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A Generalized Internal Rate of Return Depending on the Cost of Capital

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A well-known measure of the profitability of productive projects is the internal rate of return. Unfortunately, feasible projects can be partitioned into the following sets: set A_1 of projects to which no internal rate of return is attached, set A_2 of projects to which one and only one internal rate of return is attached, set A_3 of projects to which more than one internal rate of return is attached. Moreover, even when the internal rate of return is a formally viable and unambiguous measure of the projects' profitability (that is, when it concerns projects belonging to set A_2) most authors believe that such a rate is hardly able to formalize the intuitive idea of an interest rate earned by the capital (i.e. the funds) invested in a project(1).

In this paper, we propose a "new" internal rate of return such that: (i) it clearly refer to the rate of interest earned by the capital invested in a project and (ii) it be unique for any feasible project.

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(1) See E. Solomon (1956), /37/; J. Hirshleifer (1958), /13/; N. Trovato (1972), /41/; P. Puccinelli (1976), /31/. We believe that this common opinion is too pessimistic, and we think we have shown recently that any internal rate of return actually means the interest rate earned by invested capital (see S. Gronchi (1982), /11/). Unfortunately, our analysis shows that many internal rates of return also mean the interest rate paid on what we call "the capital financed by the project", so that the double meaning of these rates makes it impossible to use them as a basis for formulating both an acceptability criterion and a ranking criterion;

From a formal viewpoint, the internal rate of return we present is a function of the cost of capital (market rate of interest). Nevertheless, we shall prove that for a set of feasible projects, B_1, this function is constant, so that the internal rate of return actually depends on the cost of capital only for projects belonging to the complementary set, B_2.

An interesting point is the connection between the new internal rate of return and the traditional one. We shall prove that $B_1 \subset A_2$, and that for any project belonging to B_1, the new internal rate of return (which is constant with respect to the capital cost) is the same as the traditional one. Moreover, we shall prove that for any project belonging to set

$$(A_2 - B_1) \cup A_3,$$

the traditional internal rates of return (one or more) are the values taken by the new internal rate of return in accordance with some particular values of the cost of capital. Therefore the new internal rate of return, far from conflicting with traditional rates of return internal to a project (if any) allows for their full retrieval, and in addition it eliminates any antagonism among them in case they are more than one. Thus the internal rate of return we are presenting generalizes the traditional definition and overcomes its contradictions.

Of course on the basis of this generalized internal

rate of return it would also be possible to formulate both a criterion for establishing the acceptability of a single given project, and a criterion for ranking a number of given projects according to their degree of profitability. These two criteria and their macroeconomic implications on the aggregate investment function will be the object of a forthcoming investigation.

Our starting point is the following (very general) definition of a productive project.

Definition 1. We define a **productive project** as a vector $(a_0, \dots, a_n)$ of dated cash flows such that $a_0 < 0$, $a_n \neq 0$, $a_j > 0$ for at least one $j \in \{1, \dots, n\}$.

We give the following definition of an ordered pair of interest rates, internal to a project [2].

Definition 2. An ordered pair of interest rates (r_1,

(2) In this paper, following the common practice, we shall be concerned only with interest rates bigger than - 1 (an interest rate smaller than or equal to - 1 is considered economically meaningless).

r_2) is said to be internal to a project ($a_o,\dots,a_n$) if there exists a matrix nx2 of dated cash flows

$$B = \begin{bmatrix} b_o^1 & b_1^1 \\ \cdots & \cdots \\ b_{n-1}^n & b_n^n \end{bmatrix}$$

(the lower index indicates the date, and the higher one indicates the row to which the cash flow belongs) such that

$$(1)\quad \begin{cases} b_o^1 = a_o \\ b_j^j + b_j^{j+1} = a_j \quad (j = 1,\dots,n-1) \\ b_n^n = a_n \end{cases}$$

and that

$$(2)\quad b_j^j = \begin{Bmatrix} -\,b_{j-1}^j\,(1+r_1) & \text{if} & b_{j-1}^j > 0 \\ 0 & \text{if} & b_{j-1}^j = 0 \\ -\,b_{j-1}^j\,(1+r_2) & \text{if} & b_{j-1}^j < 0 \end{Bmatrix} \quad (j = 1,\dots,n)\ .$$

Moreover we define the interest rate r_1 of a pair (r_1, r_2) internal to a project as the borrowing rate and r_2 as the lending rate of the pair.

It is self-evident that, given a pair (r_1, r_2) internal to a project, matrix B satisfying equalities (1) and (2) is unique.

