UNIVERSITA DEGLI STUDI DI SIENA

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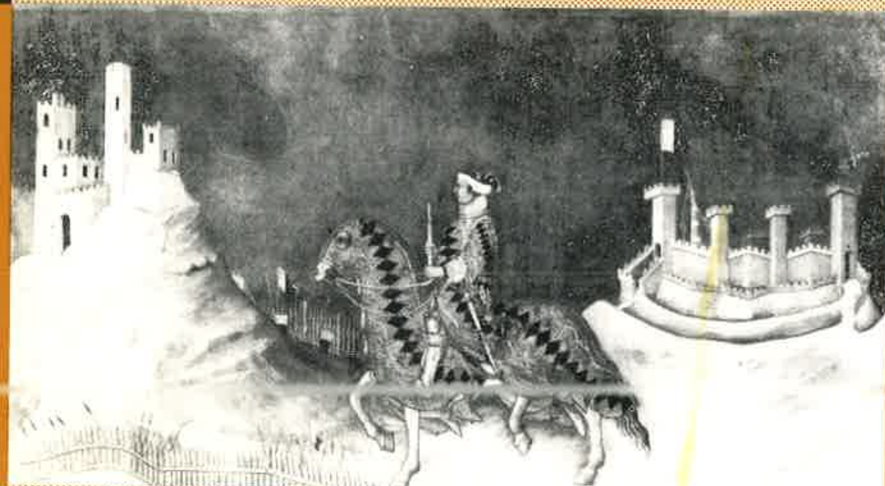
QUADERNI DEL DIPARTIMENTO
DI ECONOMIA POLITICA

NICOLA DIMITRI

GENERALIZED INFORMATION STRUCTURES: SOME THEORETICAL AND METHODOLOGICAL ISSUES



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Nicola Dimitri

GENERALIZED INFORMATION STRUCTURES: SOME THEORETICAL AND METHODOLOGICAL ISSUES

Siena, Ottobre 1992

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GENERALIZED INFORMATION STRUCTURES: SOME THEORETICAL AND METHODOLOGICAL ISSUES[1]

1 Introduction

The importance of information is unquestionable in most theories of decision making. A remarkable example is given by the Bayesian theory of choice under uncertainty, a very influential theoretical position, which explains differences in the opinions of rational agents by the differences in the information received.

Indeed, some of the most authoritative contributors in the modelling of information structures (Aumann (1976)) have been of Bayesian inspiration; as a consequence, it has been common wisdom for a long time to take the stance that agents are imparted information according to a partition of the space of all possible states of the world. This position, which goes back to Savage (1972), is well illustrated by the following argument (Aumann (1987)): if each state of the world specifies *all* aspects relevant to a decision theoretic problem of an agent, then it should also specify all other states which *every* agent can not distinguish from the true one. Since agents are supposed to know[2] the *true* model and since, being rational, they can reason counterfactually there is no room for information structures different from partitions.

A most celebrated consequence of assuming the common knowledge of partitions, rationality, and prior and posterior probabilities is the agreement result concerning the posteriors; under those circumstances these can not be different. Agreement results however have also been obtained by Bacharach (1985), Shin (1987),

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1 The first version of this paper was written while I was visiting the University of Cambridge U.K. in August 1992. I should like to thank the Comitato Nazionale per le Scienze Economiche, Sociali e Statistiche of Consiglio Nazionale delle Ricerche for financial support. I am also grateful to Frank Hahn and Stefano Vannucci for the stimulating conversations we had on some of the issues discussed in the paper, and for detailed comments on a previous version of it. The usual disclaimers apply.

2 The assumption that agents' partitions are common knowledge has been matter of debate, an account of which can be found in Shin (1987). It is interpreted, at least in Aumann's context, in an informal sense; that is to say, his model would not formally accomodate information structures as an event. Notice that when we think of rational agents reasoning counterfactually and, at the same time, with partitions different from the finest one, we are probably having in mind that some states are somehow physically indistiguishable to the agents.

Geanakoplos (1989), Samet (1990) within informational frameworks more general than partitions. Moreover, Geanakoplos (1989) and Brandenburger-Dekel-Geanakoplos (1992) have systematically explored some decision theoretic implications of working with information structures generated by *possibility correspondences*. Their formalization is based on the simple idea that individuals' information processing might be the source of various kinds of errors as forgetting or not noticing what happened, disregarding unpleasent news and others[3]. As a result, information structures may not necessarily be partitional.

However, there is another, maybe deeper, implication of such formalization. If we allow for information structures different from partitions we are, at least, implicitly assuming that agents *do not* reason counterfactually *ex post*. Agents might even *be aware* of the model *ex ante* but if they *forget* about it *ex post* they might not be able to reconstruct the true partitional information structure. Evidently, it is not difficult to think of generalized structures when agents do not know the model to begin with[4]. The line of research pursued by Geanakoplos (1989) seems to take the point of view as if agents were aware of the model *ex ante*; in that work, his efforts are largely concentrated on the comparison between standard decision theoretic results and their non-standard (non-partitional) decision theoretic counterpart from an *ex ante* perspective[5].

But this might not be the only possible interpretation of generalized structures. This paper is founded on this observation in that it takes the idea of possibility correspondences to its (perhaps not so) extreme consequences; that is to say, it is based on the assumption that agents are *boundedly rational* and are not aware of the true model . If we allow for the

3 For a fuller account of the kind of information processing errors that could be considered see Geanakoplos (1989).

4 It is immediate to produce examples where the presence of counterfactual reasoning may not be sufficient to refine the possibility correspondence into a partition; this of course might happen when the model can neither be known ex ante nor ex post.

5 A simple illustration is given by examples 2.1-2.2-2.3 in Geanakoplos (1989). There he shows how, under certain circumstances, a non-partitional information structure may generate a lower ex ante expected utility than a coarser partition. The implication of this being obvious; individuals who mishandle the information can be worse off than individuals with less information but unboundedly rational. Of course, such conclusion is unacceptable by Bayesian decision theory where it always holds true that more information can never be disadvantageous to the agent.

possibility of information processing errors specifically correlated to the state[6], then it would seem important to investigate the consequences of assuming that agents could not know a priori what kind of errors they will make; as a result, the general approach of this paper will be from the *ex post* rather than from the *ex ante* perspective. We have done so by pursuing systematically two levels of investigation; the *subjective* level and the *objective* level. In as much as one accepts this approach he/she should not forget that the modeller is the *only one* who is aware of the *true* model[7]. The paper concentrates primarily on the exploration and implications of the alleged distinction, raising and discussing theoretical as well as methodological questions.

