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A Meaningful Sufficient Condition for the Uniqueness of the Internal Rate of Return

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I. Introduction

The analysis of the most recent contributions shows that the authors' attention has definitely shifted from a concern with the significance of the internal rate of return to an interest in the more technical aspects of its existence and uniqueness (1). This shift seems to rest on the assumption that the economic meaning of the internal rate of return was

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(1) See Kaplan S.(1965), "A Note on a Method for Precisely Determining the Uniqueness or Non-uniqueness of the Internal Rate of Return for a Proposed Investment", The Journal of Industrial Engineering (January-February), vol. 16, pp. 70-71 ; C.J. Norstrom (1972), " A Sufficient Condition for a Unique Non-negative Internal Rate of Return", Journal of Financial and Quantitative Analysis (June), vol. 7, pp. 1835-1839; C.De Faro (1973), " A Sufficient Condition for a Unique Non-negative Internal Rate of Return: a Comment", Journal of Financial and Quantitative Analysis (September), vol. 8, pp. 683-684; D.L. Aucamp and W.L. EcKardt (1976), "A Sufficient Condition for a Unique Non-negative Internal Rate of Return: Comment", Journal of Financial and Quantitative Analysis (June), vol. 11, pp. 329-332; C. De Faro (1978), " A Sufficient Condition for a Unique Non-negative Internal Rate of Return: Further Comment", Journal of Financial and Quantitative Analysis (September), vol. 13, pp. 577-584; R.H. Bernhard (1979), " A More General Sufficient Condition for a Unique Internal Rate of Return", Journal of Financial and Quantitative Analysis (June), vol. 14, pp. 337-341; R.H. Bernhard (1980), "A Simplification and an Extension of the Bernhard - De Faro Sufficient Condition for a Unique Internal Rate of Return", Journal of Financial and Quantitative Analysis (March), vol. 15, pp. 201-209.

established once and for all by Solomon and by Hirshleifer in the late fifties (2). In reality these two authors were quite pessimistic in their conclusions since they denied that the internal rate of return really means the interest rate earned by funds invested in a project. This is why recent trends in the literature appear to be rather contradictory: if one denies that the internal rate of return is an adequate measure of a project's profitability (following Solomon and Hirshleifer), why should one be interested in testing its existence and uniqueness?

We believe that a logical attitude would involve adopting one of the two following approaches (which we consider mutually exclusive). The first approach would be to consider the pessimistic analysis of Solomon and Hirshleifer as conclusive, but in this view testing the existence and the uniqueness of the internal rate of return becomes a pretty useless exercise. The second approach would be to reject the Solomon-Hirshleifer analysis, but in this case one should not investigate technical properties of the internal rate of return, such as its existence and uniqueness, before assessing its conceptual validity.

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(2) See E. Solomon (1956), "The Arithmetic of Capital Budgeting Decisions", Journal of Business (April), vol. 29, pp. 124-129, and J. Hirshleifer (1958), "On the Theory of Optimal Investment Decision", Journal of Political Economy (August), vol. 66, pp. 329-352. See also P. Puccinelli (1976), "Alcuni Aspetti Controversi della Teoria della Domanda di Investimenti", Note Economiche (gennaio-febbraio), vol. 9, pp. 35-93, and M. Trovato (1972), "Sulla Validità del Tasso di Rendimento Interno come Criterio di Selezione di Progetti di Investimento", Giornale degli Economisti e Annali di Economia (settembre-ottobre), vol. 31, pp. 678-691.

