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# A disequilibrium growth cycle model with differential savings

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# A disequilibrium growth cycle model with differential savings

**Serena Sordi**

**Abstract**

This paper extends Goodwin's growth cycle model by assuming both differential savings propensities and disequilibrium in the goods market. It is shown that both modifications entail an increase in the dimensionality of the dynamical system of the model. By applying the existence part of the Hopf bifurcation theorem, the possibility of persistent and bounded cyclical paths for the resulting 4-dimensional dynamical system is then established. With the help of numerical simulation some evidence is finally given that the limit cycle emerging from the Hopf bifurcation is stable.

KEYWORDS: growth cycle, differential savings, limit cycle, disequilibrium models

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# 1 Introduction

Over the last four decades a significant body of literature has emerged inspired by Goodwin's (1967, 1972) approach to growth cycle modelling. Vela K. Velupillai (see, for example, Velupillai 1979, 1982a, 1982b, 1983, 2006 and Fitoussi and Velupillai 1987) — in his path to developing an approach to macrodynamics modelling rooted in the Cambridge tradition and in particular in Goodwin's model and Kaldor's theory of income distribution and technical progress — has contributed importantly to this literature. With respect to Goodwin's model, he has among other things concentrated on how to relax the extreme ('classical') assumption about savings behaviour originally made by Goodwin and on how to remove from the model the assumption of equilibrium in the goods market (an assumption in contrast with the spirit of previous contributions by Goodwin himself, e.g., Goodwin 1948, 1951). In doing this he has also strongly emphasised the importance of bifurcation theory – in particular of the Hopf bifurcation theorem – for a qualitative analysis of macrodynamic models of the economy with a 3-dimensional – or higher – dynamical systems.

This chapter builds on these basic recurring themes of Vela's work with the purpose of highlighting their relevance for growth cycle modelling. Our purpose is in particular to build a growth cycle model with both differential savings propensities and disequilibrium in the goods market. This is done in order to show (1) that both modifications of the model entail an increase of the dimensionality of the state-space of its dynamical system and (2) that the resulting 4-dimensional, non-linear dynamical system has a simple structure such that the Hopf bifurcation theorem can be easily applied.

The remainder of the chapter is organized as follows. In section 2 we give a brief overview of the modified version of the model. Section 3 discusses the dynamics of the model, giving both conditions for limit cycle solutions and some numerical evidence. A few concluding and summarizing results are finally given in section 4.

# 2 The model

I shall consider, as a starting point for further modifications, the generalisations of Goodwin's growth cycle model presented in Sordi (2001, 2003). The unifying element of the latter contributions is the attempt to relax one or another of the simplyfying assumptions introduced by Goodwin in order to obtain a dynamical system of the Lotka-Volterra type. In Sordi (2001) it is shown that the relaxation of the classical assumption about savings and the consideration of the differential savings hypothesis *à la* Kaldor-Pasinetti causes the dynamical system of the model to become of the third-order.[1] In Sordi (2003), on the other hand, assuming for simplicity that the workers' propensity to save is equal to zero, I have considered a different generalisation – which is obtained by introducing into the model an independent investment function and an adjustment mechanism to goods market disequilibrium – and shown that this modification too causes the dynamical system of the model to become of the third order. In both cases, the steps required to apply the Hopf bifurcation theorem to prove the possibility of periodic solutions for the model are simple ones. In the present contributions I intend to put together the two generalisations by considering a version of the model with both the assumption of differential savings and disequilibrium in the goods market. To concentrate on these two modifications, I use a version of the model that is otherwise as close as possible to Goodwin's simple original formulation.

The notation and basic assumption I use all through the chapter are listed in Table 1. In addition to the assumptions of a constant exponential growth of both labour force and productivity of labour I borrow from Goodwin (1967), the building-blocks of the present version of the model can be shortly described as follows:

- **Real-wage dynamics:**

$$\hat{w} = f(v, \hat{v}) = g(v) + \lambda \hat{v} \quad g'(v) > 0,\ g''(v) > 0 \quad \forall v,\ g(0) < 0 \quad (1)$$
$$\lim_{v \longrightarrow 1} g(v) = \infty,\ \lambda > 0$$

  This more general formulation of the Phillips curve-type dynamics for the real wage is intended to incorporate Phillips' original idea that labor's bargaining power will be higher in phases where the employment rate tends to rise, and the more so, the more the employment rate tends to rise (see Phillips 1958, p. 299). To the best of my knowledge, this formulation was first introduced into a modified version of Goodwin's model by Cugno and Montrucchio (1982, pp. 97-98) and then employed by other authors, for example by Sportelli (1995, pp. 43-44). In order to simplify and concentrate on the two above mentioned modifications

[1] This is a basic fact that appears to have been overlooked in the literature. Balducci and Candela (1982, p. 111f), for example, after having assumed the classical hypothesis justify their assumption by writing that "the different assumptions like the Kaldor-Pasinetti one do not substantially modify the content of the analysis". As far as I am aware, the only other contribution that hints to the fact that a proper consideration of workers' saving along Kaldor-Pasinetti lines leads to a model with a higher-dimensional dynamical system is the article by van der Ploeg (1984) where, however, the resulting dynamics is not analysed in details.