As an example of Definition 2, the ordered pair of interest rates (0,5 ; 0,2) is internal to the project (-20, +34, -45, +36, +2, -3) since there exists the matrix

$$\begin{bmatrix} -20 & +24 \\ +10 & -15 \\ -30 & +36 \\ 0 & 0 \\ +2 & -3 \end{bmatrix}$$

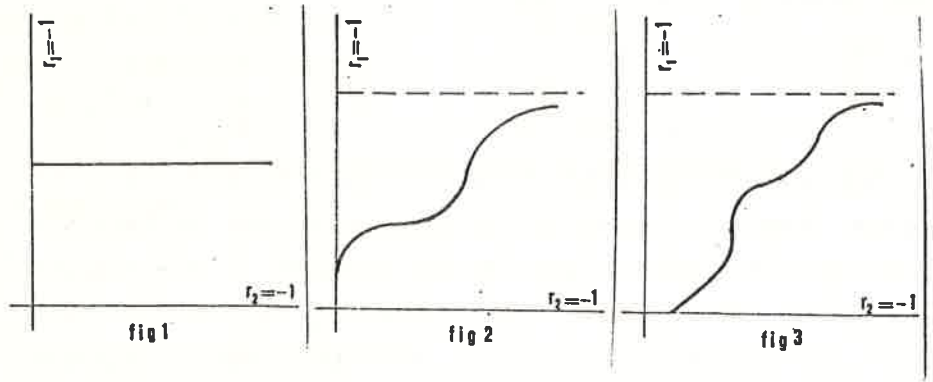
which satisfies equalities (1) and (2).

The economic meaning of Definition 2 is as follows. Given a project ($a_o,\dots,a_n$), let us consider a pair (r_1, r_2) which admits a matrix B satisfying equalities (1) and (2). Since B satisfies equalities (2), B's rows can be imagined as one-period financing operations performed at the uniform interest rate r_1, if their first cash flow is positive. On the contrary, B's rows can be imagined as one-period investment operations performed at the uniform interest rate r_2, if their first cash flow is negative. Finally, B's rows can be imagined as one-period intervals located between one (financing or investment) operation and the next one, if their first cash flow is null. We shall call these intervals one-period null operations. Since matrix B satisfies also equalities (1), the project $(a_o,\dots,a_n)$ may be correctly interpreted as the "sum", or more precisely the set of the n consecutive (financing, investment and null) one-period operations represented by B's rows.

We call R the set of all pairs (r_1, r_2) internal to a project $(a_o,\dots,a_n)$, and we state the following proposition.

Proposition 1. Set R is a continuous curve on (the economically meaningful region of [3]) plane $0-(r_1,r_2)$. If $a_n > o$, R is either a straight line which is parallel to the r_1 axis, or a strictly increasing curve approaching a horizontal asymptote. In both cases, the origin of R (which does not belong to R) is located on the straight line defined by the equation $r_1 = -1$ (see fig's 1 and 2). On the contrary, if $a_n < 0$, R is a strictly increasing curve approaching a horizontal asymptote. The origin of R (which does not belong to R as well) is located on the

(3) See footnote (2).



straight line defined by the equation $r_2 = -1$ (see fig. 3).

The proof of Proposition 1 is given in the Appendix[4].

We call i the cost of capital, and we give the following definition.

Definition 3. We define the rate of return internal

(4) It is easily proved that if R is as shown in fig 1, the same matrix B is associated to any pair $(r_1, r_2) \in R$. Moreover this matrix B is such that

$$b^j_{j-1} \leq 0$$

for every $j \in \{1,\dots,n\}$.

to a project as the lending rate r_2 of a pair $(r_1,r_2) \in R$ whose borrowing rate r_1 equals i.

The economic meaning of Definition 3 is as follows. The internal rate of return as defined above can be regarded as the rate of interest paid 'by the project' on the invested capital (in all periods when investment operations take place), following the hypothesis that the rate of interest charged 'by the project' on the financed capital (in all periods when financing operations take place) equals the interest rate prevailing on the (perfectly competitive) financial market.

We call r the internal rate of return of Definition 3, and k the abscissa of the origin of curve R. Given Definition 3, Proposition 1 implies the following proposition.

Proposition 2. The innternal rate of return r is a function of i defined for any $i > k$, and the graph of this function is the following set of points on (the economically meaningful region of) plane $0 - (i,r)$:

$$\{(i,r) : i = r_1, r = r_2, (r_1,r_2) \in R\} .$$

Now we want to compare Definition 3 with the standard definition of the internal rate of return. To this purpose, let us consider the set of interest rates $\Pi = \left\{\pi : \sum_{j=0}^{n} a_j \cdot (1+\pi)^{n-j} = 0\right\}$. It is well-known that if $\Pi \neq \emptyset$, then the elements of Π are not more than n. We state the following proposition.

Proposition 3. The set of those points on plane $0-(i,r)$ where the graph of function r meets the bisecting line equals the set of points such that $i = r = \pi$, where $\pi \in \Pi$. Written symbolically, we have:

$$\{(i,r) : r = r(i),\ r = i\} = \{(i,r) : i = r = \pi, \pi \in \Pi\},$$

where $r(i)$ means the value of function r at point i.

In order to prove Proposition 3, we shall prove that the set

$$(3) \quad \{(r_1,r_2) : (r_1,r_2) \in R,\ r_1 = r_2\}$$

equals the set

$$(4) \quad \{(r_1,r_2) : r_1 = r_2 = \pi, \pi \in \Pi\}$$

(see Proposition 2). First, we shall prove that set (3) is contained in set (4). To this purpose, let us consider a pair (r_1, r_2) belonging to set (3). Since $r_1 = r_2$, Definition 2 implies there exists a matrix B satisfying the following equalities:

(5) $b_o^1 = a_o$

(6) $b_j^j + b_j^{j+1} = a_j \qquad (j = 1,\ldots,n-1)$

(7) $b_n^n = a_n$

(8) $b_j^j = -b_{j-1}^j\,(1 + r_1) \qquad (j = 1,\ldots,n)$.