The main purpose of this article is to investigate simultaneously the significance and the uniqueness of the internal rate of return, and to show that theese two aspects are rather closely related. In Section 2 we shall prove a sufficient condition for the uniqueness of the internal rate of return. In Section 3 we shall show that this condition is crucial to understanding the real economic meaning of the internal rate of return. In fact we shall see that such a condition represents a sort of boundary between two different categories of internal rates of return which have partially different economic meanings. Despite this minor difference, we shall argue (against Solomon and Hirshleifer) that all the internal rates of return have a basic common meaning, which is the interest rate earned by invested funds (3). In Section 4 we shall discuss some major consequences of Sections 2 and 3. In Section 5 we shall summarize and criticize briefly the Solomon - Hirshleifer analysis. Finally in Section 6 we shall prove that the sufficient condition established in Section 2 generalizes Soper's well-known condition (4). Since our proof in Section 2 is much simpler than Soper's, a minor result of this article is that it provides a simpler proof for a more general proposition.

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(3) Some aspects of the ideas expressed in Section 3 can be found in embroyonic form in H.J. Bailey (1959), " Formal Criteria for Investment Decisions", Journal of Political Economy (October), vol. 67, pp. 476-488.

(4) See C.S. Soper (1959) "The Marginal Efficiency of Capital: a Further Note",The Economic Journal (March), vol. 69, pp. 174-177.

2. A Sufficient Condition for the Uniqueness of the Internal Rate of Return

The following definition of a productive project has been adopted in this paper.

Definition 1 - We define a productive project as a vector of dated cash flows $(a_o,..., a_n)$ such that $a_o < o$, $a_n \neq o$, $a_j > o$ for at leat one $j \in \{1,...,n\}$.

Moreover we have adopted the following definition of an internal rate of return attached to a project.

Definition 2 - An interest rate $r > -1$ (5) is defined as an internal rate of return attached to a project $(a_o,... a_n)$ if its transformation $R = 1 + r$ satisfies the following equality:

$$\sum_{j=0}^{n} a_j R^{n-j} = 0 . \quad (1)$$

In this section we shall prove the following proposition.

(5) Following the most general of the common practices, we consider all interest rates smaller than or equal to - 1 as economically meaningless.

Proposition 1 - An internal rate of return r attached to a project $(a_o,..., a_n)$ is unique if the following inequalities are satisfied:

$$\begin{aligned} &a_o R + a_1 \leq o \\ &a_o R^2 + a_1 R + a_2 \leq o \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &a_o R^{n-1} + a_1 R^{n-2} + ... + a_{n-2} R + a_{n-1} \leq o . \end{aligned} \quad (2)$$

Proof - Let r be an internal rate of return, attached to a project $(a_o,..., a_n)$, which satisfies inequalities (2). Its transformation R [since R satisfies equality (1)] is a distinct, real and positive root of the following n.th degree polynominal:

$$A(x) = \sum_{j=0}^{n} a_j x^{n-j} .$$

The same is true for the transformation R' of another internal rate of return r' (if any) attached to the same project $(a_o, ..., a_n)$. Therefore, in order to prove that r is unique, we need to prove that the polynominal A(x) has no distinct, real and positive root other than R. We consider the n-1.th degree polynominal $Q(x) = A(x) \div (x - R)$, whose distinct, real and positive roots (if any) are the same as the distinct, real and positive roots of A(x) other than R (6). Moreover we indicate by $q_o,...,q_{n-1}$ the coefficients

(6) See for example A. Kurosh (1971), Cours d'Algebre Superieure, Moscou: Editions Mir, esp. pp. 148-152.

of Q(x). It is well known that the following equalities hold true (7):

$$q_o = a_o$$
$$q_1 = a_oR + a_1$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$q_{n-1} = a_oR^{n-1} + a_1R^{n-2} + \ldots + a_{n-2}R + a_{n-1} .$$

Since $a_o < o$ according to Definition 1, and since by hypothesis inequalities (2) are satisfied, all Q(x)'s coefficients are non-positive. Therefore Descartes' rule of signs (8) guarantees that Q(x) has no distinct, real and positive root. The proof is thus complete.