Table 1: *Notation and basic assumptions*

| |
|---|
| for any variable $x$, $\dot{x} = dx/dt$, $\hat{x} = \dot{x}/x$ |
| $q$, output |
| $q^e$, expected output |
| $l$, employment |
| $q/l = a = a_0 e^{\alpha t}$, $\alpha > 0$, labour productivity |
| $n = n_0 e^{\beta t}$, $\beta > 0$, labour force |
| $g_n = \alpha + \beta$, natural rate of growth |
| $g$, rate of growth of output |
| $w$, real wage |
| $u = wl/q$, share of wages |
| $v = l/n$, employment rate |
| $k = k_c + k_w$, capital stock |
| $\varepsilon = k_c/k$, proportion of capital stock held by capitalists |
| $k_c = \varepsilon k$, capital stock held by capitalists |
| $k_w = (1 - \varepsilon)\, k$, capital stock held by workers |
| $k^d$, desired capital stock |
| $\sigma = k/q$, capital-output ratio |
| $P_c$, capitalists' profits |
| $P_w$, workers' profits |
| $P = P_c + P_w$ |
| $r = P/k = (1 - u)\, q/k = (1 - u)\, /\sigma$, rate of profit |
| $s_w, S_w$, workers' propensity to save and workers' savings respectively |
| $s_c, S_c$, capitalists' propensity to save and capitalists' savings respectively |
| $0 \leq s_w < s_c \leq 1$ |
| $S = S_w + S_c$ |

of the model, I retain the original formulation in real terms of wage dynamics.

- **Savings behaviour:**

$$S_c = s_c P_c = s_c r k_c = s_c (1-u) q\varepsilon \tag{2}$$

$$S_w = s_w (wl + P_w) = s_w (wl + r k_w) = s_w q - s_w (1-u) q\varepsilon \tag{3}$$

This formulation incorporates Kaldor's (1956) differential savings hypothesis according to which both capitalists and workers save a fixed proportion of their incomes with the propensity to save of the latter strictly less than the propensity to save of the former.[2]

- **Investment equations:**

$$\dot{k}_w = s_w (wl + P_w) = s_w q - s_w (1-u) q\varepsilon \tag{4}$$

$$\dot{k} = \xi \left(k^d - k\right) \qquad \xi > 0 \tag{5}$$

$$\dot{k}_c = \dot{k} - \dot{k}_w \tag{6}$$

This formulation incorporates the idea that, with regard to investment, capitalists and workers behave in rather different ways.[3] It appears reasonable to assume that workers own their share of the stock of capital only indirectly, through loans to the capitalists for a return of interest (Pasinetti 1962, p. 171). To simplify, in equation (4) I have assumed that workers loan out to the capitalists the full amount of their savings, this influencing the number but not the logic of my exercise. Capitalists, on the other hand, decide how much to invest bearing in mind the amount of output they want to produce. Thus, once they have established the desired level of production and consequently also the desired stock of capital, decide their investment – on the basis of a flexible accelerator mechanism – in order to close the gap between the desired stock of capital and the actual stock (see Goodwin 1948). This is formalised in equation (5) where $k^d$ is related to expected output and I have assumed that expectations are of the extrapolative type so that (see Gandolfo 1997, p. 219) :

$$k^d = \bar{\sigma} q^e = \bar{\sigma} (q + \tau \dot{q}) \qquad \tau > 0$$

[2] To obtain (3), ignoring risk and uncertainty, I have assumed, following van der Ploeg (1984, p. 8), that the rate of interest is equal to $r$.

[3] In doing this, I am following Vela's suggestion (1983, p. 248) according to which in a macrodynamic model in which we allow savings out of wages in addition to savings out of profits we should modify the concept of a single investment function. The two investment equations I consider, however, are different from the ones introduced by him.

Thus:

$$\dot{k} = \xi \left(\bar{\sigma} + \bar{\sigma}\tau\hat{q} - \sigma\right) q \qquad \xi > 0 \tag{7}$$

Eventually, as specified in equation (6), the variation of the amount of the stock of capital held by capitalists simply follows as a residual, after the amount of total investment has been decided by them on the basis of the flexible accelerator and workers have decided how much to save and 'loan' to capitalists.

- **Goods market adjustment mechanism:**

$$\dot{q} = g_n q + \eta \left(\dot{k} - S\right) \tag{8}$$

According to this formulation, the dynamics of output is governed by an error-adjustment mechanism such that output reacts to excess demand taking account also of a trend component.[4] Choosing the time-unit in such a way that $\xi = 1$ and substituting from (7), equation (8) becomes:

$$\hat{q} = g_n + \eta \left[\bar{\sigma} + \bar{\sigma}\tau\hat{q} - \sigma - s_w - (s_c - s_w)(1 - u)\varepsilon\right]$$

from which:

$$\hat{q} = \frac{g_n + \eta\left(\bar{\sigma} - s_w\right)}{1 - \eta\bar{\sigma}\tau} - \frac{\eta}{1 - \eta\bar{\sigma}\tau}\left[\sigma + (s_c - s_w)(1 - u)\varepsilon\right] \tag{9}$$

## 2.1 Derivation of the dynamical system of the model

We are now in a position to derive the reduced form dynamical system of the model.