The first interesting outcome that we obtain is given by the intrinsic similarity characterizing the behaviour of some important objective and subjective mappings. Furthermore, we shall also see that agreement (as defined in this model) between two agents seems to owe a lot to the capability of second guessing each other.

The structure of the work is the following; in section 2 we define the possibility and belief mappings and discuss their properties. These mappings represent the starting point for the ensuing analysis. In section 3 we propose a definition of knowledge and common knowledge which is different from the one given by the authors mentioned above. In section 4 we examine what kind of *agreeing to disagree* results could be obtained within the epistemic set up of this paper; finally, section 5 summarizes the results obtained in the work.

2 Belief and Possibility

Consider a finite state space W with generic element w; elements of W are called *states of the world*. Moreover, suppose there are two agents ,i and j, processing information[8].

6 Namely, at each state agents make different processing errors.

7 Where the true model includes also agents' information processing.

8 Unless differently specified, definitions and propositions will hold for both agents. When needed, a supercript will indicate the relevant agent.

The following definition which, among others, was given in Geanakoplos (1989) should be considered as the starting point for all the constructions and results of the paper.

Definition 1 *A possibility correspondence is a mapping $P{:}W \rightarrow 2^W-(\phi)$ which associates to each element of W a non-empty subset of W . A possibility set is $P(w)$, for all w in W.*

The interpretation of the above mapping is as follows: given that w is the true state of the world, the agent thinks one of the states in $P(w)$ to have possibly been realized. The possibility correspondence incorporates the idea of a *subjective* information structure, in the sense that an individual can make information processing errors different from other agents and, in general, different across the states of the world.

Subjectivity is further introduced by the definition below:

Definition 2 *Suppose the true state is w and consider any $A\epsilon 2^W$; then we say that an agent believes in A, at w, if $P(w)\subseteq A$ and that an agent consider A to be possible, at w, if $P(w)\cap A$ is not empty.*

An obvious consequence is that $P(w)$ itself is both believed and considered possible at w. The importance of definition 2 rests on the fact that it makes clear the distinction between the true state and the believed states. Agents can be completely wrong in their beliefs about a certain event, a case which can be captured by definition 2.

In a sense, the true states could be interpreted as the *objective* side while the believed subsets as the *subjective* side of the model. As previously anticipated, our analysis will proceed by examining both these aspects; there is a *duality* between the two views which is what we are about to explore. We shall do so by taking the point of view of the modeller, whom I assume to be both outside the model and yet to have a *complete* knowledge of it. As a further consequence, our stance will take us away from the realm of agents being aware of what they know; that is to say, agents could formulate correct beliefs[9] but, not knowing the model, they would *never* be aware on whether they are right or wrong.

9 In the sense that what they believe has indeed obtained.

Hence, the ensuing analysis aims at formalizing a framework in order to deal with the idea that agents take actions exclusively on the basis of what they *feel sure* about. Having said so we will proceed by introducing two mappings which, together with their duals, represent the most important tools of the work: the *possibility* and *belief mappings*. Even though it basically won't appear later in the *agreeing to disagree* results, we shall begin by discussing the former. In our model, the possibility mapping can not be obtained from the knowledge operator and this is why we need to provide an independent treatment[10].

Let us start with the following definitions:

Definition 3 *A possibility mapping $\pi{:}2^W \rightarrow 2^W$ for every $A\epsilon 2^W$ is defined as follows*

$$\pi(A)= \bigcup_{w \epsilon W} (P(w) \mid P(w)\cap A \neq \phi)$$

and is such that $\pi(\phi)=\phi$

Definition 4 *A dual possibility mapping $\boldsymbol{\pi}{:}2^W \rightarrow 2^W$ for every $A\epsilon 2^W$ is defined as follows*

$$\boldsymbol{\pi}(A)=(w\epsilon W \mid P(w)\cap A \neq \phi)$$

and is such that $\boldsymbol{\pi}(\phi)=\phi$

In words, $\pi(A)$ is the set of states *for which* an agent can not exclude event A as being occurred, while $\boldsymbol{\pi}(A)$ is the set of states *in which* the agent thinks it possible that A has occurred. The requirement that the two mappings take the empty set into itself seems unquestionable, and this is why we enclosed it in the definition. Mappings like these have appeared elsewhere, Monderer-Samet (1989), Binmore (1992) but in contexts where possibility correspondences were not used[11].

The dual possibility mapping represents the *objective* counterpart of agent's beliefs and, we shall see below, it basically enjoys the same properties characterizing the possibility mapping.

10 Below, we shall provide more details about it.
11 In particular, Monderer-Samet work with partitions , while Binmore has an axiomatic approach. Notice that Monderer-Samet do not speak explicitly of possibility mappings but rather of a 'beliefs with (at least) probability p' mapping.

The other mappings we shall introduce are the belief mapping and its dual:

Definition 4 *A* <u>*belief mapping*</u> $\beta:2^W\rightarrow 2^W$ *for every* $A\in 2^W$ *is defined as follows*

$$\beta(A)=\bigcup_{w\in W}(\ P(w)\ |\ P(w)\subseteq A\ )$$

and such that $\beta(\phi)=\phi$.

Definition 5 *A* <u>*dual belief mapping*</u> $\beta:2^W\rightarrow 2^W$ *for every* $A\in 2^W$ *is defined as follows*

$$\beta(A)=(\ w\in W\ |\ P(w)\subseteq A\ )$$

and such that $\beta(\phi)=\phi$.

The theoretical content of the above formalization should be clear; $\beta(A)$ represents the set of states *for which* the agent feels sure that A has occurred while the interpretation of $\beta(A)$ is analogous to that of the dual possibility mapping. The belief mappings can be considered as a generalization of the knowledge operator $K(A)$, which has been studied extensively in the literature[12].

Let us now derive the properties of these operators: we shall begin examining the mappings separately and then consider their connections.