3. The Economic Meaning of the Internal Rate of Return

Definition 2 is known to be an attempt to formalize the intuitive idea of an interest rate earned by funds invested in a project (9). In this section we shall show that this attempt is rather successfull. We shall also show that the economic meaning of the internal rate of return changes

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(7) See for example A. Kurosh, Op.cit., esp. pp. 148-152.
(8) See for example A. Kurosh, Op. cit., esp. pp. 263-267.
(9) See J.M. Keynes (1936), The General Theory of Employment, Interest and Money, London: Macmillan and Co., Limited, esp. Chapter 11.

partially depending on whether it satisfies the uniqueness condition proved in Section 2.

Our starting point is the following proposition.

<u>Proposition 2</u> - An interest rate $r > -1$ is an internal rate of return attached to a project $(a_o, \ldots a_n)$ if and only if there exists an ordered set of n pairs of dated cash flows

$$B = \left\{ \left(b_o^{(1)}, b_1^{(1)}\right), \ldots, \left(b_{n-1}^{(n)}, b_n^{(n)}\right) \right\}$$

(the higher index indicates the pair, the lower one indicates the date) such that

$$(3) \qquad \begin{array}{l} b_1^{(1)} = -b_o^{(1)} R \\ \cdots\cdots\cdots \\ b_n^{(n)} = -b_{n-1}^{(n)} R , \end{array}$$

and that

$$(4) \qquad b_o^{(1)} = a_o$$

$$(5) \qquad \left\{ \begin{array}{l} b_1^{(1)} + b_1^{(2)} = a_1 \\ \cdots\cdots\cdots \\ b_{n-1}^{(n-1)} + b_{n-1}^{(n)} = a_{n-1} \end{array} \right.$$

$$(6) \qquad b_n^{(n)} = a_n .$$

Proof – First we shall prove that the condition is sufficient. Given a project $(a_o,\dots, a_n)$, let $r > -1$ be an interest rate such that there exist a set B satisfying equalities (3) to (6). By substituting the first n - 1 equalities (3) in equalities (5), and the n.th equality (3) in equality (6), we get:

$$
(7)\qquad
\begin{aligned}
b_o^{(1)} &= a_o \\
b_1^{(2)} &= b_o^{(1)} R + a_1 \\
&\cdots\cdots\cdots\cdots \\
b_{n-1}^{(n)} &= b_{n-2}^{(n-1)} R + a_{n-1} \\
b_{n-1}^{(n)} R + a_n &= o \, .
\end{aligned}
$$

Finally, by substituting the first equality (7) in the second one, the second one (so obtained) in the third one, and so on, we get equality (1).

We now want to prove that the condition is necessary. Let r be an internal rate of return attached to a project $(a_o,\dots, a_n)$. We consider the following ordered set of n pairs of dated cash flows:

$$
(8)\qquad
\begin{aligned}
&\left(b_o^{(1)} = a_o,\ b_1^{(1)} = -a_o R\right) \\
&\left(b_1^{(2)} = a_o R + a_1,\ b_2^{(2)} = -a_o R^2 - a_1 R\right) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\left(b_{n-2}^{(n-1)} = a_o R^{n-2} + \dots + a_{n-2},\ b_{n-1}^{(n-1)} = -a_o R^{n-1} - \dots - a_{n-2} R\right) \\
&\left(b_{n-1}^{(n)} = a_o R^{n-1} + \dots + a_{n-1},\ b_n^{(n)} = -a_o R^n - \dots - a_{n-1} R\right).
\end{aligned}
$$

One can easily verify that these pairs satisfy equalities (3) to (5). Furthermore, equality (6) is satisfied as well since R satisfies equality (1) by hypothesis. The proof is now complete.

Proposition 2 is the first step toward a full understanding of the economic meaning of Definition 2. Infact, according to Proposition 2, an internal rate of return attached to a project can be redefined as an interest rate $r > -1$ such that this project could be broken down into n consecutive one-period investment, financing or null operations performed at the (uniform) interest rate r itself. These operations are represented by pairs in set B. A pair in this set represents an investment operation if its first cash flow is negative and the second one is positive. On the contrary, a pair in set B represents a financing operation if its first cash flow is positive and the second one is negative. Finally, a pair in set B represents what we call a 'null' operation if its first cash flow is null as well as the second one.