By substituting the first $n - 1$ equalities (8) in equalities (6), and the n.th equality (8) in equality (7), from equalities (5) to (8) we get the following:

(9) $b_o^1 = a_o$

(10) $b_j^{j+1} = b_{j-1}^j\,(1 + r_1) + a_j \qquad (j = 1,\ldots,n-1)$

(11) $b_{n-1}^n\,(1 + r_1) + a_n = 0$.

By substituting the first equality in the second one, the second equality (so obtained) in the third one, and so on, from equalities (9) to (11) we get the following:

$$\sum_{j=0}^{n} a_j\,(1 + r_1)^{n-j} = 0 ,$$

which implies that $r_1 = \pi \in \Pi$. Therefore, since $r_2 = r_1$, the given pair (r_1,r_2) belongs also to set (4).

Now we must prove that set (4) is contained in set (3). To this purpose, let us consider a pair (r_1,r_2) belonging to set (4). Since $r_1 = r_2 = \pi$, it is easily proved that the $n \times 2$ matrix whose j.th row is the following pair

$$\left(\sum_{i=0}^{j-1} a_i(1+\pi)^{j-i-1} \;,\; - \sum_{i=0}^{j-1} a_i(1+\pi)^{j-i} \right),$$

satisfies equalities (1) and (2). Therefore, according to Definition 2, the given pair (r_1,r_2) belongs also to set (3).

Proposition 3 implies the following corollaries.

First, set A_1 of projects such that $\Pi = \emptyset$ equals the set of projects such that the graph of function r never meet the bisecting line[5].

Second, set A_2 of projects such that Π contain one and only one element, equals the set of projects such that the graph of r meet the bisecting line once and only once. This set is the union of set B_1 of projects such that r be a constant function of i (see fig 4) and of set $A_2 - B_1$ of projects such that the graph of r be an increasing curve meeting the bisecting line once and only once (see fig's 5 and 6). For projects contained in B_1, $r = \pi$ for

(5) Since the graph of r meets the bisecting line at least once if $a_n > 0$ (see Proposition 1), this corroborates the well-known proposition that $a_n < 0$ is a necessary condition for a project to belong to set A_1.

all values of i, while for projects contained in $A_2 - B_1$, $r = \pi$ if and only if $i = \pi$. The existence of set $A_2 - B_1$ points out that the uniqueness of the traditional internal rate of return does not guarantee that such a rate has an adequate economic meaning regardless of the cost of capital. In fact, for a project contained in set $A_2 - B_1$, the one and only one existing π has the precise

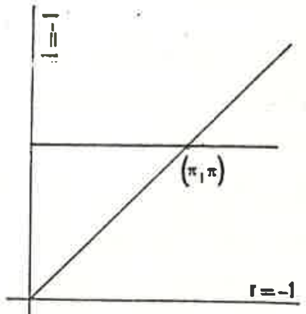


fig 4

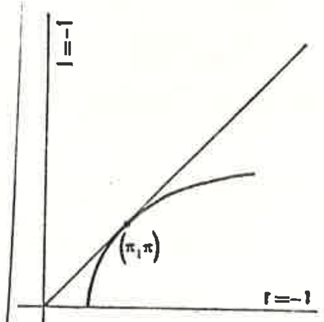


fig 5

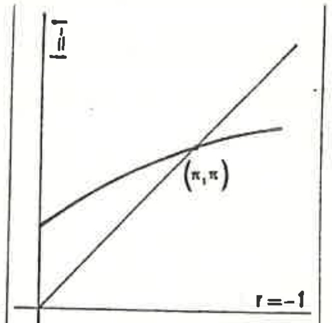


fig 6

but economically irrelevant meaning of the lending rate of a pair (internal to the project) whose borrowing rate differs from the cost of capital.

Third, set A_3 of projects such that Π contain more than one element equals the set of projects such that the graph of r (which is necessarily a strictly increasing curve) meet the bisecting line at least twice (see fig's 7 and 8). For projects contained in A_3, $r = \pi_i$ if and only

if $i = \pi_i$, where π_i means the i.th element of set Π.

The partition of feasible projects into sets A_1 (such that $\Pi = \emptyset$), A_2 (such that Π contain one and only one element), A_3 (such that Π contain at least two elements), and the partition of these same feasible projects into sets B_1 (such that r be a constant function of i), and B_2

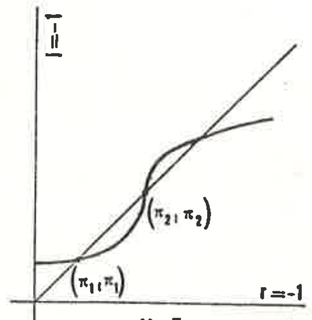


fig 7

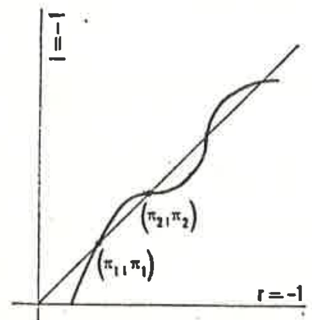


fig 8

(such that r be an increasing function of i), are superimposed in fig 9. The Venn diagram reproduced in fig 9 shows that $B_2 = (A_2 - B_1) \cup A_1 \cup A_3$.