<u>Properties of $\pi(A)$</u>

1) $\pi(\phi)=\phi$
2) $\pi(W)\subseteq W$
3) if $A\subseteq B$ then $\pi(A)\subseteq\pi(B)$
4) $\pi(A\cap B)\subseteq\pi(A)\cap\pi(B)$
5) $\pi(A\cup B)=\pi(A)\cup\pi(B)$
6) $\pi(A)\subseteq\pi\pi(A)$

<u>Properties of $\boldsymbol{\pi}(A)$</u>

1a) $\boldsymbol{\pi}(\phi)=\phi$
2a) $\boldsymbol{\pi}(W)=W$
3a) if $A\subseteq B$ then $\boldsymbol{\pi}(A)\subseteq\boldsymbol{\pi}(B)$
4a) $\boldsymbol{\pi}(A\cap B)\subseteq\boldsymbol{\pi}(A)\cap\boldsymbol{\pi}(B)$
5a) $\boldsymbol{\pi}(A\cup B)=\boldsymbol{\pi}(A)\cup\boldsymbol{\pi}(B)$

Proof

1) by definition

2) obvious

12 See, among others, Aumann (1976), Milgrom (1981), Bacharach (1985). Notice that Geanakoplos (1989) defines our dual belief mapping to be his knowledge operator. In what follows we shall see that the definition of knowledge adopted here is different from Geanakoplos' in that it requires the relevant event to have truly obtained.

3) suppose $P(w)\subseteq\pi(A)$; then $P(w)\cap A$ is not empty which implies that $P(w)\cap B$ is not empty and that $P(w)\subseteq\pi(B)$ so that $\pi(A)\subseteq\pi(B)$.

4) from 3) we know that since $A\cap B\subseteq A$ and $A\cap B\subseteq B$ we have $\pi(A\cap B)\subseteq\pi(A)$ and $\pi(A\cap B)\subseteq\pi(B)$; hence $\pi(A\cap B)\subseteq\pi(A)\cap\pi(B)$.

To see that the other direction does not hold in general it suffices to consider the following simple example. Suppose $W=(1,2,3,4)$ and that $A=(1,2)$, $B=(2,3,4)$; moreover, assume that $P(1)=A=P(2)$, $P(3)=(1,3,4)$ and $P(4)=(3,4)$. Then $\pi(A\cap B)=A$ and $\pi(A)\cap\pi(B)=W$ which proves the above claim.

5) by a reasoning analogous to the one made for property (4) it is immediate to see that $\pi(A)\cup\pi(B)\subseteq\pi(A\cup B)$. Let us now prove the other direction . If $P(w)\subseteq\pi(A\cup B)$ then $P(w)\cap(A\cup B)$ is not empty, which implies that $(P(w)\cap A)\cup(P(w)\cap B)$ is not empty. This in turn entails that either $P(w)\subseteq\pi(A)$ or else $P(w)\subseteq\pi(B)$ or both; in any case we obtain that $P(w)\subseteq\pi(A)\cup\pi(B)$.

6) obvious

1a) by definition

2a) obvious

3a) if $w\in\boldsymbol{\pi}(A)$ then $P(w)\cap A$ and, consequently, $P(w)\cap B$ are not empty; this implies that $w\in\boldsymbol{\pi}(B)$ which gives $\boldsymbol{\pi}(A)\subseteq\boldsymbol{\pi}(B)$.

4a) by (3a) it is immediate to see that $\boldsymbol{\pi}(A)\cap\boldsymbol{\pi}(B)\subseteq\boldsymbol{\pi}(A\cap B)$. That the other direction does not hold in general can be seen from the example discussed in the proof of property (4); in that case we have $\boldsymbol{\pi}(A)\cap\boldsymbol{\pi}(B)=(1,2,3)$ and $\boldsymbol{\pi}(A\cap B)=A=(1,2)$.

5a) as in (4a) it is easy to see that $\boldsymbol{\pi}(A)\cup\boldsymbol{\pi}(B)\subseteq\boldsymbol{\pi}(A\cup B)$. To prove the other direction suppose $w\in\boldsymbol{\pi}(A\cup B)$; this implies that $P(w)\cap(A\cup B)$ is not empty. In turn we obtain that $(P(w)\cap A)\cup(P(w)\cap B)$ is not empty so that either $(P(w)\cap A)$ or else $(P(w)\cap B)$ or both are not empty; such consideration leads us to conclude that $w\in\boldsymbol{\pi}(A)\cup\boldsymbol{\pi}(B)$.•

The above properties exhibit some differences and quite a few similarities.

First of all, one can easily prove that unless further specifications are introduced (2) and (2a) may indeed not coincide. This is due to the very nature of the possibility correspondences in that some states might never be considered as possible.

Moreover, the dual possibility mapping does not necessarily enjoy the analogous of property (6); in fact $\boldsymbol{\pi}\boldsymbol{\pi}(A)$ is the set of all w such that the corresponding $P(w)$ has a non-emtpy intersection with $\boldsymbol{\pi}(A)$. We can see at once that, without further specifications, nothing can be said on this event as related to $\boldsymbol{\pi}(A)$.

So much for the differences; let us now comment on the similarities. We can notice a rather remarkable fact in that all the remaining properties are exactly the same; the nature of the two mappings is intrinsically similar independently on whether the point of view is *objective* or *subjective*. If one were to confine the investigation to the structural characteristics of these mappings, it would become nearly irrelevant which mapping one is taking.

Let us now discuss briefly the above mentioned characteristics; we shall only comment on the properties of $\pi(A)$ leaving to the reader those about $\boldsymbol{\pi}(A)$. Property (1) says that observing nothing is thought to be impossibile while (2) means that an agent may not consider every state to be possible. The third is a fairly intuitive property saying that if an event is contained in another event, every time the first event is thought to be possible so is the second; this could be interpreted as a kind of *consistency* property. The fourth point formalizes how an agent can not think it possible that A and B both occurred without thinking A possible and B possible. The contrary however does not always hold; A and B can both not be excluded to have obtained and yet their *joint* realization be considered impossible.

The fifth property captures and formalizes the idea that an individual can not think an event to have possibly occurred without allowing the possibility that at least one of its components has occurred ,and viceversa.

