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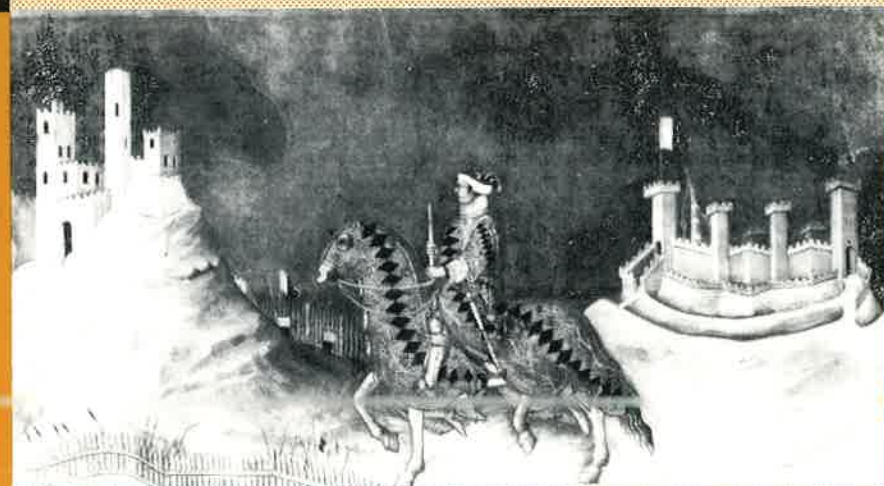
**QUADERNI DEL DIPARTIMENTO
DI ECONOMIA POLITICA**

SERENA SORDI

NONLINEAR MODELS OF THE BUSINESS CYCLE AND CHAOS THEORY: A CRITICAL ANALYSIS



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NONLINEAR MODELS OF THE BUSINESS CYCLE AND CHAOS THEORY: A CRITICAL ANALYSIS

Siena, marzo 1993

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1. Introduction*

In the last two decades or so, business cycle theory has undergone a radical and rapid change, certainly more impressive than those undergone by other fields of economic theory. As a result, business cycle theory is nowadays a very highly formalized topic to which are continuously applied "*purely technical developments*",[1] often imported from other fields of scientific research (e.g., physics, biology, ...). Given this situation, an interesting question is that of understanding the interactions between the technical developments and the (conceptual) developments achieved in business cycle theory in the last few years; in particular, it appears crucial to understand the direction of causality, if any, in such a relation.[2]

According to the dominant view, as expressed most forcefully by Lucas — surely the most representative exponent of the equilibrium approach to business cycle theory — the direction of causality runs from *purely technical developments*, "... that enlarge our abilities to construct analogue economies" (LUCAS, 1980, p.697), to conceptual innovations in business cycle theory. From this Lucasian position, it seems to follow that the attention paid by the profession to the equilibrium approach is to be mainly explained in terms of the developments achieved in the mathematical theory of stochastic difference equations.

Our intention in this contribution is to question such a rigid view, and rather to support a different position according to which the main developments recently achieved in business cycle theory are due mainly to "conceptual innovations" in economic theory. Moreover, and more importantly, our purpose is to show that, in many cases, the acritical application of given mathematical tools to economic problems may hinder rather than stimulate "conceptual developments" in business cycle theory.

Among the "theoretical innovations" in economic theory, the one which helps the most in explaining the passage between business cycle theories of the past — by authors such as Kaldor, Kalecki, Goodwin and Hicks — and business cycle theory of the present — for example, by Lucas — is the idea, now dominant, according to which, in order to obtain a "theoretically sound" aggregative economics, we must search for its *micro-foundations* in the traditional sense. As a consequence of this search, we have nowadays *macroeconomic* theories of the business cycle which are based on the analysis of the behaviour of an optimizing *representative* agent and which disregard any assumption not derived from an individual optimization process as *ad hoc*.[3]

The result of such a concentration on the problem of the micro-foundations of macroeconomics is that the "*ad hockery*" of other very special — and, as such, unsatisfactory — assumptions is rarely discussed. These as-

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1 See LUCAS, 1980, p.697, where we read that these developments include improvements in both mathematical methods and in computational capacity.

2 An important contribution which tries to answer this question is contained in VELUPILLAI, 1991.

3 See, for example, LUCAS, 1977, pp.8 and 14.

sumptions are often required to "force" the structure of a given economic model into a *pre-determined* mathematical structure. Although assumptions of this kind may allow the use of existing mathematical results which guarantee an "elegant" analysis of the model, the consequence is that often the main *economic* problem for the solution of which the model was constructed loses its priority. Such a situation appears to be particularly evident in business cycle theory and has already been the object of some discussion.[4] Our purpose here is to contribute to the discussion by analysing the problem in connection with some recent (and less recent) nonlinear models of the business cycle.[5] In doing this, the various models will be classified on the basis of the order of their final dynamical equation.

First of all (Section 2), we briefly consider two early applications of chaos theory to dynamical economic models. As we will see, in both models the final dynamical equation can be reduced to a one-dimensional difference equation. Our purpose in doing this is to show that in this case, the "technical developments" — which are sufficiently understood only for one-dimensional dynamical systems — cannot lead to relevant "theoretical developments" in business cycle theory, apart from showing that irregular paths can be generated from purely deterministic models. In order to obtain "conceptual developments" in business cycle theory, it is necessary to study higher-dimensional dynamical systems.

Then, in Section 3, the problem of the interaction between "technical developments" and "theoretical developments" in business cycle theory is analysed with regard to higher-dimensional continuous-time models. First, we analyse two nonlinear models of the business cycle formalised in terms of two-dimensional differential equations, namely, Goodwin's 1951 model and the model recently (and repeatedly!)[6] proposed by Benassy. Our purpose in doing this is to show that in these models the crucial (and very unsatisfactory) special assumptions are the ones which make it possible to use the Poincaré-Bendixson theorem and all associated concepts of planar dynamics.[7] Second, to understand what are the consequences of the relaxation of such assumptions, Goodwin's model is analysed as a "forced" oscillator.

Some conclusions are then drawn in Section 4.

2. Nonlinear Models of the Business Cycle in Discrete-Time

It is nowadays well established that very complex (chaotic) behaviour can emerge in one-dimensional, discrete-time dynamical systems.

After May's 1976 article, attention has concentrated mainly on the so-called *logistic equation*:

$$x_{t+1} = \mu x_t(1 - x_t) = f(x_t),\ x \in [0,1],\ \mu \in [0,4],$$

[4] Apart from Velupillai's contribution already quoted in footnote 1, see also VERCELLI, 1991, in particular Chs 8-9.

[5] There has recently been a renewed interest in nonlinear dynamics, stimulated by the mathematical discovery of *chaotic* dynamical systems. See the *Introduction* in LORENZ, 1989.

[6] See, for example, BENASSY, 1984a,b; 1986a,b.

[7] On this point, see FITOUSSI-VELUPILLAI, 1987, p.31.

the mathematical properties of which are sufficiently well understood, at least as compared to those of higher-dimensional systems.[8] It is not surprising, therefore, that also most of the economic models with chaotic dynamics have a final dynamical equation which can be reduced to an equation of the logistic type or of a qualitatively equivalent type.[9] However, what *is* surprising is that the pretty special assumptions, which are required to "force" the structure of a given economic model into the structure of the logistic model, are often introduced without any economic motivation, and, what is even worse, without any discussion of the extent to which the results obtained in the model depend on them.

Our purpose is now to discuss this problem with reference to two early applications of chaos theory to business cycle models.

2.1. Day's Neoclassical Growth Model

One of the first examples of economic models which can generate chaotic dynamics is contained in Day's 1982 article, where the author *reformulates in discrete time* Solow's 1956 neoclassical growth model.[10] In doing this, Day's purpose is to show that complex behaviour can emerge from quite simple economic structures.

When the standard neoclassical growth model is modified by introducing a production lag, and by *assuming that the capital stock exists for exactly one period*, we can write:

(1) $Y_t = C_t + I_t,$

(2) $I_t = K_{t+1},$

(3) $S_t = Y_t - C_t = sY_t,\ s > 0$

(4) $Y_t = F(K_t, L_t),$

(5) $L_t = (1 + n)^t L_0,\ n > 0$

where Y is income; C, consumption; I, investment; K, capital stock; S, saving; F(K,L), a linear-homogenous production function; and L, the labour employed.

From (1)-(5) we obtain the following relation in per capita terms:

$$(6)\qquad \frac{K_{t+1}}{L_{t+1}} = k_{t+1} = \frac{s}{(1+n)} F\left[\frac{K_t}{L_t}, 1\right] = \frac{s}{(1+n)} f(k_t).$$

Under the usual convexity assumption on the production function, the model possesses two equilibria: one, at the origin, is repelling; the other, at

[8] See, for example, LORENZ, 1989, pp.103-129.