For example, the interest rate $r = 0.2$ is an internal rate of return attached to the project (-10, +16, +5, -20, +24) since this project can be broken down into the following four consecutive one-period operations performed at the (uniform) interest rate $r = 0.2$ itself: $\{(-10, +12), (+4, -5), (0,0), (-20, +24)\}$.

Let us now consider an internal rate of return r attached to a project $(a_o, \dots, a_n)$. The necessity part of Proposition 2 implies that at least one set B exists which satisfies equalities (3) to (6). On the other hand, by examining equalities (3) to (6) themselves, one sees immediately that at most one set B satisfying these equalities is associated with a given interest rate. Therefore set B associated with r is unique. Moreover the procedure used to prove the necessity part of Proposition 2 also proves that such a (unique) set B is set (8). Therefore the following proposition holds true.

<u>Proposition 3</u> - Given an internal rate of return r attached to a project $(a_o, \dots, a_n)$, there exists only one set B which satisfies equalities (3) to (6), and this set is set (8).

Let us now go back to condition (2). On the basis of Proposition 2 and Proposition 3, we can rewrite condition

(2) as follows:

$$b_1^{(2)} \leq o$$

$$b_2^{(3)} \leq o$$

$$\dots\dots$$

$$b_{n-1}^{(n)} \leq o \; .$$

Since $b_o^{(1)} < o$ (see Definition 1), this rewriting of condition (2) allows us to recognize it as the condition that no operation in set B be a financing operation. If this is the case, the internal rate of return means simply the rate of interest earned by funds invested in a project. On the contrary, if condition (2) is not satisfied, then at least one operation (but not the first one) in set B is a financing operation, so that the internal rate of return means both the interest rate earned by funds invested in a project and the interest rate paid on funds financed by the project. One should note that funds invested in the project are those contained in the set $\left\{ \left| b_{j-1}^{(j)} \right| : \left(b_{j-1}^{(j)}, b_j^{(j)} \right) \in B, \; b_{j-1}^{(j)} < o \right\}$, while funds financed by the project are those contained in the set $\left\{ b_{j-1}^{(j)} : \left(b_{j-1}^{(j)}, b_j^{(j)} \right) \in B, \; b_{j-1}^{(j)} > o \right\}$.

Therefore, if we interpret mathematical condition (2) economically, it proves to be a sort of boundary between two classes of internal rates of return that have partially different economic meanings.

4. Further Comments

Section 2 and 3 can be summarized as follows:

(i) any internal rate of return means the interest rate earned by funds invested in a project;

(ii) an internal rate of return that has this economic meaning and no other one is unique;

(iii) an internal rate of return that also means the interest rate paid on funds financed by the project may be unique or may not be.

Point (iii) is due to the fact that condition (2) is sufficient, but not necessary, for the uniqueness of the internal rate of return. In order to prove this, let us consider the counter-example represented by the internal rate of return $r = 0.7$ attached to the project (-100, +270, -170). One can easily verify that such an internal rate is unique (10). Nevertheless it does not satisfy condition (2) since we have: $a_0 R + a_1 = 100$.

In this section we want to discuss some rather relevant consequences of Sections 2 and 3.

First of all, let us call A the set of all projects to which at least one internal rate of return is attached (11).

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(10) Infact the other root of the 2.nd degree polynomial $100x^2 + 270x - 170$ is economically meaningless (see footnote 5).

(11) It is well known that there are projects to which no internal rate of return is attached. These are the projects such that no root of the polynomial $\sum_{j=0}^{n} a_j x^{n-j}$ be economically meaningful (see footnote 5).