Finally, the sixth property states that every time an event is considered possible then the individual considers as possible that it is possible.

Properties similar to the above ones have been found by other authors; for example, Binmore (1992) obtains them axiomatizing first a knowledge operator $K(A)$ and then deriving the properties

of the possibility operator which he defines as $P(A)=(K(A)^c)^c$[13]. Although the theoretical context we are working with in this paper is different from Binmore's[14], we can notice an analogy between the mappings.

Before analizing the properties of the belief mappings it is worth looking at the connection between $\pi(A)$ and $\boldsymbol{\pi}(A)$. One can immediately prove that the following holds true

$$\boldsymbol{\pi}(A)\subseteq\boldsymbol{\pi}\pi(A) \qquad (*)$$

The interpretation of (*) is quite intuitive since it states that if at a certain w the agent thinks A to be possible then (at that same w) he/she also thinks to be possible that A is possible[15]. This property is just another way of looking at (6); in analogy with the case of $\boldsymbol{\pi}(A)$, nothing can be said about $\boldsymbol{\pi}\boldsymbol{\pi}(A)$.

Let us now list and discuss the properties of the belief mappings .

<u>Properties of $\beta(A)$</u>

7) $\beta(\phi)=\phi$
8) $\beta(W)\subseteq W$
9) $\beta(A)\subseteq A$, for each $A\in 2^W$
10) if $A\subseteq B$ then $\beta(A)\subseteq\beta(B)$
11) $\beta(A\cap B)\subseteq\beta(A)\cap\beta(B)$
12) $\beta(A)\cup\beta(B)\subseteq\beta(A\cup B)$
13) $\beta\beta(A)=\beta(A)$

<u>Properties of $\boldsymbol{\beta}(A)$</u>

7a) $\boldsymbol{\beta}(\phi)=\phi$
8a) $\boldsymbol{\beta}(W)=W$
9a) if $A\subseteq B$ then $\boldsymbol{\beta}(A)\subseteq\boldsymbol{\beta}(B)$
10a) $\boldsymbol{\beta}(A\cap B)=\boldsymbol{\beta}(A)\cap\boldsymbol{\beta}(B)$
11a) $\boldsymbol{\beta}(A)\cup\boldsymbol{\beta}(B)\subseteq\boldsymbol{\beta}(A\cup B)$

Proof

7) by definition
8) obvious
9) obvious
10) obvious
11) from 10) we can immediately obtain that $\beta(A\cap B)\subseteq\beta(A)\cap\beta(B)$.

To see that the other direction does not hold in general we can take the following example. Let $W=\{1,2,3,4\}$, $A=\{1,2\}$,

13 The superscript c stands for complementarity. The definition of Binmore simply says that an event is possible whenever one does not know that it did not occur. Here we are using Binmore's notation.
14 For the sake of completness it is worth specifying that, as in Binmore, we have a finite state space. Furthermore, in Binmore's set up it is always true that $A\subseteq\pi(A)$.
15 On the other (one might say that curiously enough) he/she could think possible that A is possible without necessarily thinking A to be possible. To highlight the intuition behind this sentence Frank Hahn suggested the following simple example: "It is possible that telepathy is possible but I do not think telepathy possible".

$B=\{1,3,4\}$ and $P(1)=A$, $P(2)=B$, $P(3)=W=P(4)$; then $\beta(A\cap B)=\phi$ while $\beta(A)\cap\beta(B)=\{1\}$. It is important to notice that, by 10), if either $A\subseteq B$ or $B\subseteq A$ the weak inclusion holds as an equality.

12) it is immediate to see that $\beta(A)\cup\beta(B)\subseteq\beta(A\cup B)$. To see that the other direction does not hold in general ,it suffices to consider the following example.

Let $W=\{1,2,3,4\}$, $A=\{1,2\}$ and $B=\{3,4\}$. Furthermore, suppose $P(w)=W$, for all $w\in W$, so that $\beta(A)=\phi$, $\beta(B)=\phi$ but $\pi(A)=W=\pi(B)$. Then $\beta(A\cup B)=W$ while $\beta(A)\cup\beta(B)=\phi$.

13) obvious

We omit the proofs of the properties of $\beta(A)$, except that of 10a), since they can be easily obtained in analogy with the proofs given for the properties of the dual possibility mapping.

10a) from 9a) it is clear that $\beta(A\cap B)\subseteq\beta(A)\cap\beta(B)$. To see the other direction take $w\in\beta(A)\cap\beta(B)$; then $w\in\beta(A)$ and $w\in\beta(B)$. This implies that $P(w)\subseteq A$ and $P(w)\subseteq B$, namely that $P(w)\subseteq A\cap B$ which means that $w\in\beta(A\cap B)$. Needless to say that if $\beta(A)\cap\beta(B)$ is empty the weak inclusion follows immediately. ▪

Comments on these findings could be made following the same line of reasoning as for the possibility mappings. However, it might be interesting to notice the asymmetry between 11) and 10a); the reason being that the intersection of two possibility sets may not in turn be a possibility set.

Here too, we can investigate if there are some interesting connections between the belief mappings. The answer we obtain is formalized below by expression (**)

$$\beta\beta(A)=\beta(A) \qquad (**)$$

Whenever, at a certain w, the agent believes in A then, at the same w, the agent believes he/she believes in A and viceversa.

So far we have discussed the properties of the possibility and belief mappings separately; worthy of note is however the comparison between these mathematical objects because it exhibits some differences.

In particular, the main differences concern the intersection and the union of events. To have an intuition about them one should consider the nature of the functions.

Here I only confine myself to the observation that, in a sense, the characteristics of these functions reflect the idea that beliefs are more *demanding* than possibilities. Beliefs must be contained in the relevant set! Consider the union of two events; it might be the case that an individual may neither feel sure that A occurred nor that B occurred. However, if he/she does not exclude *at least* one of them he/she can be in a position for believing that their union has indeed occurred. A compound event (according to the union operator) is more "likely" to be believed than its components. A limit case would obtain when A and B partition W so that, even though neither of them may be believed (when taken separately), their union is of course always believed. An analogous argument can be put forward for the intersection of a number of events.