[9] To replicate the qualitative features of the dynamics of the logistic equation — such as the "*period doubling route to chaos*" — one has only to assume that, in a relation of the type $x_{t+1} = F(x_t;\mu)$, the map is hump shaped with a unique maximum that increases in the "tuning" parameter μ. See BROCK-DECHERT, 1991, p.2212. For a survey of economic models with chaotic dynamics, see BAUMOL-BENHABIB, 1989; BROCK-DECHERT, 1991; GOODWIN-PACINI, 1992; and KELSEY, 1988.

[10] See also DAY, 1983; BENHABIB-DAY, 1980, 1981, 1982.

the point k^* — where k^* is such that $k^*(1 + n) = sf(k^*)$ — is asymptotically stable. In this case, taking k_0 as the initial capital-labour ratio, we obtain the graph shown in **FIGURE 1**.[11]

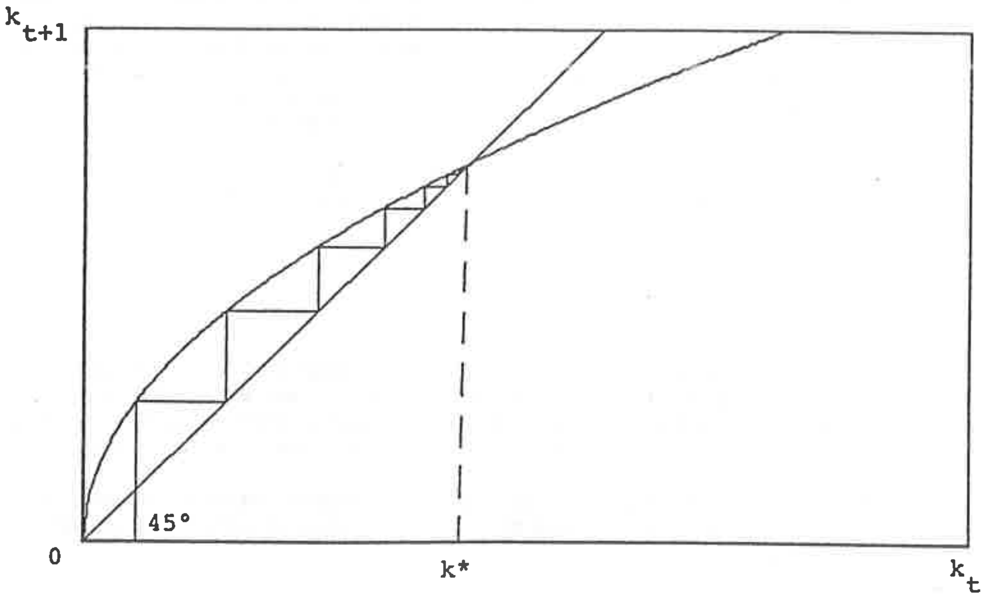

FIGURE 1

Having seen this, it is not difficult to understand how the model needs to be modified in order to obtain a dynamical equation similar — and qualitatively equivalent — to the logistic map. Indeed, we know that, under the usual assumptions on the production function, the right-hand side of (6) is a monotonically, strictly increasing function of the capital-labour ratio. In contrast, as shown by Day (1982, pp.409-411), assuming that there is a productivity inhibiting effect caused by increasing concentration of capital, such that the production function is:

$$\frac{Y_t}{L_t} = f(k_t) = Bk_t^{\beta}(m - k_t)^{\gamma}, \qquad k_t \leq m,$$

(6) becomes a unimodal map:

$$k_{t+1} = \frac{sB}{(1 + n)}k_t^{\beta}(m - k_t)^{\gamma}.^{12}$$

[11] For the simulation we have used a production function of the Cobb-Douglas type — $Y_t/L_t = f(k_t) = Bk_t^{\beta}$ — with $B = 2.8$ and $\beta = 0.5$.

[12] Day considers also other two examples to show how the standard neoclassical model can be modified to obtain the representation of chaotic dynamics, one in which there is a variable savings ratio and one in which the substitution of capital and labour is constrained by a maximal potential growth rate.

Thus, assuming that $\beta = \gamma = m = 1$, we obtain a dynamical equation formally equivalent to the logistic equation with $\mu = sB/(1 + n)$:

$$k_{t+1} = \frac{sB}{(1 + n)}k_t(1 - k_t),$$

which can be easily simulated. For example, for values of s, B and n such that the ratio $sB/(1 + n)$ is above a certain critical level, we obtain the phase diagram and the solution trajectory shown in **FIGURE 2** and **FIGURE 3** respectively.[13]

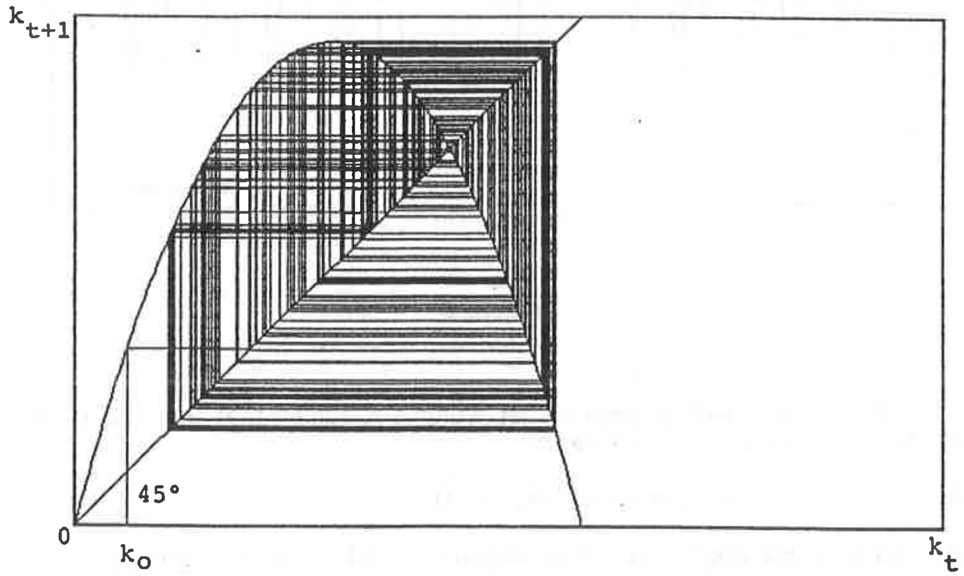

FIGURE 2

2.2. A Discrete-Time Version of Goodwin's 1967 Growth Cycle Model

Another economic model the final equation of which can be reduced to an equation of the logistic type is contained in POHJOLA (1981), where the author analyses a *discrete-time version* of Goodwin's 1967 growth cycle model.

[13] For the simulation we have taken $s = 0.4$, $B = 12.35$, and $n = 0.3$. Even for the general five-parameters case, DAY (1982, pp.409-11), by applying the Li-Yorke theorem, shows that the above map is chaotic for appropriate values of the parameters.

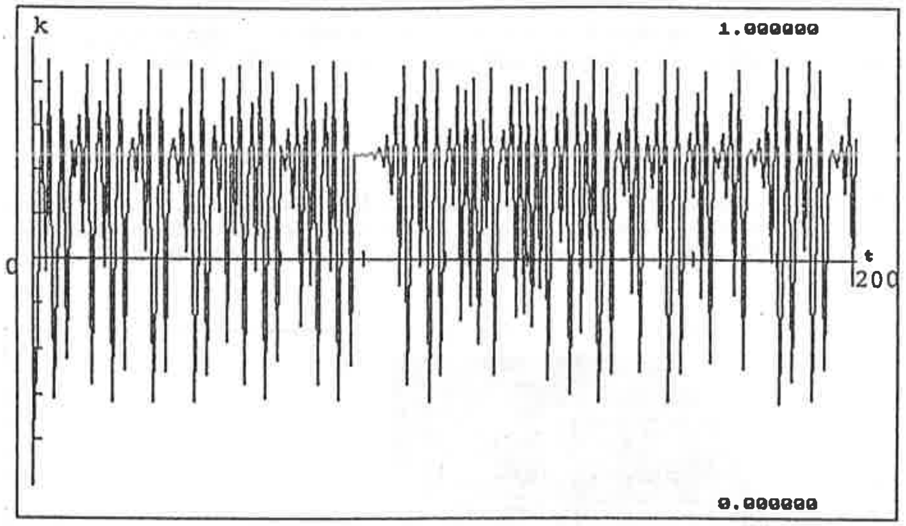

FIGURE 3

As in Goodwin's original model, Pohjola assumes that the labour supply (N_t) grows at a constant rate (n):