First, from the definition of the variable $v$, equation (9) and the assumptions about the dynamics of $n$ and $a$ (see Table 1) it follows that:

$$\hat{v} = \frac{\eta\left(\bar{\sigma} - s_w + \bar{\sigma}\tau g_n\right)}{1 - \eta\bar{\sigma}\tau} - \frac{\eta}{1 - \eta\bar{\sigma}\tau}\left[\sigma + (s_c - s_w)(1 - u)\varepsilon\right] \tag{10}$$

Second, from the definition of the variable $u$ and equation (1), we obtain:

$$\hat{u} = h(v) + \lambda\hat{v} - \alpha \tag{11}$$

[4] In making this assumption I am again following a suggestion by Vela (see, for example Velupillai 1982b, p. 81 or 2006, pp. 754-755) according to which any proper generalisation of the growth cycle model should concentrate on relaxing Goodwin's assumption of equilibrium in the goods market. As in the previous case, however, the formulation I have chosen to work with is different from the one suggested by Vela and is instead the one used in the growth cycle model advanced by Glombowski and Krüger (1988).

Moreover, from (7) and (9):

$$\hat{k} = \left(\frac{\bar{\sigma} - \sigma}{\sigma}\right) + \frac{\bar{\sigma}\tau}{\sigma}\left\{\frac{g_n + \eta\left(\bar{\sigma} - s_w\right)}{1 - \eta\bar{\sigma}\tau}\right.$$
$$\left. - \frac{\eta}{1 - \eta\bar{\sigma}\tau}\left[\sigma + \left(s_c - s_w\right)\left(1 - u\right)\varepsilon\right]\right\} \tag{12}$$

so that:

$$\begin{aligned}
\hat{\varepsilon} &= \hat{k}\left(\frac{k}{k_c} - 1\right) - \frac{\dot{k}_w}{k_c} \\
&= \hat{k}\left(\frac{1 - \varepsilon}{\varepsilon}\right) - \left[\frac{s_w - s_w\left(1 - u\right)\varepsilon}{\varepsilon\sigma}\right] \\
&= \frac{\left(\bar{\sigma} - \sigma\right)\left(1 - \varepsilon\right)}{\sigma\varepsilon} + \frac{\bar{\sigma}\tau\left(1 - \varepsilon\right)}{\sigma\varepsilon}\left\{\frac{g_n + \eta\left(\bar{\sigma} - s_w\right)}{1 - \eta\bar{\sigma}\tau}\right. \\
&\quad \left. - \frac{\eta}{1 - \eta\bar{\sigma}\tau}\left[\sigma + \left(s_c - s_w\right)\left(1 - u\right)\varepsilon\right]\right\} - \left[\frac{s_w - s_w\left(1 - u\right)\varepsilon}{\varepsilon\sigma}\right]
\end{aligned} \tag{13}$$

Finally, (9) and (12) imply that:

$$\hat{\sigma} = \left(\frac{\bar{\sigma} - \sigma}{\sigma}\right) + \left(\frac{\bar{\sigma}\tau - \sigma}{\sigma}\right)\left\{\frac{g_n + \eta\left(\bar{\sigma} - s_w\right)}{1 - \eta\bar{\sigma}\tau}\right.$$
$$\left. - \frac{\eta}{1 - \eta\bar{\sigma}\tau}\left[\sigma + \left(s_c - s_w\right)\left(1 - u\right)\varepsilon\right]\right\} \tag{14}$$

Equations (11), (10), (13) and (14) form a complete 4D-dynamical system in the four endogenous variables $v$, $u$, $\varepsilon$ and $\sigma$:

$$\dot{v} = \frac{\eta}{1 - \eta\bar{\sigma}\tau}\left\{\bar{\sigma} - s_w + \bar{\sigma}\tau g_n - \sigma - \left(s_c - s_w\right)\left(1 - u\right)\varepsilon\right\} v \tag{15}$$

$$\dot{u} = \left[h\left(v\right) + \lambda\hat{v} - \alpha\right] u = \left[G\left(v, u, \varepsilon, \sigma\right) - \alpha\right] u \tag{16}$$

$$\begin{aligned}
\dot{\varepsilon} &= \frac{\left(\bar{\sigma} - \sigma\right)\left(1 - \varepsilon\right)}{\sigma} + \frac{\bar{\sigma}\tau\left(1 - \varepsilon\right)}{\sigma\left(1 - \eta\bar{\sigma}\tau\right)}\left\{g_n + \eta\left(\bar{\sigma} - s_w\right)\right. \\
&\quad \left. - \eta\left[\sigma + \left(s_c - s_w\right)\left(1 - u\right)\varepsilon\right]\right\} - \left[\frac{s_w - s_w\left(1 - u\right)\varepsilon}{\sigma}\right]
\end{aligned} \tag{17}$$