On the other hand, if an agent thinks it possible that the union of A and B has occurred then he/she also *must* think possible that *at least* one of them occurred, and viceversa. Not being able to exclude an event is something *weaker* than feeling sure that an event occurred.

To complete the analysis of all the mappings introduced so far, we consider the properties of the compound mappings β and π, which can be obtained from a straightforward application of their definitions.

<u>Properties of $\beta\pi(A)$ and of $\pi\beta(A)$</u>

14) $\pi(A)=\beta\pi(A)$
15) $\beta(A)\subseteq\pi\beta(A)$
16) $\pi\beta(A)\subseteq\pi(A)$

So 14) says that whenever an agent believes that the occurrence of an event A can not be excluded, then he/she does not exclude it, and viceversa. Property 15) instead states that whenever an agent believes in an event then, he/she indeed can not exclude to believe that A has occurred. Finally, 16) says that to think it possible to believe in an event implies thinking possible that event to have occurred; of course, this last conclusion can always be obtained from property (3).

Properties of $\beta\pi(A)$ and $\pi\beta(A)$

17) $\pi(A)\subseteq\beta\pi(A)$
18) $\beta(A)\subseteq\pi\beta(A)$
19) $\pi\beta(A)\subseteq\pi(A)$

Notice the interesting symmetry exhibited by properties (15)-(18) and (16)-(19).

So far, in the analysis, we have not introduced any concept of knowledge; this will be done in the following section where I shall define some further important mappings.

3. Knowledge, Common Knowledge, Common Beliefs And Common Possibility

As already anticipated in this section we shall discuss the knowledge mapping and other concepts that will later be used in the *agreeing* results of the final part. However, before entering into the necessary details it is important to dwell shortly on a crucial issue.

More specifically, we notice that in the title we might have probably written about *generalized knowledge*, *generalized common knowledge* and so on. Indeed, as I have pointed out earlier , the sense in which knowledge will be defined here is different from the standard definitions of knowledge. Agents *can not* be sure whether their beliefs are correct or not; at most they could communicate each other beliefs in order to have further sources of information about relevant events. Hence, knowledge here lacks awareness on the part of the agents and for this reason a better denomination would have seemed that of *generalized knowledge*[16].

However, since the ensuing analysis can only be conducted from the point of view of the modeller, to simplify matters we have chosen to write of knowledge, common knowledge and so on[17].

16 The terminology being borrowed by that of generalized information structures. Of course, the same observation applies to all other mappings defined in this section.
17 Of course, it is perfectly legitimate to accept the idea that agents would be able to find all the results given in this paper for some abstract agent. Nevertheless, the particular model to which an agent belong can not be understood from the inside; that is to say, decisions and opinions depend only upon the state which obtains.

Definition 6 *A* knowledge mapping *$K:2^W\to 2^W$ for every $A\in 2^W$ is defined as follows*

$$K(A)=\bigcup_{w\in W}\{\,P(w)\ |\,w\in\beta(A)\cap A\,\}$$

The above definition formalizes the concept of knowledge according to the following idea; when an individual knows that an event has occurred it means that his/her beliefs are correct. If an event is believed to have occurred, and it indeed occurred, then we say that such event is known.

The rationale behind this definition should be self-evident. Within a partitional framework, and in most axiomatic approaches too, knowledge is always defined in such a way as to satisfy $K(A)\subseteq A$. That is to say, knowledge of an event necessarily implies its occurrence. Thus, according to this view knowledge can not be separated from truth. Our definition allows us to retain the essence of the above message while working with generalized information structures.

So the concept of knowledge given here is exclusively relevant to the external observer. This is not disturbing if one accepts the stance that agents act only on the basis of their beliefs[18] ; whether those beliefs are correct or not is independent of the decision of undertaking a specific action.

Of course, definition 6 has certain analogies with the ones proposed in the literature. However, once again there is an important difference which rests on the fact that an agent might be completely wrong[19] about the perception of the true state, and yet be correct about a certain event having obtained (when such event indeed obtained). Once again, this seems to be intrinsic to the nature of possibility correspondences, which could be considered essentially a *local* theory of information structure.

We can also define a *dual knowledge mapping* as we did for the possibility and the belief mappings.

Definition 7 *A* dual knowledge mapping *$K:2^W\to 2^W$ for every $A\in 2^W$ is defined as follows*

$$K(A)=\beta(A)\cap A$$

18 A point of view which seems to be shared by Geanakoplos (1992).
19 In the sense that P(w) does not include w.

Obviously, the definition of a knowledge mapping can be reformulated by using the definition of the dual knowledge mapping.

A complete characterization of knowledge, as introduced in this section, can be given by the following list of properties.

<u>Properties of $K(A)$</u>	<u>Properties of $K(A)$</u>
20) $K(\phi)=\phi$	20a) $K(\phi)=\phi$
21) $K(W)\subseteq W$	21a) $K(W)=W$
22) $K(A)\subseteq \beta(A)\subseteq A$, for all $A\epsilon 2^{W}$	22a) $K(A)\subseteq \beta(A)$, for all $A\epsilon 2^{W}$
23) if $A\subseteq B$ then $K(A)\subseteq K(B)$	23a) if $A\subseteq B$ then $K(A)\subseteq K(B)$
24) $K(A\cap B)\subseteq K(A)\cap K(B)$	24a) $K(A\cap B)=K(A)\cap K(B)$
25) $K(A)\cup K(B)\subseteq K(A\cup B)$	25a) $K(A)\cup K(B)\subseteq K(A\cup B)$
26) $KK(A)\subseteq K(A)$	

Proof As a sample we only give the proof for 23). The remaining proofs will be omitted.

23) if $P(w)\subseteq K(A)$ then $w\epsilon K(A)$. Since $A\subseteq B$, it is true that $\beta(A)\subseteq \beta(B)$ which gives $K(A)\subseteq K(B)$ and consequently $w\epsilon K(B)$. Finally, we obtain $P(w)\subseteq K(B)$ which proves the property.•

Notice that all these properties, except (26), are exactly the same as the ones obtained for the belief mappings; this should not be surprising if one recalls that, by definition, $K(A)$ is a subset of $\beta(A)$. On the other hand, there is one important difference between other formulations and ours; in particular, property (26) says that if one knows that A is known then A is known.