$$N_{t+1} = (1 + n)N_t,\ n \geq 0; \tag{7}$$

that the technical progress is labour-augmenting at a constant rate (σ):

$$\frac{Y_{t+1}}{L_{t+1}} = (1 + \sigma)\frac{Y_t}{L_t},\ \sigma \geq 0; \tag{8}$$

and that the capital-output ratio is constant (μ):

$$\frac{K_t}{Y_t} = \mu,\ \mu > 0. \tag{9}$$

However, in other respects, Pohjola's model deviates crucially from Goodwin's original one. Indeed, not only does he consider a discrete-time version of the model, but he also takes a different specification of the way in which income distribution reacts to the employment situation: Goodwin's original specification — a Phillips-curve in which the *relative change* of real wages depends on the employment rate (E = L/N) — is replaced by KUH's (1967) specification which makes the *level* of wages depend on E:

$$w_t = h(E_t)\frac{Y_t}{L_t},\ h' > 0,$$

or, taking a linear approximation of $h(E_t)$:

$$w_t = (-\alpha + \beta E_t)\frac{Y_t}{L_t}.\text{[14]} \tag{10}$$

Although this modification of the model is introduced by Pohjola without any further discussion, it is not difficult to understand that this is the crucial assumption which ensures that the final dynamical equation of the model is one-dimensional.[15]

From equations (7)-(10), assuming, as in Goodwin's model, a saving function of the classical type, we can write the condition for the equilibrium in the product market as:

$$K_{t+1} - K_t = (1 - A_t)Y_t, \tag{11}$$

where A is the workers' share of total output:

$$A_t = \frac{w_t L_t}{Y_t} = -\alpha + \beta E_t. \tag{12}$$

Thus, we obtain:

$$\frac{E_{t+1} - E_t}{E_t} = \left(\frac{L_{t+1}}{N_{t+1}}\right)\left(\frac{N_t}{L_t}\right) - 1 =$$
$$= \frac{1}{(1 + n)(1 + \sigma)}\left(\frac{Y_{t+1}}{Y_t}\right) - 1,$$

from which:[16]

$$E_{t+1} = E_t\left[1 + r\left(1 - \frac{E_t}{E^*}\right)\right], \tag{13}$$

where:

$$r = \frac{1 - \mu g + \alpha}{\mu(1 + g)},$$
$$g = \sigma + n + n\sigma,$$
$$E^* = \frac{1 - \mu g + \alpha}{\beta}.\text{[17]}$$

[14] See POHJOLA, 1981, p.29.

[15] A discrete-time version of Goodwin's original specification would contain the term $(w_{t+1} - w_t)/w_t$ on the left-hand side.

[16] See POHJOLA, 1981, pp.29-30.

[17] Moreover, it is assumed that $1 - \mu g + \alpha > 0$, $-\alpha + \beta > 1 + \mu$, $r < 3$, and $r > 0$. From these assumptions, it follows that $0 < E_t < 1$ and $-\alpha < A_t < 1 + \mu$ in all periods.

Finally, from (13), writing x_t for $rE_t/(1 + r)E^*$, we obtain the following equation of the logistic type:

$$x_{t+1} = (1 + r)x_t(1 - x_t). \tag{14}$$

As was the case for Day's model, also for Pohjola's model it is not difficult to find values of the parameters such that the solution has a chaotic behaviour. For example, with r = 2.94 we obtain the phase diagram and the solution trajectory of the model shown in FIGURE 4 and FIGURE 5 respectively.

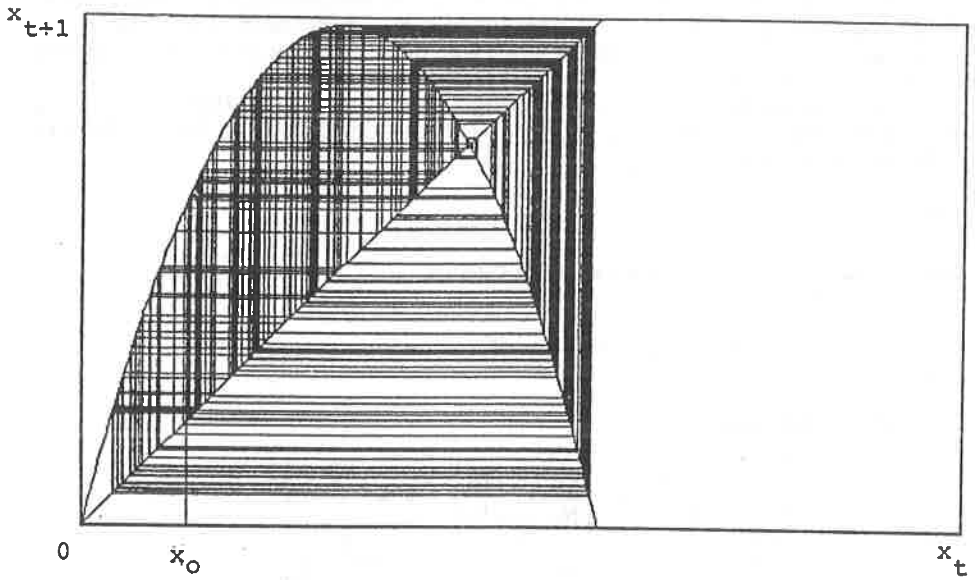

FIGURE 4

Thus, the discrete-time version of Goodwin's 1967 model, with the modification introduced by Pohjola, is capable of generating chaotic growth paths.

2.3. Concluding Remarks

The models we have analysed in this **Section** — which we have taken as significant, although not exhaustive, examples of the application of chaos theory to economic modeling — were proposed right at the beginning of the Eighties, when economic investigations of chaotic dynamics were still in their infancy. Thus, they had, together with a few other contributions,[18] the im-

[18] An earlier model which gives rise to chaotic dynamics is contained in STUTZER, 1980, where the author analyses a *discrete-time version* of Haavelmo's 1954 growth model.

portant role of showing to economists that purely deterministic models can generate very irregular behaviours, strongly resembling random processes.[19] However, besides that, these models do not seem to offer us the possibility of improving our understanding of the dynamics of capitalist economies. On the contrary, as we have seen, they show very clearly that strong (*ad hoc*) assumptions are required to "force" the structure of a given business cycle model into a pre-determined mathematical structure.

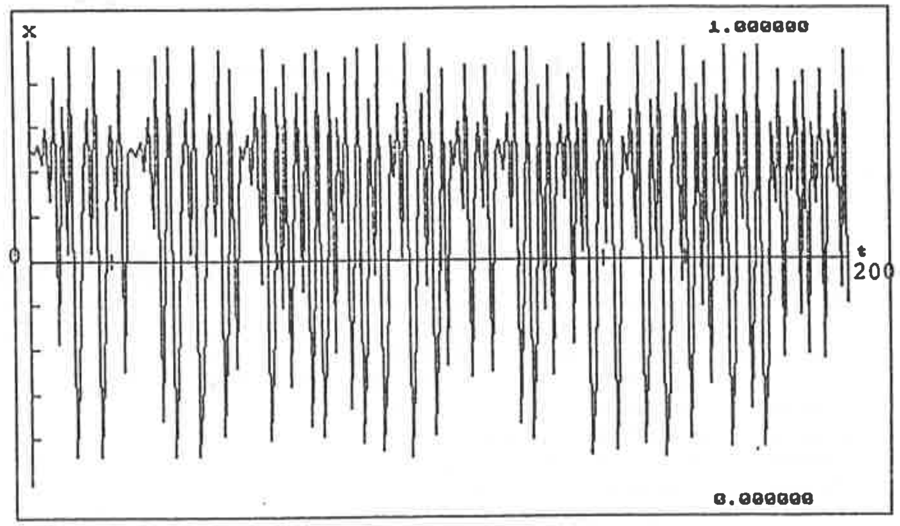

FIGURE 5

Both Solow's 1956 growth model and Goodwin's 1967 growth cycle model, with the modifications introduced by Day and Pohjola, have a final dynamical equation of the logistic type to which all known techniques can be easily applied. For example, once the final dynamical equation has been reduced to an equation of the logistic type, one can easily find the values of the parameters for which the equilibrium is stable, the values for which there are stable cycles of period 2^k, or the values for which there is chaotic behaviour. However, although the use of these techniques makes a very "elegant" analysis of the models possible, it is important to understand that the modifications of the original models have above all a mathematical motivation and seem rather unsatisfactory from an economic theory point of view. First of all, it is necessary to consider discrete-time versions of models that, as was the case for the two models we have considered above, were originally framed in continuous-time. Second, it is necessary to ensure that the map is one-dimensional, even in the cases in which the original version of the mod-

Also in this case, the final dynamical equation is a nonlinear, one-dimensional difference equation qualitatively equivalent to the logistic equation.