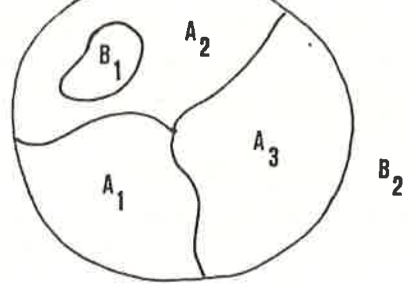


fig 9

Appendix*

The purpose of this appendix is to prove Proposition 1 of this paper.

Given a project $(a_o,\ldots,a_n)$, let us consider set S of ordered pairs of positive numbers (s_1,s_2) such that there exist a matrix

$$B = \begin{bmatrix} b_o^1 & & b_1^1 \\ \cdots & \cdots & \cdots \\ b_{n-1}^n & & b_n^n \end{bmatrix}$$

and a number b_n^{n+1} satisfying the following equalities:

(A.1) $b_o^1 = a_o$

(A.2) $b_j^{j+1} = a_j - b_j^j \qquad (j=1,\ldots,n)$

(A.3) $b_n^{n+1} = 0$

(*) I would like to thank Prof. A. Pasini of Siena University's Mathematics Department for having suggested mathematical induction as a tool for determining the zero contour line of surface:

$$b_n^{n+1}(s_1,s_2)$$

(see below). I would like to thank him also for his useful comments on the first draft of this appendix.

(A.4) $$b_j^j = \begin{cases} -\,s_1\, b_{j-1}^j & \text{if } b_{j-1}^j > 0 \\ -\,s_2\, b_{j-1}^j & \text{if } b_{j-1}^j \leq 0 \end{cases} \qquad (j=1,\ldots,n).$$

It is self-evident that

(A.5) $$R = \left\{ (r_1,r_2) : r_1 = s_1 - 1,\ r_2 = s_2 - 1,\ (s_1,s_2) \in S \right\}.$$

Therefore, in order to determine set R, we shall determine set S.

For any given pair (s_1,s_2), there exist one and only one matrix B and one and only one number b_n^{n+1} which satisfy equalities (A.1), (A.2) and (A.4) [excluding equality (A.3)]. Let us call this matrix and this number respectively, $B(s_1,s_2)$ and $b_n^{n+1}(s_1,s_2)$. We shall refer to the elements of matrix $B(s_1,s_2)$ as $b_j^i(s_1,s_2)$, for $i = 0,\ldots,n$ and $j = i-1,i$.

Matrix $B(s_1,s_2)$ and number $b_n^{n+1}(s_1,s_2)$ also satisfy equality (A.3) if and only if $b_n^{n+1}(s_1,s_2)$ is null. Therefore

(A.6) $$S = \left\{ (s_1,s_2) : b_n^{n+1}(s_1,s_2) = 0 \right\},$$

so that we can think of S as the zero contour line of surface $b_n^{n+1}(s_1,s_2)$.

Let us note that we have:

$$\text{(A.7)} \qquad b_1^2(s_1,s_2) = a_1 + a_o s_2$$

$$\text{(A.8)} \qquad b_j^{j+1}(s_1,s_2) = a_j + \begin{cases} s_1\, b_{j-1}^j(s_1,s_2) & \text{if } b_{j-1}^j(s_1,s_2) > 0 \\ s_2\, b_{j-1}^j(s_1,s_2) & \text{if } b_{j-1}^j(s_1,s_2) \le 0 \end{cases}$$

$(j = 2,\dots,n)$

[substitute equalities (A.4) and (A.1) in equalities (A.2)].

Given equality (A.7), since $a_o < 0$ (see Definition 1), the following proposition is true.

Proposition A.1. For any given value $\bar{s}_1$ of s_1, function $b_1^2(\bar{s}_1,s_2)$ is continuous, strictly decreasing, and we have

$$\lim_{s_2 \to +\infty} b_1^2(\bar{s}_1,s_2) = -\infty .$$

Moreover, for any given value $\bar{s}_2$ of s_2, function $b_1^2(s_1,\bar{s}_2)$ is constant.

The following Propositions A.2 - A.6 (together with Proposition A.1) are the steps for reaching the concluding Proposition A.7.

Proposition A.2. For $j \in \{2,\dots,n\}$, let us suppose the following hypotheses. For any given value $\bar{s}_1$ of s_1, function $b_{j-1}^j(\bar{s}_1,s_2)$ is continuous, strictly decreasing, and we have

$$\lim_{s_2 \to +\infty} b_{j-1}^j(\bar{s}_1,s_2) = -\infty .$$

Moreover, there exists a value $s_2(b_{j-1}^j)$ of s_2 such that for any $\bar{s}_2 \in \left(0 , s_2(b_{j-1}^j)\right)$, function $b_{j-1}^j(s_1,\bar{s}_2)$ be continuous, strictly increasing, and we have

$$\lim_{s_1 \to +\infty} b_{j-1}^j(s_1,\bar{s}_2) = +\infty ,$$

while for any $\bar{s}_2 \in \left[s_2(b_{j-1}^j), +\infty\right)$, function $b_{j-1}^j(s_1,\bar{s}_2)$ be constant. Under these hypotheses, for any given value $\bar{s}_1$ of s_1, function $b_j^{j+1}(\bar{s}_1,s_2)$ is continuous, strictly decreasing, and we have

