the individual evaluating the probabilities of outcomes in ambiguous contexts. To take the degree of ambiguity into account Ellsberg proposed the weighted average of the expectation of the most reliable ("best guess") probability distribution, amongst a set of plausible probability distributions, and the maximin solution. The idea came from Hodges and Lehmann (1952), who had worked with a set of plausible probability distributions as priors, rather than a unique probability, and had discussed how the decision-maker could use them. Wald's maximin criterion being extremely conservative, even in a context of complete ignorance where conservatism may make good sense, Hodges and Lehmann contended that a decision-maker who fails to pick a single distribution out of a set as acceptable, may nevertheless regard one of them as the most reliable, and use it to ponder Wald's maximin. Accordingly the decision rule adopted by Ellsberg (1961, p. 664) was to associate with each act x the index $\rho E(x) + (1-\rho) \min(x)$, and then choose the act associated with the maximum value of the index.[23]

Two main aspects of Ellsberg's proposal are worth stressing. The first one concerns the class of criteria Ellsberg's is part of. To a large extent Ellsberg's criterion has been incorporated in the maximin expected utility model of Gardenfors and Sahlin (1982), and indeed it has just recalled that Gardenfors and Sahlin's proposal was motivated by the aim to account for Keynes's weight of argument and Ellsberg's degree of ambiguity. Gardenfors and Sahlin argued that different sets of priors could represent different degrees of epistemic

[22] In the thesis, Ellsberg first recognised that his notion of ambiguity is akin to Keynes's weight (the 1961 article introducing the paradox does not refer to Keynes), then discussed specific instances of non-additive probabilities which could explain actual choices in urn-problems left unexplained in terms of additive probability priors, and finally examined what criterion for choice would be appropriate in an ambiguous context.

[23] Where E(x) is the expected payoff to the act corresponding to the best guess distribution, min(x) is the minimum expected payoff to the act as the probability distribution ranges in the set of non-unacceptable distributions, like in Wald (1945), and ρ represents the degree of confidence in the best guess distribution. Following Hodges and Lehmann, Ellsberg called the criterion "restricted Bayes." The adjective restricted hints of course at the set of plausible priors.

reliability of the amount and quality of the information available. Since the subjective probability is not unique there is a set of expected utilities for each action. Gardenfors and Sahlin then proposed the following criterion: an action *a* is preferred to *b* if and only if the minimum possible value of the expected utility of *a* is greater than the minimum expected value of *b*.[24] Mostly by virtue of the axiomatic foundations provided by Gilboa and Schmeidler (1989), this model is the most referred to in the current literature on decision making under uncertainty. The appeal of this model is that it rejects the fundamental Bayesian assumption of a unique probability measure in a simple and tractable way. Further, the representation of the decision-maker's knowledge by means of a set of additive probability measures has a simple cognitive interpretation: one can imagine the reasoning process which is involved in the decision.

A similar cognitive interpretation seems to be absent, for instance, in the process of integration of Choquet expected utility. However, there is a close link between maximin expected utility and non-additive probability theory which needs emphasising. Gilboa and Schmeidler (1994) showed that there is an isomorphism between a non-additive probability measure and a convex set of additive probability measures. If the convex capacity that represents the decision-maker's beliefs has a non-empty core of additive measures, then the Choquet expected utility with respect to the capacity equals the maximin expected utility of the set of additive measures. As a result, under uncertainty aversion there is no difference between maxmin expected utility and Choquet expected utility and those who do not find the intuition under the non-additive prior convincing can use the multiple prior hypothesis as well. This aspect has not been noted even by the more thorough Keynesian interpreters. The Keynesian derivation of the representation of decision under uncertainty through multiple priors has been recently praised even among Keynesian interpreters (Runde 2001 and Carabelli 2002).[25] In particular, Runde has correctly argued that Keynes's distinction between a judgement of probability and the confidence with which it is made turns out to be

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[24] Maximin expected utility equals subjective expected utility with respect to the less favourable probability distribution in the set of additive probability distributions. And indeed if the set of probabilities consists only of a single probability, maximin expected utility coincides with subjective expected utility. If, on the other hand, it consists of all possible probability distributions it coincides with Wald's maximin.