[19] Perhaps, an even more influential article in this sense was that of GRANDMONT (1985), which showed to economists that this is true also for *general equilibrium* models.

els had a higher-dimensional dynamical system. Finally, it is necessary — introducing very special assumptions — to ensure that the map is unimodal. All this seems to reduce considerably the relevance and *generality* of the results obtained analysing models of this kind. Thus, this is an example in which the use of "technical developments" — which are sufficiently understood and widely known only for one-dimensional dynamical systems — does not seem to lead to relevant developments in business cycle theory.

3. Nonlinear Models of the Business Cycle in Continuous-Time

There are at least two good reasons why, to analyse higher-dimensional dynamical systems, we prefer to consider models of the business cycle framed in continuous- rather than discrete-time. First of all, doing this, we can take account of an important *interaction* between purely "technical developments" and "theoretical innovations" in business cycle theory: in 1951, Richard Goodwin — who had learnt most of the necessary theory of oscillations from the physicist Ph. Le Corbeiller[20] — published his celebrated continuous-time, nonlinear accelerator model, the dynamical system of which can be reduced to an equation of Lord Rayleigh's type (LRT) or, equivalently, of van der Pol's type (VDPT).[21] Second, complex behaviour in continuous-time dynamical systems requires an at least three-dimensional state space. Continuous-time, therefore, appears to be the natural framework in which to study higher-dimensional nonlinear systems in the same way as discrete-time is the natural framework to study one-dimensional systems.

What is relevant for us here is that, in the opinion of the exponents of the dominant approach(es) to macroeconomics, the problem with Goodwin's model is that its macro-behavioural relations lack any micro-foundations in the traditional sense and must therefore be disregarded. Jean-Pascal Benassy, for example, has recently criticized, for the *ad hockery* of their assumptions, all models — and thus also Goodwin's — with a sigmoid-shaped investment function or Phillips curve.[22] As an alternative, he has then proposed a (nonlinear) non-Walrasian model with the purpose of showing that it is possible to obtain the representation of endogenous, persistent fluctuations assuming only *traditional shapes* for the various functions.

To appreciate the relevance of Benassy's criticism, it is necessary to understand the role which, in a model of the business cycle, is played by the choice of a given nonlinearity. In the attempt to answer this question we first analyse, in **Section 3.1**, the problem of the conditions for a limit cycle in Goodwin's model. Then, in **Section 3.2**, we consider the same problem with regard to Benassy's model.

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[20] Goodwin and Le Corbeiller were colleagues during the Forties at Harvard University. See PALAZZI, 1982, pp.18-20.

[21] In this case, the causality does not seem to run from purely "technical developments" to "theoretical innovations" in business cycle theory (i.e., from the van der Pol oscillator to the nonlinear accelerator!): Goodwin's purpose was that of representing endogenously persistent economic fluctuations; had he not had this purpose, his relation with Ph. Le Corbeiller would not have resulted in such an important innovation in business cycle theory.

[22] See Benassy, 1984a,b; 1986a,b.

3.1. Goodwin's Nonlinear Accelerator Model

Our purpose is now to show, with reference to Goodwin's 1951 article, the role which the choice of a given nonlinearity plays in a continuous-time model of the business cycle. In order that the attention of the reader is not diverted from the main point we want to make, however, the mathematical steps, when necessary, are presented in the Mathematical Appendix (from now on referred to as **MA**).

In his celebrated 1951 model, Goodwin obtains, as a final equation describing the dynamics of the economic system, the following (non-autonomous) nonlinear equation (see **MA.i**):

$$\text{(15)} \qquad \varepsilon\theta\ddot{Y} + \{(\varepsilon + s\theta)\dot{Y} - \phi(\dot{Y})\} + sY = A(t+\theta),$$

where Y is national income; ε, the time-lag between income and expenditure; θ, the time-lag between investment orders and investment expenditures; s, the marginal propensity to save; $\phi(\dot{Y})$, induced investment; $A(t+\theta)$, autonomous investment; and where $\dot{Y} = dY/dt$, and $\ddot{Y} = d^2Y/dt^2$.

The induced investment is assumed to react nonlinearly to changes in output and to be such that:

$$\phi(o) = 0,$$
$$\frac{d\phi(o)}{dt} = v,$$
$$\lim_{\dot{Y}\to-\infty} \phi(\dot{Y}) = -I^{**},$$
$$\lim_{\dot{Y}\to\infty} \phi(\dot{Y}) = I^{*},$$

where I^* is a positive constant corresponding to the maximum rate of investment allowed by existing productive capacity, and $-I^{**}$ a negative constant corresponding to zero gross investment.

Equation (15) is a nonlinear differential equation of Lord Rayleigh type (LRT). Assuming for the moment, *as Goodwin does*, that the autonomous component of investment is constant and equal to A^* for all t, taking the derivative of both sides, and writing y for $(Y - A^*/s)$ and x for $\dot{y}$, we obtain the equivalent equation of van der Pol's type (VDPT):[23]

$$\text{(16)} \qquad \ddot{x} + f(x)\dot{x} + g(x) = 0,$$

where we have chosen the time-unit so as to have $\theta = 1$, and where:

$$f(x) = \frac{1}{\varepsilon}\{(\varepsilon + s) - \phi'(x)\},$$
$$g(x) = \left[\frac{s}{\varepsilon}\right]x.$$

---

[23] For a discussion of the relation between these two types of equations, see LE CORBEILLER, 1960.

Using an intuitive attitude to the question, we can assemble qualitative evidence that, in the case in which the equilibrium is unstable (see **MA.ii**), there is a limit cycle.

To this end, let us use an analogy from mechanics[24] and consider equation (16) as an equation describing the displacement — x — of a particle of unit mass under a force system containing a conservative element — g(x) — and a dissipative or energy-generating component — $f(x)\dot{x}$. From energy considerations (see **MA.iii**), i.e., studying the exchange of energy along a phase path, it is not difficult to understand that the shape chosen for the investment function, together with the relation among parameters implied by the stability condition, is such that when $|x|$ is "small", x must increase, whereas when $|x|$ is "large", x must decrease. This means that, although the origin is locally unstable and thus in the region near the origin $|x|$ increases, sooner or later a region will be reached in which $\phi'(x) < (\varepsilon + s)$ so that $|x|$ decreases.

This appraisal of the regions of the phase plane in which energy loss and energy gain take place gives grounds for expecting a limit cycle and makes it possible to understand the role played by the choice of that nonlinearity: the nonlinearity — in this case that of the investment function — is such that, over a given trajectory, the exact balance of regions in which the energy decreases and regions in which the energy increases is guaranteed at least once. In this sense, we can say that, in order to obtain in the phase plane a limit cycle solution, an *ad hoc* nonlinearity is always required. Keeping this in mind, it is now interesting to turn our attention to Benassy's model.

3.2. A Standard Criticism of the (Nonlinear) Accelerator Model

In his contribution, Benassy's purpose is that of achieving a synthesis between models of the non-Walrasian equilibrium type and traditional Keynesian models of the IS-LM type with fixed wage and flexible price.

In such a framework, the IS curve is obtained while taking account of the fact that aggregate demand, which is always equal to sales, is equal to:

(17) $$Y = C(Y,p) + I(X,r),$$

where Y is income (output); p, the price; X, the expected demand; r, the interest rate; and where all usual assumptions on the signs of the derivatives of the various functions are made, namely, $C_y > 0$, $C_p < 0$, $I_x > 0$, and $I_r < 0$.

Firms, which are assumed to be not rationed at all, have a demand for labour equal to their notional demand and, as a consequence, also a short-run supply of goods equal to their notional supply. Thus, given the short-run production function with decreasing returns which, *for a given level of the capital stock*, relates output to the level of employment — $Y = F(L)$, with $F' = dF/dL > 0$ and $F'' = d^2F/dL^2 < 0$ — we have:

(18) $$Y = F\left[F'^{-1}\left(\frac{w}{p}\right)\right] = S(w,p),$$

[24] See, for example, JORDAN-SMITH, 1987, pp.16-20.

where w is the money wage and where (see **MA.iv**):

$$S_w < 0,\ S_p > 0.$$

With regard to the LM curve, it is assumed that it is of the form:

(19) $$L(Y,r,p) = \bar{M},$$

where L_y, $L_p > 0$, $L_r < 0$ and where $\bar{M}$ is the (fixed) quantity of money in the economy.