$$\lim_{s_2 \to +\infty} b_j^{j+1}(\bar{s}_1,s_2) = -\infty .$$

Moreover, there exists a value $s_2(b_j^{j+1})$ of s_2 such that for any $\bar{s}_2 \in \left(0,\ s_2(b_j^{j+1})\right)$, function $b_j^{j+1}(s_1,\bar{s}_2)$ be continuous, strictly increasing, and we have

$$\lim_{s_1 \to +\infty} b_j^{j+1}(s_1,\bar{s}_2) = +\infty ,$$

while for any $\bar{s}_2 \in \left[s_2(b_j^{j+1}),\ +\infty\right)$, function $b_j^{j+1}(s_1,\bar{s}_2)$ be constant.

In order to prove the first part of Proposition A.2, that is the part concerning function $b_j^{j+1}(\bar{s}_1,s_2)$, let us first note that equation $b_{j-1}^{j}(\bar{s}_1,s_2) = 0$ has one (economically meaningful, i.e.positive) solution at most. In fact, by hypothesis function $b_{j-1}^{j}(\bar{s}_1,s_2)$ is strictly decreasing, and we have:

$$\lim_{s_2 \to +\infty} b_{j-1}^{j}(\bar{s}_1,s_2) = -\infty .$$

Let us suppose that a solution exists, and let us call it k. We have $b_{j-1}^{j}(\bar{s}_1,s_2) > 0$ for $s_2 \in (0,\ k)$, and $b_{j-1}^{j}(\bar{s}_1,s_2) \leq 0$ for $s_2 \in [k\ ,\ +\infty)$. Therefore, according to

equalities (A.8), we have:

$$b_j^{j+1}(\bar{s}_1,s_2) = a_j + \begin{cases} \bar{s}_1 \, b_{j-1}^{j}(\bar{s}_1,s_2) & \text{for } s_2 \in (0\ ,\ k) \\ s_2 \, b_{j-1}^{j}(\bar{s}_1,s_2) & \text{for } s_2 \in [k\ ,\ +\infty) \end{cases}$$

Let us now suppose that a solution does not exist since we have $b_{j-1}^{j}(\bar{s}_1,s_2) < 0$ for all values of s_2. According to equalities (A.8), we have:

$$b_j^{j+1}(\bar{s}_1,s_2) = a_j + s_2 \, b_{j-1}^{j}(\bar{s}_1,s_2)$$

for all values of s_2. This implies the first part of Proposition A.2, since a function remains continuous, strictly decreasing, and approaching $-\infty$, if it is multiplied by its independent variable when it is non-positive, and by a positive constant if and when it is positive.

We must now prove the second part of Proposition A.2, that is the part concerning function $b_j^{j+1}(s_1,\bar{s}_2)$. We shall first consider the case where $\bar{s}_2 \in \left(0\ ,\ s_2(b_{j-1}^{j})\right)$. Equation $b_{j-1}^{j}(s_1,\bar{s}_2) = 0$ has one (economically meaningful , i.e. positive) solution at most. Infact, by hypothesis, function $b_{j-1}^{j}(s_1,\bar{s}_2)$ is strictly increasing, and we have:

$$\lim_{s_1 \to +\infty} b_{j-1}^{j}(s_1,\bar{s}_2) = +\infty .$$

Let us suppose that a solution exists, and let us call it h. We have $b^j_{j-1}(s_1,\bar{s}_2) \le 0$ for $s_2 \in (0\ ,\ h]$, and $b^j_{j-1}(s_1,\bar{s}_2) > 0$ for $s_2 \in (h\ ,\ +\infty\ )$. Therefore, according to equalities (A.8), we have:

$$b^{j+1}_j(s_1,\bar{s}_2) = a_j + \begin{cases} s_1\ b^j_{j-1}(s_1,\bar{s}_2) & \text{for} \quad s_1 \in (h\ ,\ +\infty\ ) \\ \bar{s}_2\ b^j_{j-1}(s_1,\bar{s}_2) & \text{for} \quad s_1 \in (0\ ,\ h] \end{cases} .$$

Let us now suppose that a solution does not exist since we have $b^j_{j-1}(s_1,\bar{s}_2) > 0$ for all values of s_1. According to equalities (A.8), we have:

$$b^{j+1}_j(s_1,\bar{s}_2) = a_j + s_1\ b^j_{j-1}(s_1,\bar{s}_2)$$

for all values of s_1. This implies that for $\bar{s}_2 \in \left(0, s_2(b^j_{j-1})\right)$ function $b^{j+1}_j(s_1,\bar{s}_2)$ is continuous, strictly increasing, and we have

$$\lim_{s_1 \to +\infty} b^{j+1}_j(s_1,\bar{s}_2) = +\infty \ .$$

In fact a function remains continuous, strictly increasing and approaching $+\infty$, if it is multiplied by its independent variable when it is positive and by a positive constant

if and when it is non-positive.