$$\begin{aligned}
\dot{\sigma} &= \left(\bar{\sigma} - \sigma\right) + \frac{\left(\bar{\sigma}\tau - \sigma\right)}{1 - \eta\bar{\sigma}\tau}\left\{g_n + \eta\left(\bar{\sigma} - s_w\right)\right. \\
&\quad \left. - \eta\left[\sigma + \left(s_c - s_w\right)\left(1 - u\right)\varepsilon\right]\right\}
\end{aligned} \tag{18}$$

where the function $G(v,u,\varepsilon,\sigma)$ in equation (16) is such that:

$$G_v(v,u,\varepsilon,\sigma) = h'(v) > 0 \qquad G_u(v,u,\varepsilon,\sigma) = \frac{\lambda\eta(s_c - s_w)\varepsilon}{1-\eta\bar{\sigma}\tau}$$

$$G_\varepsilon(v,u,\varepsilon,\sigma) = -\frac{\lambda\eta(s_c - s_w)(1-u)}{1-\eta\bar{\sigma}\tau} \qquad G_\sigma(v,u,\varepsilon,\sigma) = -\frac{\lambda\eta}{1-\eta\bar{\sigma}\tau}$$

Letting $E^+ \equiv (v^+, u^+, \varepsilon^+, v^+)$ stand for the unique positive equilibrium point of the model, from (16) it follows that:

$$v^+ = h^{-1}(\alpha)$$

Moreover, we must have:

$$\bar{\sigma} - s_w + \bar{\sigma}\tau g_n - \sigma^+ - (s_c - s_w)\left(1-u^+\right)\varepsilon^+ = 0 \tag{19}$$

$$\begin{aligned}\left(\bar{\sigma}-\sigma^+\right)\left(1-\varepsilon^+\right) + \frac{\bar{\sigma}\tau(1-\varepsilon^+)}{1-\eta\bar{\sigma}\tau}\{g_n + \eta(\bar{\sigma}-s_w)\\ -\eta\left[\sigma^+ + (s_c-s_w)\left(1-u^+\right)\varepsilon^+\right]\} - \left[s_w - s_w\left(1-u^+\right)\varepsilon^+\right] = 0\end{aligned} \tag{20}$$

$$\begin{aligned}\left(\bar{\sigma}-\sigma^+\right) + \frac{(\bar{\sigma}\tau-\sigma^+)}{1-\eta\bar{\sigma}\tau}\{g_n + \eta(\bar{\sigma}-s_w)\\ -\eta\left[\sigma^+ + (s_c-s_w)\left(1-u^+\right)\varepsilon^+\right]\} = 0\end{aligned} \tag{21}$$

Substituting from (19), (20) and (21) gives

$$\left(\bar{\sigma}-\sigma^+\right)\left(1-\varepsilon^+\right) + \bar{\sigma}\tau\left(1-\varepsilon^+\right)g_n - \left[s_w - s_w(1-u)\varepsilon\right] = 0 \tag{22}$$

$$\left(\bar{\sigma}-\sigma^+\right) + \left(\bar{\sigma}\tau-\sigma^+\right)g_n = 0 \tag{23}$$

Thus, from (23):

$$\sigma^+ = \frac{\bar{\sigma}(1+\tau g_n)}{(1+g_n)} \tag{24}$$

Then, substituting (24) in (19) and (22),

$$\bar{\sigma} - s_w + \bar{\sigma}\tau g_n - \frac{\bar{\sigma}(1+\tau g_n)}{(1+g_n)} - (s_c-s_w)\left(1-u^+\right)\varepsilon^+ = 0$$

$$\left[\bar{\sigma} - \frac{\bar{\sigma}(1+\tau g_n)}{(1+g_n)}\right]\left(1-\varepsilon^+\right) + \bar{\sigma}\tau\left(1-\varepsilon^+\right)g_n - \left[s_w - s_w\left(1-u^+\right)\varepsilon^+\right] = 0$$

we get the equilibrium values for $\varepsilon$ and $u$. which are given by:[5]

$$\begin{aligned}\varepsilon^+ &= 1 - \frac{s_w\left[s_c(1+g_n) - \bar{\sigma}g_n(1+\tau g_n)\right]}{(s_c-s_w)g_n\bar{\sigma}(1+\tau g_n)}\\ &= \frac{s_c\left[g_n\bar{\sigma}(1+\tau g_n) - s_w(1+g_n)\right]}{(s_c-s_w)g_n\bar{\sigma}(1+\tau g_n)}\end{aligned} \tag{25}$$