This property involves *iterated knowledge* and is usually axiomatized by the opposite inclusion. It should be borne in mind however, that most of the alternative definitions incorporate the assumption of agents being aware of their information processing, which is not the case in our paper. Hence, we need to produce an appropriate interpretation of the above property. Indeed, what we have is the following: when, at a certain state w, an agent has correct beliefs about the set which describes his/her correct beliefs about an event A, then he/she also has correct beliefs about A . Of course, the agent *is not* aware of it but this does not have anything to do with the property itself.

Another important difference with standard formulations of knowledge is that in this context the *axiom of wisdom* does not necessarily hold[20]; in general it will be the case that $K(A)\subseteq (K(K(A))^{c})^{c}$. This inclusion can be rewritten as $K(K(A))^{c}\subseteq (K(A))^{c}$ and the interpretation is the standard one: if an individual knows that he/she does not know the event A then the agent does not know it.

Having defined the knowledge mapping I can now proceed to define the common knowledge mapping ; to do so I shall use the formalization of common knowledge adopted, among others, by Brandenburger-Dekel (1987) and which, for the sake of completeness, I report below .

Consider the two agents, agent i and agent j, and let $K^{i}(A)$ identify the event *for which* agent i's beliefs about the event A are correct[21]. Define now the mappings ${}^{k}C^{i}(A)$ and ${}^{k}C^{j}(A)$ as

$$ {}^{k}C^{i}(A)= K^{i}(A)\cap K^{i}K^{j}(A)\cap K^{i}K^{j}K^{i}(A)\cap\ldots $$
$$ {}^{k}C^{j}(A)= K^{j}(A)\cap K^{j}K^{i}(A)\cap K^{j}K^{i}K^{j}(A)\cap\ldots $$

and the mappings ${}^{k}C(A)$ and ${}^{k}C(A)$ as

$$ {}^{k}C(A)={}^{k}C^{i}(A)\cap {}^{k}C^{j}(A) $$
$$ {}^{k}C(A)={}^{k}C^{i}(A)\cap {}^{k}C^{j}(A) $$

where ${}^{k}C^{i}(A)= K^{i}(A)\cap K^{i}K^{j}(A)\cap K^{i}K^{j}K^{i}(A)\cap\ldots$ and ${}^{k}C^{j}(A)= K^{j}(A)\cap K^{j}K^{i}(A)\cap K^{j}K^{i}K^{j}(A)\cap\ldots$

Then we define common knowledge[22] as follows:

Definition 8 *An event A is said to be common knowledge at w if $w\epsilon {}^{k}C(A)$.*

Keeping the terminology adopted so far we call ${}^{k}C(A)$ the *common knowledge mapping* and ${}^{k}C(A)$ the *dual common knowledge mapping;* from the properties of the knowledge operators it is easy to obtain those of the common knowledge operators.

So far we have not introduced any structure on the possibility correspondences. However, it is exactly in the spirit of the work by Geanakoplos to taxonomize the departures from partitional structures in such a way as to be able to *qualify* the degree of bounded rationality, according to which certain decision theoretic results obtained with perfect rationality

20 For a discussion on this point see, among others, Binmore-Brandenburger (1990), Samet (1990).

21 Accordingly, $K^{j}(A)$ identifies the set of states for which j's beliefs are correct.

22 Needless to say that agents are not aware of each other possibility correspondences.

still hold. One of the most important properties that we consider below is the 'Knowing That You Know' (KTYK) property[23] ; a consequence of KTYK, on the common knowledge operators, is given by the following proposition.

Proposition 1 *Suppose P^i and P^j both satisfy KTYK for all $w\in W$; then ${}^{k}C(A)={}^{k}C(A)$.*

Proof We first see that ${}^{k}C(A)\subseteq{}^{k}C(A)$. Indeed, if one of the two sets is empty the other will be empty too and the claim is proved. Suppose instead that they are not empty and take ${}^{k}C^{i}(A)=K^i(A)\cap K^iK^j(A)\cap K^iK^jK^i(A)\cap\ldots$ This, by the properties of the knowledge mapping, is equal to $K^i(A\cap K^j(A)\cap K^jK^i(A)\cap..)=K^i(A\cap{}^{k}C^{j}(A))$ from which we obtain that ${}^{k}C^{i}(A)=K^i(A\cap{}^{k}C^{j}(A))=K^i({}^{k}C^{j}(A))$.

Thus, ${}^{k}C(A)={}^{k}C^{j}(A)\cap{}^{k}C^{i}(A)={}^{k}C^{j}(A)\cap K^i({}^{k}C^{j}(A))=K^i({}^{k}C^{j}(A))={}^{k}C^{i}(A)$ and, by symmetry, ${}^{k}C(A)=K^j({}^{k}C^{i}(A))={}^{k}C^{j}(A)$ so that the inclusion follows immediately. Finally, suppose A is common knowledge at w; namely $w\in{}^{k}C(A)$. Hence $P(w)\subseteq{}^{k}C(A)$ and by KTYK $P(w')\subseteq P(w)$ for each $w'\in P(w)$. Since ${}^{k}C(A)$ can be expressed as the union of possibility sets (of both P^i and P^j) it follows that for each $w*\in{}^{k}C(A)$ we have $P(w*)\subseteq{}^{k}C(A)$, and this proves the proposition.•

A different way of stating the above result is to say that ${}^{k}C(A)$ is a public event[24] which, whenever it occurs, is known to have obtained by both agents. Of course, not only A, but any event E which contains ${}^{k}C(A)$ will be common knowledge at each state in ${}^{k}C(A)$. When KTYK holds the events *for which* the two agents commonly know A, and the event *in which* they commonly know A coincide. In this case, our definition of common knowledge coincides with the one usually given in the literature (Geanakoplos (1989), Monderer-Samet (1989)): an event is common knowledge if it contains a public event.

23 The KTYK property is defined as follows; suppose $w'\in P(w)$ then $P(w')\subseteq P(w)$. In other words, if w' is possible at w then each state possible at w' is also considered to be possible at w. In our model the terminology should be recast as KTYB (Knowing That You Believe), since there is no presumption that $w\in P(w)$ (in which case $P(w)$ would be known at w) while for $w'\in P(w)$ we have $P(w')\subseteq P(w)$ (implying $P(w)$ to be known at w'). To keep things simple we have chosen to maintain the standard terminolgy.