We shall now consider the case where $\bar{s}_2 \in \left[s_2(b^j_{j-1}),\ +\infty\right)$. We need to distinguish between the two following sub-cases. In the first sub-case there exists a value $s^*_2 > s_2(b^j_{j-1})$ of s_2 such that $b^j_{j-1}(s_1,\bar{s}_2) > 0$ for any $\bar{s}_2 \in \left[s_2(b^j_{j-1}),\ s^*_2\right)$, while $b^j_{j-1}(s_1,\bar{s}_2) \le 0$ for any $\bar{s}_2 \in \left[s^*_2,\ +\infty\right)$. If $\bar{s}_2 \in \left[s_2(b^j_{j-1}),\ s^*_2\right)$, we have:

(A.9) $$b^{j+1}_j(s_1,\bar{s}_2) = a_j + s_1\ b^j_{j-1}(s_1,\bar{s}_2)$$

[see equalities (A.8)], which implies that function $b^{j+1}_j(s_1,\bar{s}_2)$ is continuous, strictly increasing, and we have:

$$\lim_{s_1 \to +\infty} b^{j+1}_j(s_1,\bar{s}_2) = +\infty \ .$$

However, if $\bar{s}_2 \in \left[s^*_2,\ +\infty\right)$, we have:

(A.10) $$b^{j+1}_j(s_1,\bar{s}_2) = a_j + \bar{s}_2\ b^j_{j-1}(s_1,\bar{s}_2) \ ,$$

which implies that function $b^{j+1}_j(s_1,\bar{s}_2)$ is constant [exactly like function $b^j_{j-1}(s_1,\bar{s}_2)$]. In the second sub-case we have $b^j_{j-1}(s_1,\bar{s}_2) \le 0$ for any $\bar{s}_2 \in \left[s_2(b^j_{j-1}),\ +\infty\right)$. Therefore, for any $\bar{s}_2 \in \left[s_2(b^j_{j-1}),\ +\infty\right)$ equality (A.10) holds true, which implies that function $b^{j+1}_j(s_1,\bar{s}_2)$ is constant.

The second part of proposition (A.2) follows immediately from the two previous paragraphs, if we refer either to s_2^* or to $s_2(b_{j-1}^j)$ as $s_2(b_j^{j+1})$ (6). The proof is thus complete.

Proposition A.3. For $j \in \{2,\dots,n\}$ let us suppose the following hypotheses. For any given value $\bar{s}_1$ of s_1, function $b_{j-1}^j(\bar{s}_1,s_2)$ is continuous, strictly decreasing, and we have

$$\lim_{s_2 \to +\infty} b_{j-1}^j(\bar{s}_1,s_2) = -\infty \ .$$

Moreover, for any given value $\bar{s}_2$ of s_2, function $b_{j-1}^j(s_1,\bar{s}_2)$ is constant. Under these hypotheses, for any given value $\bar{s}_1$ of s_1, function $b_j^{j+1}(\bar{s}_1,s_2)$ is continuous, strictly decreasing, and we have

$$\lim_{s_2 \to +\infty} b_j^{j+1}(\bar{s}_1,s_2) = -\infty \ .$$

Moreover, if $a_{j-1} \leq 0$, for any given value $\bar{s}_2$ of s_2, function $b_j^{j+1}(s_1,\bar{s}_2)$ is constant. On the contrary, if $a_{j-1} > 0$, there exists a value $s_2(b_j^{j+1})$ of s_2 such that for any $\bar{s}_2 \in \left(0,s_2(b_j^{j+1})\right)$, function $b_j^{j+1}(s_1,\bar{s}_2)$ be

(6) As a corollary we also have $s_2(b_j^{j+1}) \geq s_2(b_{j-1}^j)$.

continuous, strictly increasing, and we have

$$\lim_{s_1 \to +\infty} b_j^{j+1}(s_1,\bar{s}_2) = +\infty \ ,$$

while for any $\bar{s}_2 \in \left[s_2(b_j^{j+1}), +\infty\right)$, function $b_j^{j+1}(s_1,\bar{s}_2)$ be constant.

The first part of proposition A.3 can be proved in the same way as the first part of proposition A.2. Thus we need to prove only the second part. Since we have:

$$\lim_{(s_1,s_2) \to (0,0)} b_{j-1}^j(s_1,s_2) = a_{j-1} \ ,$$

[see equality (A.7) and the first $n-3$ equalities (A.8)], in case $a_{j-1} \leq 0$, for any given value $\bar{s}_2$ of s_2 we have $b_{j-1}^j(s_1,\bar{s}_2) < 0$ (see fig's A.1 and A.2). Therefore, for any given value $\bar{s}_2$ of s_2, equality (A.10) holds true, which implies that function $b_j^{j+1}(s_1,\bar{s}_2)$ is constant. In case $a_{j-1} > 0$, there exists a value s_2^* of s_2 such that $b_{j-1}^j(s_1,\bar{s}_2) > 0$ for any $\bar{s}_2 \in (0,s_2^*)$, and that $b_{j-1}^j(s_1,\bar{s}_2) \leq 0$ for any $\bar{s}_2 \in \left[s_2^*,+\infty\right)$ (see fig. A.3). Therefore, if $\bar{s}_2 \in (0,s_2^*)$ equality (A.9) holds true, which implies that function $b_j^{j+1}(s_1,\bar{s}_2)$ is continuous, strictly increasing, and we have

$$\lim_{s_1 \longrightarrow +\infty} b_j^{j+1}(s_1,\bar{s}_2) = +\infty \ .$$

However, if $\bar{s}_2 \in \left[s_2^*,+\infty\right)$, equality (A.10) holds true, which implies that function $b_j^{j+1}(s_1,\bar{s}_2)$ is constant. The proof is thus complete.