[5] When $s_w = 0$, we obtain $\varepsilon^+ = 1$, i.e., all the capital stock is held by capitalists whereas the opposite case, with $\varepsilon^+ = 0$, is obtained when $s_c = 0$.

and

$$u^{+} = 1 - \frac{g_n \bar{\sigma} \left(1 + \tau g_n\right)}{s_c \left(1 + g_n\right)}$$

All these expressions simplify notably when expectations are such that $\tau = 1$, which we assume to be the case from now on. Thus, the positive equilibrium point of the dynamical system is given by:

$$E^{+} \equiv \left(v^{+}, u^{+}, \varepsilon^{+}, \sigma^{+}\right) = \left(h^{-1}\left(\alpha\right), 1 - \frac{g_n \bar{\sigma}}{s_c}, \frac{s_c \left(g_n \bar{\sigma} - s_w\right)}{\left(s_c - s_w\right) g_n \bar{\sigma}}, \bar{\sigma}\right)$$

where, given the economic meaning of the variables, we must have:

$$0 < h^{-1}\left(\alpha\right) < 1 \qquad\qquad 0 < 1 - \frac{g_n}{s_c} < 1$$

$$0 < \frac{s_c \left(g_n \bar{\sigma} - s_w\right)}{\left(s_c - s_w\right) g_n \bar{\sigma}} < 1 \qquad 0 < \bar{\sigma}$$

The first and the last of these conditions are always satisfied. Thus, the crucial conditions are the second and the third one. It is straightforward to show that they are both satisfied when

$$0 \leq s_w < \bar{\sigma} g_n < s_c \leq 1 \tag{26}$$

It is a matter of simple algebra to show that the positive equilibrium point $E^{+}$ guarantees results that, as should be expected, have both a Pasinettian-Kaldorian and a Goodwinian 'flavour'. First, substituting in (9) and (12), it is easy to show that the coordinates of $E^{+}$ are such to guarantee a steady-state growth of output and of the capital stock at a rate equal to the natural rate

$$\left(\hat{q}\right)^{+} = \left(\hat{k}\right)^{+} = \frac{g_n}{1 - \eta \bar{\sigma}} - \frac{\eta g_n \bar{\sigma}}{1 - \eta \bar{\sigma}} = g_n$$

Second, substituting in the definition of the rate of profit, we have that in the steady state the Cambridge equation is satisfied:

$$r^{+} = \frac{1 - u^{+}}{\sigma^{+}} = \frac{g_n}{s_c}$$

Third, substituting in (1), it follows that in the steady state the real wage grows at the same rate as labour productivity:

$$\left(\hat{w}\right)^{+} = h\left(v^{+}\right) = \alpha$$

Fourth, in order to be economically meaningful it requires that condition (26) is satisfied and this guarantees that the 'Pasinetti case' holds. Finally, and most importantly, it is possible to show that there exist parameter values for which $E^+$ is locally unstable and that, when this happens, the model may have a limit cycle solution. This final result can be easily illustrated by using the Hopf Bifurcation Theorem. In order to do that, as a preliminary step, we need to carry out a local stability analysis.

# 3 Local stability analysis and the emergence of limit cycle behaviour

The Jacobian matrix $\mathbf{J}|_{E^+}$ of the dynamical system (15)-(17) linearized at $E^+$ is:

$$\mathbf{J}|_{E^+} = \begin{bmatrix} 0 & j_{12} & j_{13} & j_{14} \\ j_{21} & j_{22} & j_{23} & j_{24} \\ 0 & j_{32} & j_{33} & j_{34} \\ 0 & 0 & 0 & j_{44} \end{bmatrix}$$