24 An event A is said to be self-evident for an agent if $P(w)\subseteq A$ for each $w\in A$. An event A is a public event if it is self-evident for both agents. Notice that when KTYK holds we also have $K(A)=K(A)$, for all $A\in 2^W$.

To define the common beliefs mapping ${}^{\beta}C(A)$ and the dual common beliefs mapping ${}^{\beta}C(A)$ it suffices to replace the knowledge operators by the belief operators in the expressions for ${}^{k}C(A)$ and ${}^{k}C(A)$; accordingly, we can also obtain ${}^{\pi}C(A)$ and ${}^{\pi}C(A)$ the common possibility mappings. It is easy to see that ${}^{\beta}C(A)$ (if not empty) is also expressed as the union of possibility sets of both agents while ${}^{\pi}C(A)$, in general, may not. The following lemma formalizes the relationship among the above mappings.

Lemma 1 *It is always true that ${}^{k}C(A)\subseteq{}^{\beta}C(A)\subseteq{}^{\pi}C(A)$ and ${}^{k}C(A)\subseteq{}^{\beta}C(A)\subseteq{}^{\pi}C(A)$.*

Proof Omitted. •

When agents commonly believe an event there is no presumption (by definition) that their beliefs are correct. Unlike the knowledge mappings, it is simple to check that if KTYK holds for both agents we have ${}^{\beta}C(A)\subseteq{}^{\beta}C(A)$; that is to say, there can be states outside ${}^{\beta}C(A)$ *in which* A is commonly believed. Moreover,in this case, ${}^{\beta}C(A)$ and ${}^{\beta}C(A)$ would be public events. Of course, this will necessarily imply that ${}^{\beta}C(A)-{}^{\beta}C(A)$ can not be public.

A coincidence between common knowledge and common beliefs can be obtained if we further impose the property of *non-delusion*[25] to the possibility correspondences. The following lemma formalizes this observation.

Lemma 2 *Suppose P^i and P^j both satisfy KTYK and non-delusion; then ${}^{\beta}C(A)={}^{\beta}C(A)={}^{k}C(A)={}^{k}C(A)$.*

Proof From KTYK we know that ${}^{\beta}C(A)\subseteq{}^{\beta}C(A)$. If ${}^{\beta}C(A)$ is empty then ${}^{\beta}C(A)$ is empty, and so are ${}^{k}C(A)$ and ${}^{k}C(A)$. In this case there would be nothing to prove. Suppose ${}^{\beta}C(A)$ is not empty, assume the above inclusion holds strictly and consider $w\in{}^{\beta}C(A)-{}^{\beta}C(A)$. From the definition of ${}^{\beta}C(A)$ we have $P(w)\subseteq{}^{\beta}C(A)$; since by non-delusion $w\in P(w)$, we deduce that w can not belong to ${}^{\beta}C(A)-{}^{\beta}C(A)$, which contradicts the initial hypothesis.•

The above result is quite intuitive; when the true state is never mis-perceived, agents' beliefs are always correct and coinciding with knowledge.

25 A possibility set is non-deluded at w if $w\in P(w)$. The definition extends to a possibility correspondence when the above holds true for all $w\in W$.

We conclude this section with a comment on $^{\pi}C(A)$ and $^{\pi}C(A)$. From lemma 2 it should be quite clear that $^{\pi}C(A)$ might not be expressed as the union of possibility sets; that is to say, it might not be believed by the agents. To put it differently, if an event is commonly possible the set of states *for which* this occurs could itself *only* considered as possible, so that $^{\pi}C(A)$ might not be public[26].

Having given the above definitions we ask now the following question. Working within a *local* approach, one in which agents are not aware of the model, what sort of implications on the possibility correspondences should we expect to be entailed by the common knowledge/beliefs/possibility of an event? This question naturally addresses *agreeing to disagree* type of issues, an example of which could be specifically formulated in the following way.

Suppose an event *A* is common knowledge at *w*; under what circumstances, if any, agents do not agree to disagree ? This point will be discussed at some lenght in the following section.

4 Agreeing About Possibilities

The *agreeing to disagree* propositions that we shall present do not deal with probabilities and so, in this sense, do not have the same nature as in the formulation given by Aumann (1976) and Samet (1990). As a matter of fact, below we define agreement as the coincidence, at a given state *w*, of possibility sets;

Definition 9 *We say that the two agents do not agree to disagree at w if $P^i(w)=P^j(w)$.*

Since the only individual aware of an event *A* being commonly known/believed/possible is the modeller, under what circumstances such an unknown (to the agents) property implies agreement? In what follows we give a very simple sufficient condition on $^{\beta}C(A)$ which guarantees agreement according to definition 9[27].

26 It could even be the case that $^{\pi}C(A)$ is believed by both agents but still not commonly believed.
27 Notice that our definition would encompass Aumann's result if one introduces common prior subjective probabilities.

Proposition 2 *Suppose $^{\beta}C(A)$ has a single element; then if A is commonly believed the two agents can not agree to disagree.*

Proof Let *w* be the only element of $^{\beta}C(A)$; since $^{\beta}C(A)$ must be expressed as the union of both agents' possibility sets, it holds true that $P^j(w')=(w)=P^i(w')$ for all $w'\epsilon^{\beta}C(A)$ which proves the result.•

Proposition 2 seems to suggest that agreement has to do mainly with beliefs rather than knowledge. The requirement of beliefs being correct is not playing any crucial role in the above proposition and, what is more, the result could hold even if $^{\kappa}C(A)$ were empty[28]. However, the proposition has very little generality and the conclusion seems to depend upon *brute force* in that an explanation is not really provided as to when we should expect to have a singleton. Below we shall see a more sophisticated, and perhaps, more convincing agreement result. We conclude by noticing that the above is by no means a necessary condition nor the only sufficient condition.

The celebrated agreeing to disagree result, originally proposed by Aumann (1976) within an informational partition framework, has recently been obtained by other authors. In particular, Samet (1990) shows that Aumann's proposition could be proved even when information structures are not partitional; he shows that possibility correspondences need only satisfy non-delusion and KTYK[29].