Inside set A, let us distinguish between set A_1 of projects to which one and only one internal rate of return is attached, and set A_2 of projects to which al least two internal rates of return are attached. According to points (ii) and (iii), there may be two alternative economic interpretations for an internal rate of return attached to a project contained in A_1. Such a rate may have the simple meaning of an interest rate earned by invested funds. On the contrary, such a rate may have the double meaning of an interest rate earned by invested funds and paid on financed funds. Moreover points (ii) and (iii) also imply that any internal rate of return attached to a project contained in A_2 has the double meaning of an interest rate earned by invested funds and paid on financed funds. Infact, let us suppose that at least one internal rate of return attached to a project contained in A_2 mean only the interest rate earned by invested funds. According to point (ii), this internal rate should be unique; and this contradicts the hypothesis that the project belonges to A_2.

Since no two internal rates of return having (partially) different economic meanings can be attached to the same project, set A may be partitioned into set A^1 of investment projects (whose internal rate of return, necessarily unique, means only the interest rate earned by invested funds), and set A^2 of investment-financing projects (whose internal rates of return, one or more, mean both the interest rate earned by invested funds and the interest rate paid on financed funds). We already know that $A^1 \subset A_1$ and that $(A_1-A^1) \cup A_2 = A^2$.

Another point we want to make clear is the following. Our analysis implies that it is sensible to apply both the so-called acceptability criterion and the so-called ranking criterion only within set A^1. For projects contained in set $A_1 - A^1$ some conceptual difficulties arise from the double meaning of their (still unique) internal rate of return (12). Therefore the uniqueness of the internal rate of return is a necessary, but not sufficient, condition forthe viability of these two criteria.

Finally we want to discuss briefly a point concerning projects contained in set A_2. It is well known that choosing among the internal rates of return attached to a project contained in A_2 is an unsolved problem. Let us now consider sets $B_1, \ldots, B_k$ [satisfying equalities (3) to (6)] associated respectively with the internal rates of return $r_1, \ldots, r_k$ attached to a project contained in A_2. Proposition 3 implies that for any two indices i and j contained in $\{1, \ldots, k\}$ we have $B_i \neq B_j$. Therefore to choose one internal rate of

(12) One should note that if the internal rate of return attached to a project contained in $A_1 - A^1$ meant only the interest rate paid on funds financed by the project, both the acceptability criterion and the ranking criterion (adequately adjusted) would still be applicable. The acceptability criterion should be adjusted as follows: a project should be accepted if its internal rate of return is smaller than the market interest rate. Similarly, the ranking criterion should be adjusted as follows: a project should be preferred to another project if its internal rate of return is smaller.

return among feasible rates $r_1, \ldots, r_k$ is the same as to choose one way of breaking down the given project among feasible ways $B_1, \ldots, B_k$.

For example let us consider the two internal rates of return $r_1 = 1$ and $r_2 = 2$ attached to the project (-1, +5, -6) (13). Set B_1 associated with r_1 is $\{(-1, +2), (+3, -6)\}$, while set B_2 associated with r_2 is $\{(-1, +3), (+2, -6)\}$. We may choose between the following two statements. The first is that funds invested in and financed by the project yield 100%. The second statement is that funds invested in and financed by the project yield 200%. Nevertheless if we choose the first statement, to be consistent, we must admit that the project absorbs one dollar at time zero, and that it finances three dollars at time one. On the contrary, if we choose the second statement, we must admit that the project absorbs one dollar at time zero and that it finances two dollars at time one (14).

(13) The project considered here is discussed in J.H. Lorie and L.J. Savage (1955), "Three Problems in Rationing Capital", Journal of Business (October), vol. 28, pp. 229-239.

(14) We want to remark that we are not suggesting how to choose among $r_1, \ldots, r_k$. On the contrary we are simply discussing some aspects of this choice.