Let us call a_p the first positive cash flow in the ordered set $\left\{a_1,\dots,a_n\right\}$ (see Definition 1). The following proposition is easily deduced from Propositions A.1, A.2, A.3 by mathematical induction.

Proposition A.4. For any $j \in \left\{2,\dots,n\right\}$ function $b_j^{j+1}(\bar{s}_1,s_2)$ is continuous, strictly decreasing, and we have

$$\lim_{s_2 \longrightarrow +\infty} b_j^{j+1}(\bar{s}_1,s_2) = -\infty$$

for any given value $\bar{s}_1$ of s_1. Moreover, for any $j\in\left\{2,\dots,p\right\}$ function $b_j^{j+1}(s_1,\bar{s}_2)$ is constant for any given value $\bar{s}_2$ of s_2. On the contrary, for any $j \in \left\{p+1,\dots,n\right\}$ there exists a value $s_2(b_j^{j+1})$ of s_2 such that for any $\bar{s}_2 \in \left(0,s_2(b_j^{j+1})\right)$,

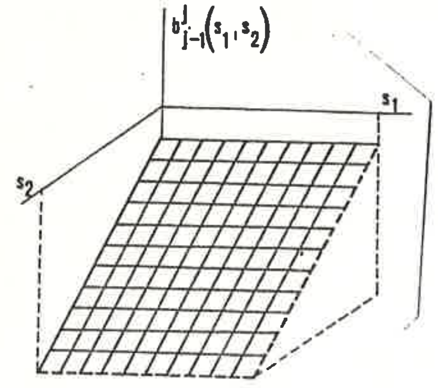


fig A1

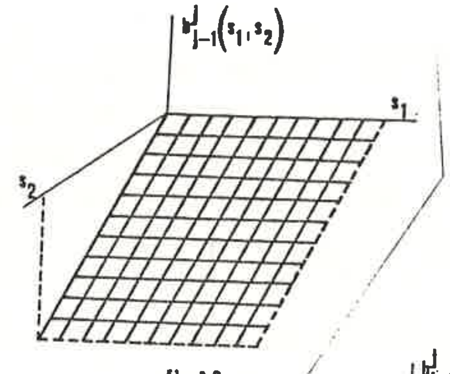


fig A2

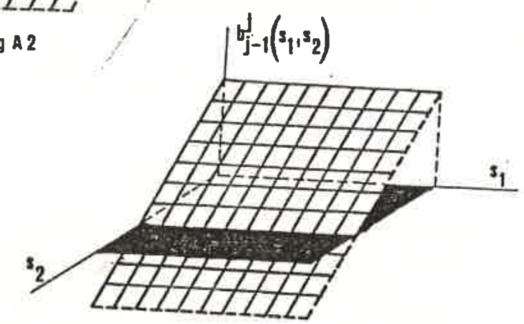


fig A3

function $b_j^{j+1}(s_1,\bar{s}_2)$ be continuous, strictly decreasing, and we have

$$\lim_{s_1 \to +\infty} b_j^{j+1}(s_1,\bar{s}_2) = +\infty ,$$

while for any $\bar{s}_2 \in \left[s_2(b_j^{j+1}), +\infty\right)$, function $b_j^{j+1}(s_1,\bar{s}_2)$ be constant.

One should note that in case $p = 1$, set $\{2,\ldots,p\}$ is empty, so that for no $j \geq 2$ function $b_j^{j+1}(s_1,\bar{s}_2)$ is constant for any given value $\bar{s}_2$ of s_2. On the other hand, in case $p = n$, set $\{p+1,\ldots,n\}$ is empty, so that for any $j \geq 2$ function $b_j^{j+1}(s_1,\bar{s}_2)$ is constant for any given value $\bar{s}_2$ of s_2.

The following proposition is a corollary of Proposition A.4.

Proposition A.5. For any given value $\bar{s}_1$ of s_1, function $b_n^{n+1}(\bar{s}_1,s_2)$ is continuous, strictly decreasing, and we have

$$\lim_{s_2 \to +\infty} b_n^{n+1}(\bar{s}_1,s_2) = -\infty .$$

Moreover we distinguish between the two following cases:

(i) if $p \in \{1,\ldots,n-1\}$, there exists a value $s_2(b_n^{n+1})$ of s_2 such that for any $\bar{s}_2 \in \left(0,s_2(b_n^{n+1})\right)$, function $b_n^{n+1}(s_1,\bar{s}_2)$ be continuous, strictly increasing, and we have

$$\lim_{s_1 \to +\infty} b_n^{n+1}(s_1,\bar{s}_2) = +\infty ,$$

while for any $s_2 \in \left[s_2(b_n^{n+1}), +\infty\right)$, function $b_n^{n+1}(s_1,\bar{s}_2)$ be constant;

(ii) if $p = n$, for any given value $\bar{s}_2$ of s_2, function $b_n^{n+1}(s_1,\bar{s}_2)$ is constant.