Hence, a natural question to ask is the following: assume that possibility correspondences satisfy non-delusion and KTYK (as in Samet); is it going to be the case that in our framework too common knowledge/beliefs entail agreement? The answer is, in general, *no* and below we produce a very simple example which should clarify the point.

Suppose $^{\kappa}C(A)=(w,w')$; then if *A* is common knowledge, say at *w*, the following possibility sets would satisfy non-delusion and

28 Requiring $^{\pi}C(A)$ to be a singleton would not guarantee agreement about possibility sets.
29 It is worth mentioning that Samet's result has the same nature as Aumann's in that it concerns the equality of the posterior probabilities. Shin (1987) takes an approach similar to that by Samet, in the sense that he uses a propositional framework, but obtains a more general agreement proposition working with the broader concept of a decision rule, rather than confining himself to posterior probabilities. For more on this approach see also Bacharach (1985).

KTYK for both agents. Indeed, assume $P^i(w')=\{w'\}$, $P^j(w)=\{w\}$ and $P^i(w)={}^kC(A)=P^j(w')$; the above information structures comply with both properties but do not entail agreement at w.

In principle, there does not seem to be any reason why non-delusion and KTYK should entail an agreement result. What seems to be missing here is some cross-condition between the possibility correspondences of the two agents. In particular, one could consider the following property

Definition 10 *We say that Knowing That (the) Other (agent) Knows (KTOK) holds at* $w \in W$, *if, for each* $w' \in P^i(w)$, *we have* $P^j(w') \subseteq P^i(w)$ *and, for each* $w'' \in P^j(w)$, *we have* $P^i(w'') \subseteq P^j(w)$.

The interpretation of KTOK is quite intuitive being a straightforward generalization of KTYK; when, at w, agent i has a possibility set given by $P^i(w)$ then, for all states belonging to $P^i(w)$, the events that agent j thinks possible are included in $P^i(w)$ itself, and viceversa. In a sense, the above property could be interpreted as a kind of *consistency* property between the possibility correspondences of the two agents. Since possibility correspondences formalize individuals' information processing, the above requirement captures the idea that the processing reveals some kind of analogy between the two agents. A better understanding of such analogy could be obtained by interpreting KTOK as interactive knowledge; when that holds true agents second guess each other.

Notice that KTOK and KTYK are two independent properties; hence, for example, if KTOK is satisfied at each state in ${}^kC(A)$ this, in general, would not entail its being public.

What sort of implications can KTOK have when associated with common knowledge/beliefs? The proposition below gives a sufficient condition for an agreement result to occur[30].

Proposition 3 *Let* $w \in {}^{\beta}C(A)$ *and assume that* P^i *and* P^j *both satisfy KTOK at* w *and at any* $w' \in P^i(w)$. *Then there exists at least one* $w^* \in {}^kC(A)$ *such that* $P^i(w^*)=P^j(w^*)$.

Proof If $P^i(w)=P^j(w)$ the result trivially holds. Suppose instead that $P^i(w)$ differs from $P^j(w)$. By KTOK we know that if $w^1 \in P^i(w)$

30 Here we shall only consider agent i, bearing in mind that a similar conclusion would hold for agent j as well.

then $P^j(w^1) \subseteq P^i(w)$ for all such w^1. The same must be true for any $w^2 \in P^j(w^1)$ giving rise to the following inclusion $P^i(w^2) \subseteq P^j(w^1)$ and so on. Obviously, since W is finite and the possibility sets are non-empty, the nested sequence $P^i(w)$, $P^j(w^1)$, $P^i(w^2)$,...must converge to a non-empty subset of $P^i(w)$, and the proposition follows immediately.•

An important consequence of proposition 3 is the non-delusion of $P^i(w^*)$[31]; however, assuming non-delusion from the beginning gives rise to the following rather strong conclusion.

Corollary 1 *Let KTOK and non-delusion hold for both agents at some* w ; *then* $P^i(w)=P^j(w)$.

Proof Obvious

Of course, corollary 1 could be considered *too* strong and of very little interest; common knowledge/beliefs would not have any role in the agreement and, unlike proposition 3, we would not even need to assume that KTOK holds for all states in $P^i(w)$ and $P^j(w)$.

The main message coming out of proposition 3 seems to contradict that of proposition 2; agreement about possible states is mainly governed by knowledge rather than beliefs. When agents are *sufficiently* consistent in processing the information, non-delusion would be guaranteed and beliefs would be correct. Hence, under the conditions of proposition 3 ${}^kC(A)$ and, consequently, ${}^kC(A)$ can not be empty. A crucial role in the proof is evidently played by KTOK; it is intuitively clear that when agents second guess each other, one should expect them to reach an agreement about possibility sets, which would also be non-deluded.

Unlike Aumann's decision theoretic framework, and in spite of possibility sets being correct, agreement here does not need awarness about the model on the part of the agents, but rather an appropriate degree of consistency in processing the information. This conclusion sounds intuitive to us in as much as the relevant issue is given by how agents come to have similar opinions, and

31 Of course the same conclusion applies to $P^j(w^*)$.

issue is given by how agents come to have similar opinions, and not whether they are aware of the conditions under which an agreement is reached.

5 Conclusions

In this paper we have explored some theoretical, as well as methodological, consequences of recent proposals concerning the modelling of information structures. In particular, we have argued that if one allows for information processing errors to have a role in a decision theoretic problem, then the appropriate formalization of information structures should take into account two aspects: a *subjective* and an *objective* one. This is intrinsic to the nature of the possibility correspondences which are a more general, and then more flexible, tool for modelling information structures than the standard partitional formalization.

The point of view that we have taken in this work about the nature of possibility correspondences has been extreme in the following sense. If information processing errors are to be considered specific to each possible state, then agents *can not be aware* of the true model. Moreover, we defined knowledge in such a way as to keep together the essence of possibility correspondences and that of a Bayesian framework. The latter only allows non-deluded partitional information structures, while the former is much more general. We said that an agent knows an event when his/her beliefs about the event are non-deluded.

According to this approach, the concept of knowledge and common knowledge become exclusively relevant to the modeller, the only one aware of the true model. In the paper we have also proposed some *agreeing to disagree* results which, under our interpretation of possibility correspondences, have been conceived as *agreeing about possible states* results.

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