where

$$\begin{aligned}
j_{12} &= \left.\frac{\partial \dot{v}}{\partial u}\right|_{E^+} = \frac{\eta\left(s_c - s_w\right)\varepsilon^+ v^+}{1-\eta\bar{\sigma}} = \frac{\eta s_c\left(g_n\bar{\sigma} - s_w\right)v^+}{\left(1-\eta\bar{\sigma}\right)g_n\bar{\sigma}} \\
j_{13} &= \left.\frac{\partial \dot{v}}{\partial \varepsilon}\right|_{E^+} = -\frac{\eta\left(s_c - s_w\right)\left(1-u^+\right)v^+}{1-\eta\bar{\sigma}} = -\frac{\eta\left(s_c - s_w\right)g_n\bar{\sigma}v^+}{\left(1-\eta\bar{\sigma}\right)s_c} \\
j_{14} &= \left.\frac{\partial \dot{v}}{\partial \sigma}\right|_{E^+} = -\frac{\eta v^+}{1-\eta\bar{\sigma}} \\
j_{21} &= \left.\frac{\partial \dot{u}}{\partial v}\right|_{E^+} = h'\left(v^+\right)u^+ = \frac{h'\left(v^+\right)\left(s_c - g_n\bar{\sigma}\right)}{s_c} \\
j_{22} &= \left.\frac{\partial \dot{u}}{\partial u}\right|_{E^+} = \frac{\lambda\eta\left(s_c - s_w\right)\varepsilon^+ u^+}{1-\eta\bar{\sigma}} = \frac{\lambda\eta\left(g_n\bar{\sigma} - s_w\right)\left(s_c - g_n\bar{\sigma}\right)}{\left(1-\eta\bar{\sigma}\right)g_n\bar{\sigma}} \\
j_{23} &= \left.\frac{\partial \dot{u}}{\partial \varepsilon}\right|_{E^+} = -\frac{\lambda\eta\left(s_c - s_w\right)\left(1-u^+\right)u^+}{1-\eta\bar{\sigma}} \\
&= -\frac{\lambda\eta\left(s_c - s_w\right)g_n\bar{\sigma}\left(s_c - g_n\bar{\sigma}\right)}{\left(1-\eta\bar{\sigma}\right)s_c^2} \\
j_{24} &= \left.\frac{\partial \dot{u}}{\partial \sigma}\right|_{E^+} = -\frac{\lambda\eta u^+}{1-\eta\bar{\sigma}} = -\frac{\lambda\eta\left(s_c - g_n\bar{\sigma}\right)}{\left(1-\eta\bar{\sigma}\right)s_c} \\
j_{32} &= \left.\frac{\partial \dot{\varepsilon}}{\partial u}\right|_{E^+} = \frac{\eta}{1-\eta\bar{\sigma}}\left(s_c - s_w\right)\left(1-\varepsilon^+\right)\varepsilon^+ - \frac{s_w}{\bar{\sigma}}\varepsilon^+ \\
&= \frac{s_w s_c\left(g_n\bar{\sigma} - s_w\right)\left(\eta s_c - g_n\right)}{\left(1-\eta\bar{\sigma}\right)\left(s_c - s_w\right)g_n^2\bar{\sigma}^2}
\end{aligned}$$

$$
\begin{aligned}
j_{33} &= \left.\frac{\partial \dot{\varepsilon}}{\partial \varepsilon}\right|_{E^+} = g_n - \frac{\eta\left(s_c - s_w\right)}{1 - \eta\bar{\sigma}}\left(1 - \varepsilon^+\right)\left(1 - u^+\right) + \frac{s_w\left(1 - u^+\right)}{\bar{\sigma}} \\
&= -\frac{\left(s_c - s_w\right) g_n - \eta s_c\left(g_n\bar{\sigma} - s_w\right)}{\left(1 - \eta\bar{\sigma}\right) s_c} \\
j_{34} &= \left.\frac{\partial \dot{\varepsilon}}{\partial \sigma}\right|_{E^+} = -\left[\frac{1 + \left(1 - \eta\bar{\sigma}\right) g_n}{\bar{\sigma}\left(1 - \eta\bar{\sigma}\right)}\right]\left(1 - \varepsilon^+\right) + \frac{\left[s_w - s_w\varepsilon^+\left(1 - u^+\right)\right]}{\left(\bar{\sigma}\right)^2} \\
&= -\frac{s_w\left(s_c - \bar{\sigma} g_n\right)}{\left(1 - \eta\bar{\sigma}\right)\left(s_c - s_w\right) g_n\bar{\sigma}^2} \\
j_{44} &= \left.\frac{\partial \dot{\sigma}}{\partial \sigma}\right|_{E^+} = -\left(1 + g_n\right)
\end{aligned}
$$

Under the following two assumptions, which can be shown to hold for a wide range of plausible parameter values,:

**Assumption 1:** The desired capital-output ratio is high enough and such that:

$$\bar{\sigma} > 1/\eta \tag{27}$$

**Assumption 2:** The natural growth rate is high enough to satisfy:

$$g_n > -\frac{\eta s_w s_c}{s_c\left(1 - \eta\bar{\sigma}\right) - s_w} > 0 \tag{28}$$

the signs of all non-zero elements of $\mathbf{J}|_{E^+}$ are uniquely determined and such that:

$$
\begin{array}{llll}
j_{12} < 0 & j_{13} > 0 & j_{14} > 0 & \\
j_{21} > 0 & j_{22} < 0 & j_{23} > 0 & j_{24} > 0 \\
j_{32} < 0 & j_{33} < 0 & j_{34} > 0 & \\
j_{44} < 0 & & &
\end{array}
$$

Then, simple calculation show that the characteristic equation is given by:

$$\left(\lambda - j_{44}\right)\left[\lambda^3 + A\lambda^2 + B\lambda + C\right] = 0$$

where

$$
\begin{aligned}
A &= -\left(j_{22} + j_{33}\right) > 0 \\
B &= j_{22}j_{33} - j_{23}j_{32} - j_{12}j_{21} > 0 \\
C &= j_{12}j_{21}j_{33} - j_{13}j_{21}j_{32} > 0
\end{aligned}
$$

Thus, one characteristic roots is certainly negative and equal to

$$\lambda_1 = j_{44} < 0$$

whereas the other three are the roots of:

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0$$

Given that $A$, $B$ and $C$ are all positive, from the necessary and sufficient Routh-Hurwitz conditions for all characteristic roots to be negative if real or have a negative real parts if complex it follows that the equilibrium point is locally asymptotically stable if and only if:

$$AB - C > 0$$

Selecting the propensity to save out of wages as the bifurcation parameter, it is possible to show that:

$$\begin{aligned} &A(s_w)\,B(s_w) - C(s_w) \\ &\quad = \frac{1}{(1-\eta\bar{\sigma})^2 g_n^2 \bar{\sigma}^2} \left\{ -\left[ s_w(\eta s_c - g_n)\left(1-u^+\right) - \lambda\eta\,(g_n\bar{\sigma} - s_w)\,u^+ \right.\right. \\ &\qquad\qquad \left. + (1-\eta\bar{\sigma})\,g_n^2\bar{\sigma}\right] \eta s_c\,(g_n\bar{\sigma} - s_w)\,u^+ \left[\lambda g_n + v^+ h'\left(v^+\right)\right] \\ &\qquad\qquad\qquad\qquad \left. + (1-\eta\bar{\sigma})\,g_n^2\bar{\sigma}\eta s_c\,(g_n\bar{\sigma} - s_w)\,v^+ h'\left(v^+\right)u^+ \right\} \end{aligned}$$

so that

$$A(s_w)\,B(s_w) - C(s_w) \gtreqless 0$$

according to whether:

$$\begin{aligned} &-\left[ s_w(\eta s_c - g_n)\left(1-u^+\right) - \lambda\eta\,(g_n\bar{\sigma} - s_w)\,u^+ \right. \\ &\left. + (1-\eta\bar{\sigma})\,g_n^2\bar{\sigma}\right] \eta s_c\,(g_n\bar{\sigma} - s_w)\,u^+ \left[\lambda g_n + v^+ h'\left(v^+\right)\right] \\ &+\eta s_c\,(g_n\bar{\sigma} - s_w)\,(1-\eta\bar{\sigma})\,g_n^2\bar{\sigma} v^+ h'\left(v^+\right)u^+ \gtreqless 0 \end{aligned}$$

which simplifies to:

$$\begin{aligned} F(s_w) \equiv \left[ -s_w(\eta s_c - g_n)\left(1-u^+\right) + \lambda\eta\,(g_n\bar{\sigma} - s_w)\,u^+ \right] \left[\lambda g_n + v^+ h'\left(v^+\right)\right] \\ - (1-\eta\bar{\sigma})\,g_n^3\bar{\sigma}\lambda \gtreqless 0 \end{aligned}$$

When workers do not save, so that $s_w = 0$, Assumption 1 guarantees that we have:

$$F(0) > 0 \leftrightarrow A(0)\,B(0) - C(0) > 0$$

namely, that all the Routh-Hurwitz conditions for stable roots are satisfied. The same holds for sufficiently small $s_w > 0$ by continuity. We have thus proved the following:

**Proposition 1** *Under Assumptions 1 and 2, if the propensity to save out of wages is sufficiently low and such that*

$$F(s_w) > 0 \tag{29}$$

*the positive equilibrium $E^*$ of the dynamical system (15)-(18) is locally asymptotically stable.*

The function $F(s_w)$ is a decreasing function of $s_w$. As a consequence, when $s_w$ is further increased, the difference $A(s_w)B(s_w) - C(s_w)$ sooner or later becomes negative so that there exists a parameter value $s_{wH}$ such that $A(s_{wH})B(s_{wH}) - C(s_{wH}) = 0$ and $F'(s_{wH}) \neq 0$. We have thus established the possibility of persistent cyclical paths of the variables of the model:

**Proposition 2** *Under Assumptions 1 and 2, there exists a value of the propensity to save out of wages*

$$s_{wH} = \frac{g_n\bar{\sigma}\lambda\eta u^+ \left[\lambda g_n + v^+ h'(v^+)\right] - (1-\eta\bar{\sigma})\, g_n^3\bar{\sigma}\lambda}{\left[\lambda g_n + v^+ h'(v^+)\right]\left[(\eta s_c - g_n)(1-u^+) + \lambda\eta u^+\right]} > 0$$

*at which the dynamical system (15)-(18) undergoes an Hopf bifurcation.*

Proposition 2, however, proves only the existence part of the Hopf bifurcation theorem and says nothing on the uniqueness and stability of the closed orbits. With the help of numerical simulation, however, it is not difficult to find values of the parameters of the model that satisfy all conditions required by the analysis we have performed and that are such that when the positive equilibrium point looses local stability a stable closed orbit emerges. For example, using for simplicity a linear generalised Phillips curve-type dynamics for real wages, such that:

$$\hat{w} = h(v) + \lambda\hat{v} \approx -\gamma + \rho v + \lambda\hat{v}$$

and the following set of parameter values:

$$\begin{array}{llll} \eta = 1 & s_c = 0.8 & \bar{\sigma} = 2.57 & \alpha = 0.0221 \\ \beta = 0.0037 & \gamma = 0.92 & \rho = 1 & \lambda = 0.1 \\ s_w = 0.0354 & & & \end{array}$$

all trajectories converge to a limit cycle. More precisely, as appears from Figures 1 and 2, and as was to be expected from local analysis, the capital-output ratio converges monotonically and rapidly to its equilibrium value, whereas the other three variables persistently fluctuate around their equilibrium values.