Thus, the short-run equilibrium values of the variables are determined by the following system:

(20) $$Y = C(Y,p) + I(X,r),$$
(21) $$Y = S(w,p),$$
(22) $$L(Y,r,p) = \bar{M}.$$

In the case considered by Benassy, with fixed wage and flexible price, the above system determines the equilibrium values of Y, r, and p for given x and w. Thus, its solution in Y is a relation of the type:

(23) $$Y = z(X,w),$$

such that (see **MA.v**):

$$\frac{\partial Y}{\partial X} = z_x > 0,$$
$$\frac{\partial Y}{\partial w} = z_w < 0.$$

The crucial step is now to specify the two equations governing the behaviour in time of the money wage and of the expected demand..

With regard to the nominal wage, Benassy assumes that it evolves according to a Phillips Curve of the type:

(24) $$\dot{w} = H(u),\ H'(u) < 0,$$

where, writing L_0 for the (inelastic) supply of labour, the level u of unemployment is given by:

$$u = L_0 - F^{-1}(Y),$$

and where the function H is assumed to be such that:[25]

$$\lim_{u \to 0} H(u) = +\infty,$$

[25] The formulation chosen by Benassy takes account of the assumption according to which the firm is not constrained on the labour market, i.e., that full employment is never reached.

$$H(\overline{u}) = 0.$$

From (24), we obtain a relation between the increase in nominal wage and output:

(25) $$\dot{w} = H(u) = H(L_0 - F^{-1}(Y)) = \\ = G(Y),\ G'(Y) > 0,$$

where, writing Y_0 for the full employment output $F(L_0)$ and $\overline{Y}$ for $F(L_0 - \overline{u})$, the function G is such that:

$$\lim_{Y \to Y_0} G(Y) = \lim_{Y \to Y_0} H(L_0 - F^{-1}(Y)) = +\infty,$$
$$G(\overline{Y}) = H(L_0 - F^{-1}(\overline{Y})) = H(\overline{u}) = 0.$$

Finally, with regard to the dynamics of expected demand, Benassy assumes that it "adapts" to actual demand with a speed of adjustment equal to μ, i.e.:

(26) $$\dot{X} = \mu(Y - X),\ \mu > 0.$$

Thus, the system describing the dynamics of the economy is the following:

(27) $$\dot{w} = G(Y),$$
(28) $$\dot{X} = \mu(Y - X),$$

where:

(29) $$Y = z(X,w).$$

Writing (Y^*,X^*,w^*) for the long-run equilibrium values of the variables, we find that:

$$Y^* = X^* = \overline{Y},$$

and that w^* is the value of w which satisfies the relation:

$$z(\overline{Y},w^*) = \overline{Y}.$$

Benassy's contention is that this (long-period) equilibrium — which exists and is unique — is either stable or unstable, in which case there will be at least one limit cycle.

The equilibrium point is locally unstable or locally stable according as to whether (see **MA.vi**):

(30) $$\mu(z_x - 1) \gtrless -G'(\overline{Y})z_w,$$

On the basis of the assumptions made by Benassy, we can conclude that the partial derivative z_x is positive. However, it is not possible to say anything about its absolute value, namely, whether it is less or greater than

one. Thus, there are different, alternative cases which must be considered. For example, when z_x is less than one, the left-hand side of the condition is negative so that the equilibrium point is stable. However, when z_x is greater than one, there are two possibilities depending on whether the positive difference $\mu(z_x - 1)$ is less or greater than the right-hand side of the conditions we have obtained; i.e.:

$$z_x - 1 > 0,$$

is a necessary — although not sufficient — condition for the equilibrium point to be unstable.

If this is the case, Benassy shows, using the *Poincaré-Bendixson Theorem*, that there exists at least a limit cycle.[26]

Having seen this, however, it is not too difficult to understand that in Benassy's model, the crucial role in the generation of limit cycles is played by the (*ad hoc*) specification of the nonlinear dependence of the equilibrium level of income on expected demand. In order to concentrate our attention on such a crucial nonlinearity, we consider a *simplified version* of the model.

To this end, we replace the Phillips curve with its linear approximation, obtained by expanding the function G(Y) in Taylor's series at the equilibrium point and then neglecting all terms of order higher than the first.

In doing this, the dynamical system determining the evolution of the economy becomes:

(31) $$\dot{w} = \sigma(Y - \overline{Y}),$$
(32) $$\dot{X} = \mu(Y - X),$$

where $Y = z(X,w)$ and $\sigma = G'(\overline{Y}) > 0$.

From (31)-(32), we obtain (see **MA.vii**):

(33) $$\ddot{x} + \{\mu(1 - z_x) - \sigma z_w\}\dot{x} - \mu\sigma z_w x = 0.$$

To simplify still further, we assume that all functions (20), (21) and (22) are linear in all arguments except the investment function which is nonlinear in the expected demand, and such that:

$$\frac{\partial^2 I}{\partial x^2} = I_{xx} \neq 0,$$
$$\frac{\partial^2 I}{\partial x \partial r} = \frac{\partial^2 I}{\partial r^2} = 0.$$

From this assumption, it follows that the short-run equilibrium value of Y is such that its partial derivative with respect to X depends only on X whereas its partial derivative with respect to w is constant. As a consequence, we can write:

$$z_x = Z'(X) > 0,$$
$$z_w = \bar{z}_w < 0 \text{ (constant)},$$
$$z_{xx} = Z''(X) \neq 0,$$

[26] See, for example, Benassy, 1984a, pp.86-88.

$$z_{ww} = z_{xw} = 0.$$[27]

Thus, equation (33) becomes:

(34) $$\ddot{x} + f_2(x)\dot{x} + g_2(x) = 0,$$

where:

$$f_2(x) = \mu[1 - Z'(x+X^*)] - \sigma\bar{z}_w,$$
$$g_2(x) = -\mu\sigma\bar{z}_w x,$$

which is an equation of (generalized) van der Pol's type.

In spite of Benassy's claim, also for his model we can easily understand the role played by the choice of that nonlinearity.

To see this, let us write equation (34) as:

(35) $$\dot{x} = u,$$
(36) $$\dot{u} = -f_2(x)x - g_2(x)$$

For this dynamical system, the total energy is given by:

$$E = \frac{u^2}{2} + G_2(x),$$

where:

$$G_2(x) = \int_0^x g_2(\tau)d\tau,$$

so that:

$$\frac{dE}{dt} = -f_2(x)u^2 \gtrless 0,$$

according as to whether:

27 In the case in which all functions are linear in all arguments except the investment function which is nonlinear in the expected demand variable, we obtain:

$$J \cdot \begin{bmatrix} d^2y \\ d^2r \\ d^2p \end{bmatrix} = \begin{bmatrix} I_{xx}dx^2 \\ 0 \\ 0 \end{bmatrix}$$

where J is the same matrix of coefficients we have considered above. Thus, $\partial^2 y/\partial x^2$ is different from zero and such that:

$$\partial^2 y/\partial x^2 = z_{xx} = -I_{xx} I_r S_p/|J|$$

For an analogous simplifying assumption, see SCHINASI, 1981.

$$f_2(x) = \mu[1 - Z'(x+X^*)] - \sigma\bar{z}_w \lesseqgtr 0.$$

Having seen this, it is possible to understand that the assumption implicitly made by Benassy according to which there exists a value of X — say X^{**} — for which $Z'(X^{**}) = 1$ is crucial.[28] Given this assumption, we know that there exists an $X = X^{**} < X^*$ for which we have:

$$f_2(X^{**}) = -\sigma\bar{z}_w > 0,$$

so that:

$$\frac{dE}{dt} < 0.$$

On the other hand, we have seen above that the condition for the equilibrium point to be unstable requires that:

$$\mu[1 - Z'(X^*)] - \sigma\bar{z}_w < 0,$$

i.e.:

$$f_2(X^*) < 0,$$

so that:

$$\frac{dE}{dt} > 0.$$

Thus, the proof of the existence of a limit cycle offered by Benassy by using the Poincaré-Bendixson theorem simply implies that the forms chosen for the various functions of the model are such that these positive and negative exchanges of energy compensate each other exactly at least once over a trajectory. As a consequence, although Benassy is surely correct in saying that in his model he does not need to assume a sigmoid shape for any of the functions involved, his solution is subject to the same type of criticism.

Some conclusions can be drawn from the analysis we have developed.

In Goodwin's model a crucial role is played by the specification of the investment function. This should hardly be surprising given the crucial role that factors explaining the investment behaviour have in determining the dynamics of an economy. In particular, Goodwin gives an explanation for the reasons behind the assumption of a sigmoid shape for the investment function which are plausible if seen as part of the very simplified framework (*"closed economy model with no role for government"*) in which they are made.