5. The Common Interpretation of the Internal Rate of Return

Solomon and Hirshleifer investigated the economic meaning of Definition 2 (15). The Solomon-Hirshleifer analysis can be summarized as follows. Let r be an internal rate of return attached to a project

$$(9) \qquad (a_o, \dots, a_n) \; .$$

Transformation R = 1+r satisfies equality (1) according to Definition 2. But equality (1) in the same as the following:

$$a_o R^n = \sum_{j=1}^{n} a_j R^{n-j} \; .$$

Thus r can be interpreted as the interest rate earned by funds invested in the following project:

$$(10) \quad \left(b_o = a_o, \; b_1 = o, \dots, b_{n-1} = o, \; b_n = \sum_{j=1}^{n} a_j R^{n-j}\right) .$$

One should note that funds invested in project (10) are limited to $/a_o/$, which is invested at time zero. One should also note that project (10) is obtained from project (9) by postponing intermediate cash flows $a_1, \dots, a_{n-1}$ to time n by means of an interest rate exactly equal to r (16).

So far we perfectly agree with Solomon and Hirshleifer,

(15) See E. Solomn, Op.cit., and J. Hirshleifer, Op. cit.
(16) It is quite obvious that project (10) is the same as project (9) in the special case where all intermediate cash flows of project (9) are null. Therefore in this special case r can be immediately interpreted as the interest rate earned by funds invested in project (9).

but these authors claim they go much further. Infact, since r can be interpreted as the interest rate earned by funds invested in project (10), Solomon and Hirshleifer 'deduce' that r cannot be interpreted as the interest rate earned by funds invested in project (9) unless external opportunities allow for the transformation of the latter project into the former, and unless such opportunities be actually taken by the investor.

Hirshleifer asserts that the proposition according to which r is the interest rate earned by funds invested in project (9) is based upon "...the implicit assumption...that all intermediate cash flows are reinvested (or borrowed, if cash flows are negative) at the rate r itself" (17). Hirshleifer is even more pessimistic in the following passage where he recognizes that this assumption is hardly realistic:

> There will not normally be other investment opportunities arising for investment of intermediate cash proceeds at the rate r, nor is it generally true that intermediate cash inflows (if required) must be obtained by borrowing at the rate r. The rate r...will only by rare coincidence represent relevant economic alternatives (18).

The analysis we have made in Section 3 shows that r can be interpreted correctly as the interest rate earned by funds invested in project (9) independently of any external opportunity

(17) See J. Hirshleifer, Op.cit., p. 351.
(18) See J. Hirshleifer, Op.cit., p. 350.

6. Soper's Sufficient Condition for the Uniqueness of the Internal Rate of Return

A well known sufficient condition for the uniqueness of the internal rate of return was proved by Soper (19). Put into our notation, Soper's condition can be rewritten as follows:

$$
(11)\quad
\begin{aligned}
& a_o + a_1 R^{-1} < o \\
& a_o + a_1 R^{-1} + a_2 R^{-2} < o \\
& \ldots\ldots\ldots\ldots\ldots\ldots \\
& a_o + a_1 R^{-1} + a_2 R^{-2} + \ldots + a_{n-1} R^{1-n} < o .
\end{aligned}
$$

Since R is a positive real number (see Definition 2), inequalities (11) are the same as the following:

$$
(12)\quad
\begin{aligned}
& a_o R + a_1 < o \\
& a_o R^2 + a_1 R + a_2 < o \\
& \ldots\ldots\ldots\ldots\ldots\ldots \\
& a_o R^{n-1} + a_1 R^{n-2} + \ldots + a_{n-2} R + a_{n-1} < o .
\end{aligned}
$$

Inequalities (12) show that Soper's condition is dominated by our condition (2) established in Section 2.

(19) See C.S. Soper, Op.cit. In Soper's article a sufficient condition for the existence of the internal rate of return is also established. This condition is that the last cash flow of a project be null.

One should note that in the light of Proposition 2 Soper's condition (11) can be interpreted to mean that all operations in set B, satisfying equalities (3) to (6), must be investment operations.

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