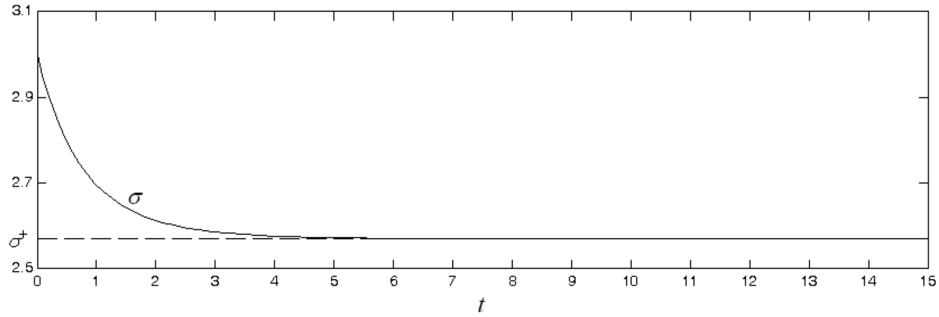


Figure 1: Convergence of the capital-output ratio to its equilibrium value $\sigma^+ = \bar{\sigma}$

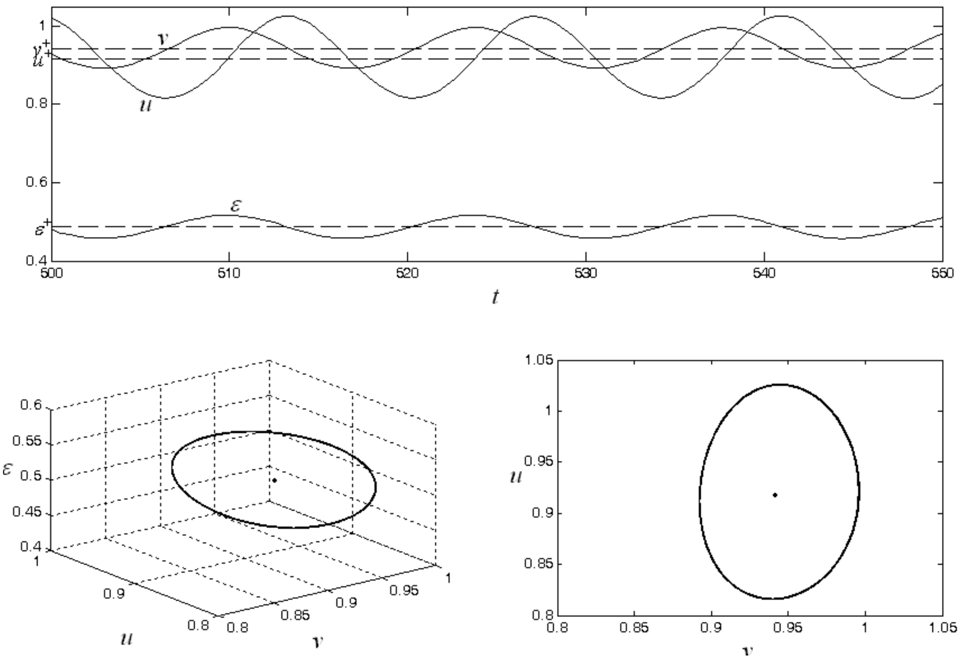


Figure 2: Visualisation of the limit cycle in 2D and 3D

## 4 Conclusions

In this chapter we have investigated a generalisation of the growth cycle model obtained by taking a lead from some Vela's suggestions. We have arrived at a version of the model with a four-dimensional dynamical system in the employment rate, the share of wages, the proportion of capital held by capitalists and the capital-output ratio. First, we have utilized the HBT to prove the existence of persistent and self-sustained growth cycles and, second, we have given numerical evidence that the emerging limit cycle is stable. Thus, we can conclude that our modification of the growth cycle model represents a novel case in which that model gives rise to persistent fluctuations of the variables (to be added to those already existing in the economic literature).

What we have done in this paper, however, provides only a basic understanding of the dynamics of the model. Much remains to be done in terms of local and global bifurcation analysis. Moreover, to concentrate on the effects of the introduction into the modelo of the differential savings hypothesis and disequilibrium in the goods market, I have used the simple linear flexible accelarator as an explanation of capitalists' investment and neglected many other possible extensions of the model, well established in the existing literature. However, given that the existence part of the Hopf bifurcation theorem and some useful numerical techniques such as two-parameter bifurcation diagrams and chaos plots can be applied to higher dimensional dynamical systems as well, there seems to be room for a closer analysis of the dynamics of the present model and for further generalisation along the lines here suggested and sketched.

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