On the contrary, assumptions are often not explicitly made by Benassy. In particular, he considers it an advantage of his model that cyclical fluctuations in economic activity are obtained without having to assume any *ad hoc* shape for the various functions involved. However, on the basis of

28 That such an assumption is made by Benassy is implicit in the phase diagram where he draws, for the unstable case, the graph of the locus $\dot{X} = 0$. See BENASSY, 1986a, pp.182-183.

the analysis we have developed above, there are good grounds for disagreeing with him. For example, we have seen that, for the existence of limit cycles, the partial derivative of Y with respect to X must be positive but less than one for "low" levels of expected demand and greater than one for "high" levels of expected demand. Although Benassy says nothing about this matter, this implies that, to have a limit cycle, we must have $\partial^2 Y/\partial X^2 = z_{xx} > 0$. In the simplified model we have considered, as well as in the original model, this is the case when $I_{xx} > 0$. However, to require that I_{xx} is positive for all levels of expected demand is an (*ad hoc*) assumption which would seem to deserve, at least, some discussion.[29]

The lack of attention given to the analysis of the investment process makes itself felt also in another important respect. As we have seen, Benassy's model has the "traditional" structure of *short run* non-Walrasian equilibrium which is not abandoned even when *long run* considerations are introduced into the model. For example, given the assumed short run structure, the production function relates output to the level of employment for a *given level of the capital stock*. The resulting short run equilibrium level of output (income) is then used, together with the two equations describing the behaviour in time of w and X, to obtain the dynamical system which governs the dynamics of the economy. The problem is that, although this is surely a "traditional" assumption for the short run, it is difficult to understand what it means to maintain it in the long run. As a result, we have an investment function which has an important role in determining some features of the dynamics of the economy but which has no effect on the level of the stock of capital and, therefore, not even on the level of output.

To justify his assumption, Benassy says that:

> "... (we) could have added an equation depicting the evolution of the capital stock, so that investment would have effects on both the demand and supply sides... The inclusion of these variables, and possibly other relevant ones, *would have however increased the dimensionality of our dynamic system beyond that which can be* elegantly *handled by available mathematical cycle theory.*"
>
> (BENASSY, 1986b, p.144; italics added).

Thus, we again have assumptions which are made to "force" the structure of the model into a given mathematical structure. Such assumptions — which make it possible to employ the Poincaré-Bendixson theorem and all associated concepts of *planar* dynamics — allow an "elegant" analysis of the model but certainly do not open the way to relevant developments in business cycle theory. Developments of this kind seem to require the study of higher-dimensional dynamical systems. Also this can be easily understood considering Goodwin's nonlinear accelerator model.

3.3. Goodwin's Model as a Forced Oscillator

As we have seen, Goodwin, in his nonlinear accelerator model, obtains the following final dynamical equation:

[29] An analogous comment is made in GABISCH-LORENZ, 1987, p.138.

$$\ddot{Y} + \frac{1}{\varepsilon}\{(\varepsilon + s)\dot{Y} - \phi(\dot{Y})\} + \frac{s}{\varepsilon}Y = \frac{1}{\varepsilon}A(t+1), \tag{37}$$

but analyses the equation:

$$\ddot{y} + \frac{1}{\varepsilon}\{(\varepsilon + s)\dot{y} - \phi(\dot{y})\} + \frac{s}{\varepsilon}y = 0, \tag{38}$$

where y represents deviations from the equilibrium level.

To obtain equation (38) from equation (37), Goodwin assumes that the autonomous component of investment is constant and it is important to notice that this assumption, which is the crucial one, is not even mentioned by Benassy — perhaps because it has to do with the dimensionality of the resulting dynamical system.

In equation (37), time enters explicitly in the right-hand side and this causes the *dimensionality of the system to increase*.[30] Thus, although Goodwin introduces the assumption of constant autonomous outlays without discussing its implications,[31] it is easy to understand that this is the crucial (*ad hoc*) simplification which allows Goodwin to write the equation in terms of deviations from equilibrium as in (38) and then to perform the subsequent analysis using the Poincaré-Liénard method of graphical integration. However, this assumption — which seems to have above all a mathematical motivation — is not very satisfactory from an economic theory point of view.

To understand this, let us stress that innovational investment is part of A(t+1), i.e., of that component of investment outlays which in the model is taken as constant. A better alternative is to assume — as in GOODWIN, 1946 — that innovations are a periodic function of time and write:

$$A(t+1) = b\sin(c(t+1)).$$

In this case, the (forced) equation describing the dynamics of the economic system is equivalent to the following three-dimensional system:

$$Y_1' = Y_2, \tag{39}$$

$$Y_2' = -\frac{1}{\varepsilon}\{(\varepsilon + s)Y_2 - \phi(Y_2)\} - \frac{s}{\varepsilon}Y_1 + \frac{b}{\varepsilon}\sin cY_3, \tag{40}$$

$$Y_3' = 1. \tag{41}$$

After having specified the investment function $\phi(\dot{Y})$ (see **MA.viii**), it is possible to simulate the solution trajectory of the dynamical system.

With $b = 0$ — the "unforced" case — equations (39)-(41), when the equilibrium is unstable, describe a stable limit cycle as is shown in **FIGURE 6**.

However, when $b \neq 0$, we may have an infinite set of new phenomena in addition to the equilibrium point and the (stable) limit cycle which can appear from the planar theory of nonlinear oscillations.[32]. For example, simu-

[30] For an introduction to the study of "forced" equations, see Ch.16 — "Dimension Two and One Half" — in Hale-Koçak, 1991, pp.497-509.

[31] See GOODWIN, 1951, p.12.

[32] See GUCKENHEIMER-HOLMES, 1983, pp.ix-xii, and 66-82. See also GOODWIN, 1990, pp.117-130; KOÇAK, 1989, pp.7-8 and 143-144; JORDAN-SMITH, 1987, Chapters 5, 7 and 12; and LORENZ, 1988. Goodwin's model as a forced oscillator is studied in LORENZ, 1987, and LORENZ, 1989, pp.155-157.

lating the system for $\varepsilon = 0.5$, $s = 0.4$, $v = 1.2$, $b = 1$ and $c = 1$, we obtain the phase diagram shown in FIGURE 7. Alternatively, simulating the equivalent VDPT equation, we obtain the phase diagram and the solution trajectory shown in FIGURE 8 and FIGURE 9 respectively. In particular, the latter figure shows that the model, with the values chosen for the parameters, exhibits the so-called *sensitive dependence on initial conditions* — a distinctive feature of chaotic dynamical systems.[33]

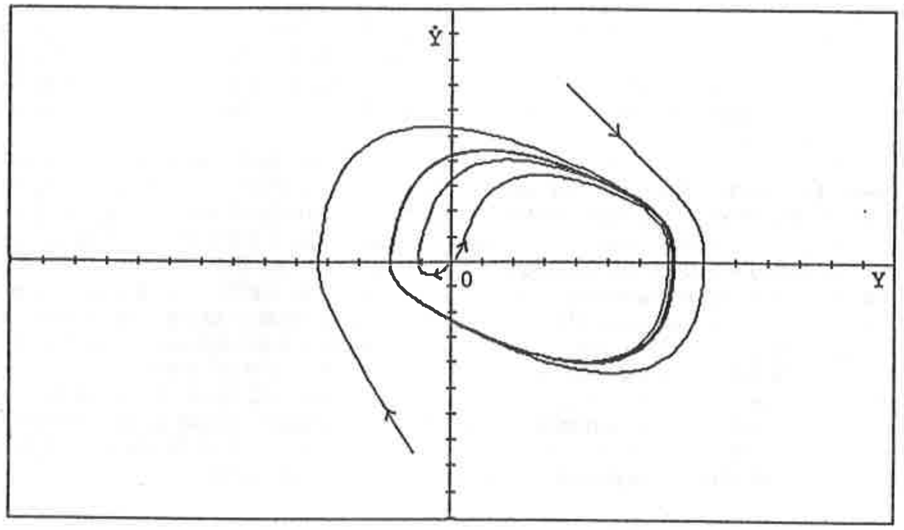

FIGURE 6

As we see, the limit cycle behaviour of the "unforced" case is transformed into more complex dynamics. Thus, the impression one has is that it is only by abandoning all very special assumptions which make it possible to use the concepts of planar dynamical systems, that we can improve our understanding of the dynamics of the capitalist economy.

4. Conclusions

This paper has been an attempt to understand and clarify the use of special assumptions in nonlinear models of the business cycle.