[25] An alternative approach to uncertainty which makes use of multiple priors is the one proposed by Bewley (1986). In Bewley, preferences over contingent goods are incomplete, a fact which is modelled through a set of probability priors over the state space. A bundle (act) x is preferred to a bundle y if and only if the expected utility of x is larger than the expected utility of y for all the probability distributions in the set of priors. This makes the ordering among acts partial and the status quo a typical outcome. Runde (2001) discusses Bewley's approach from a Keynesian viewpoint.

incompatible with strict Bayesianism, since "it drives a wedge into the link between choice and degrees of belief on which it is founded" (Runde 2001, p. 147). However, in the light of the isomorphism stressed by Gilboa and Schmeidler, this does not hold for the recent attempts to represent beliefs through a non-additive real valued function on the states space, and Runde's objection to consider non-additive measures as a possible representation of Keynes's viewpoint on uncertainty does not seem to stand.

To conclude let us come to the second aspect of interest with regard to Ellsberg's criterion. In representing the degree of ambiguity by the parameter expressing the degree of confidence, Ellsberg neither was precise about the relationship between the additive probability distribution that serves as best estimate and the probabilities in the entire set, nor provided an axiomatic characterisation of his decision rule. However the reading of the multiple prior approach provided by the non-additive developments show that the simple capacity introduced earlier can play this role. In particular, Eichberger and Kelsey (1999, pp. 123-124) proved that the Choquet expected utility of an action with respect to a simple capacity is the Ellsberg's 1961 suggested representation.[26]

5. Conclusion

This paper has discussed some elements of Keynes's probability theory from the viewpoint of modern decision theory. It has been argued, first, that the measurability issue Keynes put forward to contrast frequency probability – a criticism that can be levelled against Savage's subjective probability approach as well – has been elaborated in the non-additive probability approach, and, second, that the notion of weight can be accommodated in such a theoretical setting. Further, it has been pointed out that the search for alternative criteria in current decision making under ambiguity and uncertainty addresses a specific Keynesian theme, the

[26] It should be added that in the 1962 thesis Ellsberg also discussed a different criterion, which he admitted he did not pay enough attention to in the essay published the year before, the Hurwicz's criterion (see Arrow and Hurwicz 1972). Hurwicz (1951) too had introduced a parameter intended to obviate to the pessimistic attitude of Wald's suggestion to concentrate only on states having the worst consequences. Hurwicz's criterion selects the minimum, m, and the maximum, M, payoff to each given action and then associates the index $\alpha M+(1-\alpha)m$ to each action. Of any two actions, the one with the maximum index is preferred, with the parameter α representing the degree of optimism of the decision-maker evaluating alternative acts. Even as regards this criterion, which has striking similarities with Shackle's (1949), the intuition behind it has been discussed and axiomatised in terms of non-additive probabilities (see Ghirardato, Maccheroni and Marinacci 2004).

rejection of mathematical expectation as a criterion for decision making, which was endorsed by Ellsberg in an attempt to explain his well-known paradox.

Modern decision theory under uncertainty, and the related experimental evidence, can be seen as an attempt to account for the notion of ambiguity introduced by Ellsberg. This paper has shown that a closer examination of Ellsberg's rejection of Savage's viewpoint reveals that Ellsberg's approach has a clear origin in Keynes's insights on probability theory, a fact that suggests the Keynesian derivation of most current work in individual decision making.

The paper also aims at giving substance to the view that Keynes's analysis of uncertainty has continuing relevance as a contribution to decision theory. As recently argued in a contribution to this journal, "Keynes's analysis can provide the foundations for a more general theory of decision making under uncertainty that can encompass both orthodox risk-based theories such as subjective expected utility theory and more recent non-risk-based approaches" (Fontana and Gerrard 2004, p.621); we have put forward additional evidence on the parallels between Keynes's approach and certain radical developments in modern decision theory, chiefly originated in experimental economics and psychology.

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