Since we have

$$\lim_{(s_1,s_2) \to (0,0)} b_n^{n+1}(s_1,s_2) = a_n$$

[see the $n-2$. th equality (A.8)], and since $a_n \neq 0$ (see Definition 1), Proposition A.5 implies the following proposition.

Proposition A.6. In case $p \in \{1,\ldots,n-1\}$, surface $b_n^{n+1}(s_1, s_2)$ is as shown in fig. A.5 or A.6 if $a_n > 0$, while it is as shown in fig. A.7 if $a_n < 0$. On the other hand, in case $p = n$, surface $b_n^{n+1}(s_1,s_2)$ is as shown in fig. A.4.

One should be warned that broken lines r', r'', r''', etc in fig's A.5, A.6, A.7 represent strictly decreasing curves, while straight lines t', t'', t''', etc represent strictly increasing curves (following Proposition A.5). Moreover, broken line r and straight line t (which are not included in the surface, though they form its border) represent, respectively, a non-increasing curve and a non-decreasing curve. Indeed, the fact that curves r', r'', r''', etc are strictly decreasing and that curves t', t'', t''', etc are strictly increasing, does not contradict the fact that curves r and t are stationary at times.

Finally, Proposition A.6 implies the following concluding proposition.

Proposition A.7. Set S is a continuous curve in the (economically meaningful region of) plane $0 - (s_1,s_2)$. In case $a_n > 0$, S is either a straight line which is parallel

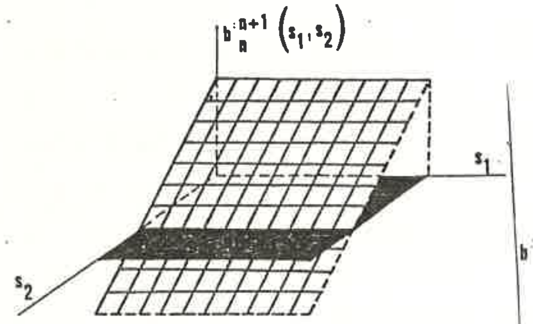


fig A4

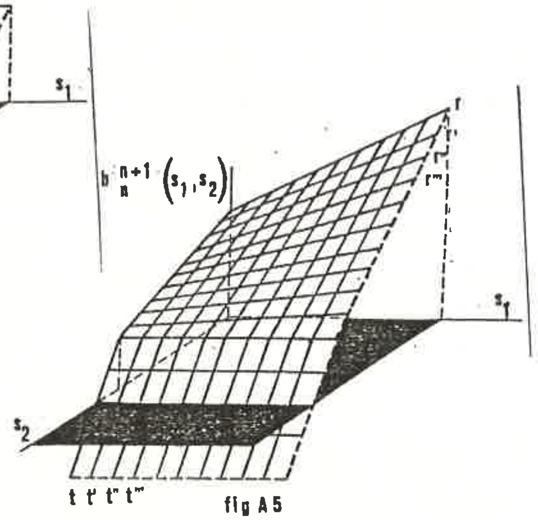


fig A5

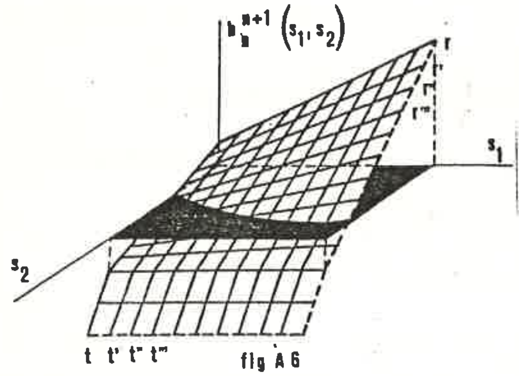


fig A6

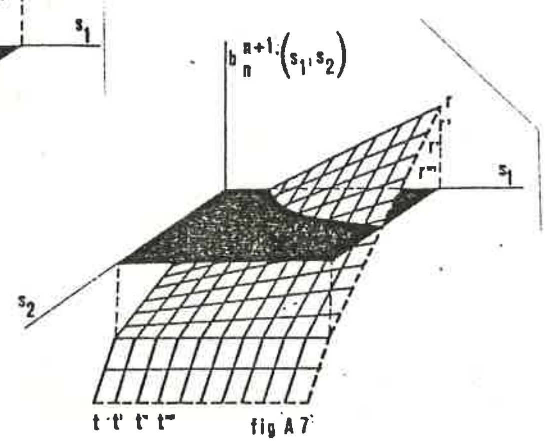


fig A7

to s_1 axis or a strictly increasing curve approaching a horizontal asymptote, and S's origin (which is not included in S itself) is located on s_2 axis (see fig's A.4, A.5 and A.6). On the other hand, in case $a_n < 0$, S is a strictly increasing curve approaching a horizontal asymptote, but S's origin (which is not included in S as well) is located on s_1 axis (see fig A.7).

Since equality (A.5) holds true, Proposition A.7 implies Proposition 1 of the paper.

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Abstract

Compared to the standard IRR (π), the IRR presented here (r) refers clearly to the rate of growth of invested capital and is unique, for any feasible project, despite a certain loss of "internality". In fact r is no longer defined independently of the market interest rate (i). The study of the connection between r and π shows that values of π (if any) attached to a project are the values taken by r in accordance with particular values of i. Thus r generalizes π and overcomes any problem of choice among its values in case they are more than one.

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