We have seen that there are models in which, on the one hand, assumptions are criticized as *ad hoc* because of their lack of micro-foundations in the traditional sense, and, on the other, very special assumptions are introduced in order to "exploit" the advantages offered by a given and well

[33] The two initial conditions used in Figure 8 are (5,5,0) and (5.1,5,0).

known mathematical structure. In these cases, even if the "mathematical elegance" is guaranteed by the use of theorems like the Poincaré-Bendixson Theorem (as in Benassy's model), the robustness of the results is inevitably weakened. In particular, all assumptions which serve the purpose of keeping "low" the order of the resulting dynamical system seem to reduce considerably the relevance of a model.

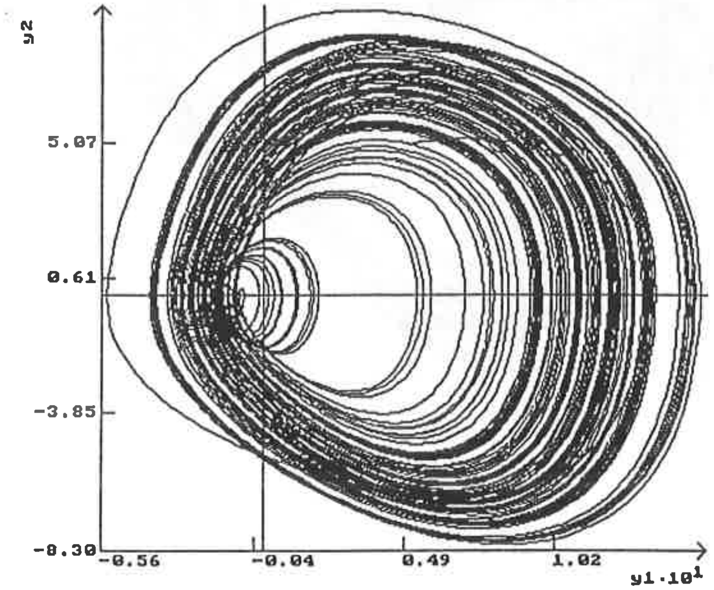

FIGURE 7

The model we have analysed in **Section 3.3** is very simple and, *as such*, it does not represent a better alternative to models formalized in terms of one-dimensional or two-dimensional dynamical systems. However, it suggests to us that "forced" models, as stressed by Goodwin himself (1990, pp.121-123), may have a considerable relevance to economics. First of all, "forced" models allow us to take account of the fact that the economy consists of a very large number of separate and distinct parts and that, as a consequence, the various parts are subject to continual exogenous forces. Second, and more importantly, "forced" models offer us a way of studying the interaction between cycle and trend. Indeed, although the superposition of a trend to a linear model of the business cycle is certainly a very unsatisfactory way of dealing with the problem, the situation is very different when we consider a nonlinear model. In this case, the superposition of an exogenous trend may have unforeseeable consequences. For example, in the simplified example we have considered, the limit-cycle behaviour of the "unforced" system is transformed into chaotic dynamics.

However, as stressed recently by HAHN (1991), it is possible that the analysis of dynamical systems of this kind will make analytical methods impossible and that, as a consequence, we shall need simulation.

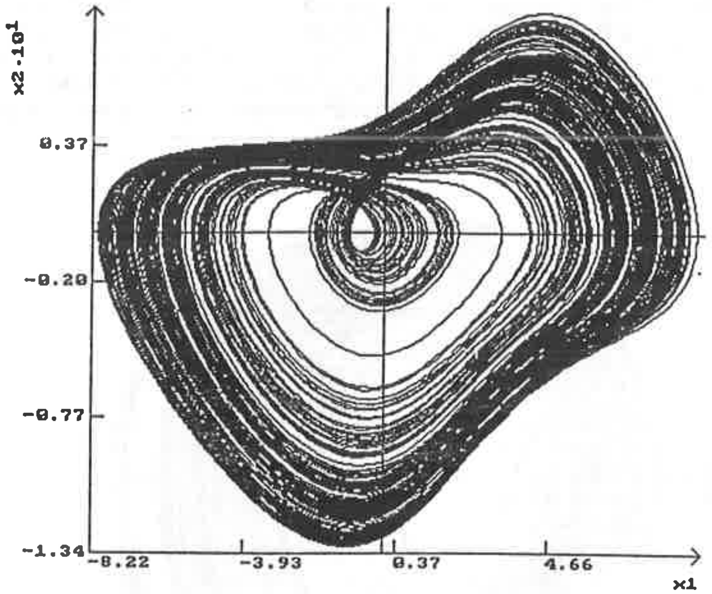

FIGURE 8

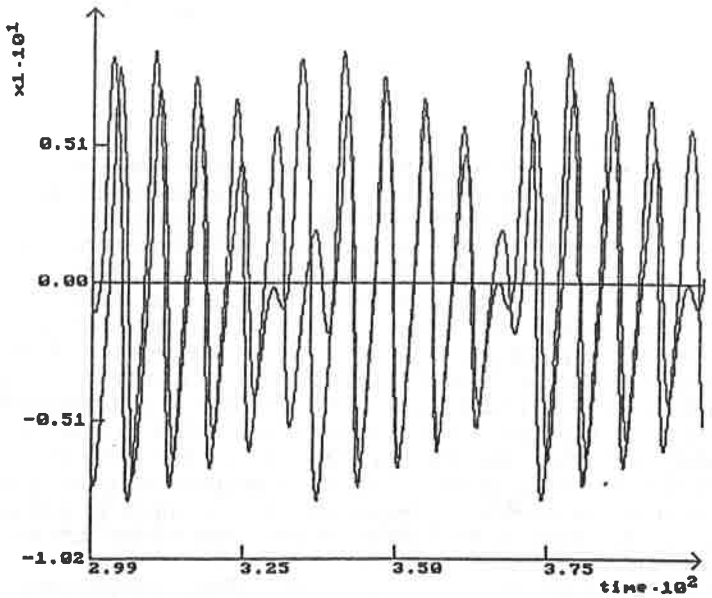

FIGURE 9

Mathematical Appendix

MA.i. GOODWIN (1951, pp.11-12) obtains equation (15) as an approximation to his original final equation which is an equation of the mixed differential-difference type. To obtain equation (15) directly, we can reformulate the model in terms of adjustments to desired levels and write:

(i.1) $$\dot{Y} = \frac{1}{\varepsilon}\{(C + I) - Y\},$$

(i.2) $$\dot{I} = \frac{1}{\theta}\{\phi(\dot{Y}) - [I - A(t+\theta)]\},$$

where, apart from the notation already introduced in the text, C is the level of consumption; I, the level of investment; $\dot{Y} = dY/dt$, and $\dot{I} = dI/dt$. The system (i.1)-(i.2) can be rewritten as:

(i.3) $$(\varepsilon D + 1)Y = C + I,$$

(i.4) $$(\theta D + 1)I = \phi(\dot{Y}) + A(t+\theta),$$

where $D = d/dt$. From (i.4):

$$I = \frac{1}{(\theta D + 1)}\{\phi(\dot{Y}) + A(t+\theta)\},$$

and, inserting in (i.3):

$$(\varepsilon D + 1)Y = C + \frac{1}{(\theta D + 1)}\{\phi(\dot{Y}) + A(t+\theta)\},$$

from which, ignoring the constant part of consumption, we obtain (15).

MA.ii. Linearizing the homogeneous part of (i.3)-(i.4) around the equilibrium point $(y,I) = (0,0)$, we obtain:

(ii.1) $$\begin{bmatrix} \dot{y} \\ \dot{I} \end{bmatrix} = \begin{bmatrix} -\frac{s}{\varepsilon} & \frac{1}{\varepsilon} \\ -\frac{vs}{\varepsilon} & \frac{v}{\varepsilon} - 1 \end{bmatrix} \begin{bmatrix} y \\ I \end{bmatrix} = \mathbf{A}\begin{bmatrix} y \\ I \end{bmatrix},$$

where $y = Y - \overline{Y}$ and where the matrix **A** is such that:

$$\mathrm{Det}(A) = \frac{s}{\varepsilon} > 0,$$

$$\mathrm{Tr}(A) = -\frac{s}{\varepsilon} + \frac{v}{\varepsilon} - 1 \lesseqgtr 0 \Leftrightarrow v \lesseqgtr s + \varepsilon.$$

Thus, if $v > s + \varepsilon$, as Goodwin assumes to be the case, the equilibrium is unstable.

MA.iii. The total energy in equation (16) is defined by:

$$E = T + U = \left(\frac{1}{2}\right)\dot{x}^2 + G(x),$$

where $T = (1/2)\dot{x}^2$ is the kinetic energy and $U = G(x) = \int_0^x g(\tau)d\tau$, the potential energy. In our case, as a particular motion progresses, we have:

$$\begin{aligned}\dot{E} &= \frac{d}{dt}\left\{\left(\frac{1}{2}\right)\dot{x}^2 + G(x)\right\} = \\ &= \dot{x}[\ddot{x} + g(x)] = \\ &= \dot{x}^2[\phi'(x) - (\varepsilon + s)].\end{aligned}$$

Integrating this expression with respect to time from $t = \tau_0$ to $t = \tau$, we obtain:

$$E(\tau) - E(\tau_0) = \int_{\tau_0}^{\tau} \{\phi'(x) - (\varepsilon + s)\}\dot{x}^2 dt,$$

where x(t) is the solution of (16) on the chosen path. Given the shape chosen by Goodwin for the nonlinear investment function, we have:

(a) $\{\phi'(x) - (\varepsilon + s)\} > 0 \Leftrightarrow E(\tau) > E(\tau_0)$ for $|x|$ "small"; i.e., the energy is increasing and $f(x)\dot{x}$ has the effect of an internal source injecting energy into the system;
(b) $\{\phi'(x) - (\varepsilon + s)\} < 0 \Leftrightarrow E(\tau) < E(\tau_0)$ for $|x|$ "large"; i.e., the energy decreases and $f(x)\dot{x}$ has a damping effect, contributing to a general decrease in amplitude.

When, over a trajectory, these counteracting forces cancel out, we have a limit cycle. Thus we can write the following (sufficient) condition, for the existence of a limit cycle:

$$\oint \{\phi'(x) - (\varepsilon + s)\}\dot{x}^2 dt = 0,$$

where the curvilinear integral is taken along the trajectory. This means that it is always the choice of the function $\{\phi'(x) - (\varepsilon + s)\} = -\varepsilon f(x)$ — and in particular of the nonlinearity of this function — such that this condition holds that guarantees the limit cycle solution.

MA.iv. The partial derivatives of the function S(w,p) are easily obtained by considering the maximization problem solved by the firm. Writing Π for profits, this problem is the following:

$$\text{Max } \Pi = pF(L) - wL.$$

The first order condition for a maximum is:

$$\frac{w}{p} = F'(L).$$

Given that $F'' < 0$ for all L, we can use the inverse function rule and write:

$$L = F'^{-1}\left(\frac{w}{p}\right) = H(w,p).$$

From the first order condition, we find:

$$\begin{aligned}\frac{\partial(w/p)}{\partial w} &= \frac{1}{p} = \left(\frac{dF'(L)}{dL}\right)\left(\frac{\partial L}{\partial w}\right) = F''(L)H_w,\\ \frac{\partial(w/p)}{\partial p} &= -\frac{w}{p^2} = \left(\frac{dF'(L)}{dL}\right)\left(\frac{\partial L}{\partial p}\right) = F''(L)H_p,\end{aligned}$$

from which:

$$\begin{aligned}H_w &= \frac{1}{pF''(L)} < 0,\\ H_p &= -\frac{w}{p^2F''(L)} > 0.\end{aligned}$$

Thus, the short-run equilibrium production function is:

$$Y = F\left[F'^{-1}\left(\frac{w}{p}\right)\right] = F(H(w,p)) = S(w,p),$$

such that:

$$\frac{\partial Y}{\partial w} = \left(\frac{dF}{dL}\right)\left(\frac{\partial H}{\partial w}\right) = S_w < 0,$$

and:

$$\frac{\partial Y}{\partial p} = \left(\frac{dF}{dL}\right)\left(\frac{\partial H}{\partial p}\right) = S_p > 0.$$

MA.v. In (20)-(22), we have:

$$\begin{aligned}&Y - C(Y,p) - I(X,r) = F_1(Y,r,p;X,w) = 0,\\ &Y - S(w,p) = F_2(Y,r,p;X,w) = 0,\\ &L(Y,r,p) - M = F_3(Y,r,p;X,w) = 0.\end{aligned}$$

Thus, taking the total differential of both sides for each of the three equations, we obtain relations of the type:

$$\left[\frac{\partial F_i}{\partial Y}\right]dY + \left[\frac{\partial F_i}{\partial r}\right]dr + \left[\frac{\partial F_i}{\partial p}\right]dp = - \left[\left[\frac{\partial F_i}{\partial X}\right]dX + \left[\frac{\partial F_i}{\partial w}\right]dw\right],\ i = 1,2,3.$$

Calculating all partial derivatives, we obtain the following system:

$$\begin{aligned} &(1 - C_y)dY - I_r dr - C_p dp = I_x dX, \\ &dY - S_p dp = S_w dw, \\ &L_y dY + L_r dr + L_p dp = 0, \end{aligned}$$

the Jacobian of which is:

$$|J| = \begin{vmatrix} 1-C_y & -I_r & -C_p \\ 1 & 0 & -S_p \\ L_y & L_r & L_p \end{vmatrix} = \\ = I_r L_p - C_p L_r + S_p(1 - C_y)L_r + S_p I_r L_y < 0.$$

Supposing, for example, that $dx \neq 0$ and $dw = 0$, the system becomes:

$$\begin{bmatrix} 1-C_y & -I_r & -C_p \\ 1 & 0 & -S_p \\ L_y & L_r & L_p \end{bmatrix} \begin{bmatrix} \frac{\partial Y}{\partial X} \\ \frac{\partial r}{\partial X} \\ \frac{\partial p}{\partial X} \end{bmatrix} = \begin{bmatrix} I_x \\ 0 \\ 0 \end{bmatrix},$$

from which, using the Cramer's rule, we obtain:

$$\frac{\partial Y}{\partial X} = \frac{\partial z(X,w)}{\partial X} = z_x = \frac{\begin{vmatrix} I_x & -I_r & -C_p \\ 0 & 0 & -S_p \\ 0 & L_r & L_p \end{vmatrix}}{|J|} = \\ = \frac{I_x L_r S_p}{|J|} > 0.$$

Analogously, assuming that $dw \neq 0$ and $dX = 0$, we find:

$$\frac{\partial Y}{\partial w} = z_w = - \frac{S_w(-I_r L_p + C_p L_r)}{|J|} < 0.$$

MA.vi. The linearized system is:

$$\begin{aligned} \dot{w} &= G'(\overline{Y})z_w(w - w^*) + G'(\overline{Y})z_x(X - X^*), \\ \dot{X} &= \mu z_w(w - w^*) + \mu(z_x - 1)(X - X^*), \end{aligned}$$

the characteristic roots of which are:

$$\lambda_{1,2} = \frac{1}{2}\{Tr(A) \pm [Tr^2(A) - 4Det(A)]^{1/2}\},$$

where:

$$A = \begin{bmatrix} G'(\overline{Y})z_w & G'(\overline{Y})z_x \\ \mu z_w & \mu(z_x - 1) \end{bmatrix},$$

so that:

$$\begin{aligned} Tr(A) &= G'(\overline{Y})z_w + \mu(z_x - 1) > 0, \\ Det(A) &= G'(\overline{Y})z_w \mu(z_x - 1) - G'(\overline{Y})z_x \mu z_w = -G'(\overline{Y})z_w > 0. \end{aligned}$$

Thus, given that $Det(A)$ is always positive, the stability of the equilibrium point is determined by the sign of $Tr(A)$: if $Tr(A) < 0$ (> 0), the equilibrium is locally stable (unstable).

MA.vii. Taking the time derivative of both sides of (30), we obtain:

(vii.1) $$\begin{aligned} \dot{Y} &= z_x\dot{X} + \bar{z}_w\dot{w} = \\ &= \mu z_x(Y - X) + \sigma\bar{z}_w(Y - \overline{Y}) = \\ &= \mu z_x(y - x) + \sigma\bar{z}_w y, \end{aligned}$$

where $y = Y - \overline{Y}$ and $x = X - X^*$. Thus, inserting in (vii.1) the expression for y given by (33), we obtain equation (34).

MA.viii. Following ALLEN (1967, pp.378-380), and given that the investment function must satisfy four conditions, we have chosen a form of the type:

$$\phi(\dot{y}) = \left\{\frac{a}{b\exp(-c\dot{y}) + 1} - 1\right\}d,$$

where the four coefficients a, b, c, and d are determined as solutions of the following system:

(viii.1) $\phi(0) = 0,$
(viii.2) $\phi(-\infty) = -I^{**},$
(viii.3) $\phi(+\infty) = I^*$

(viii.4) $\phi'(0) = v.$

Following this procedure, we have obtained the following specification for the investment function:

$$\phi(\dot{y}) = \left\{ \frac{I^* + I^{**}}{I^* \exp\left[- \left[\frac{I^* + I^{**}}{I^* I^{**}} \right] v\dot{y} \right] + I^{**}} - 1 \right\} I^{**}$$

which we have then employed for the computer simulation.

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