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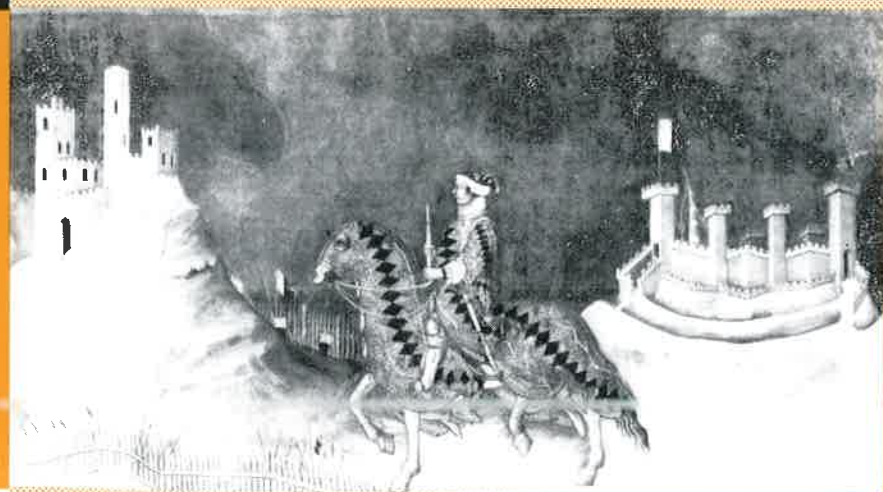
QUADERNI DEL DIPARTIMENTO DI ECONOMIA POLITICA

Graciela Chichilnisky

A UNIFIED PERSPECTIVE ON RESOURCE ALLOCATION:

LIMITED ARBITRAGE IS NECESSARY AND SUFFICIENT FOR THE EXISTENCE OF A COMPETITIVE EQUILIBRIUM, THE CORE AND SOCIAL CHOICE



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Siena, maggio 1995

Graciela Chichilnisky, Columbia University and University of Siena.

A Unified Perspective on Resource Allocation: Limited Arbitrage is Necessary and Sufficient for the Existence of a Competitive Equilibrium, the Core and Social Choice

Graciela Chichilnisky*

Dedicated to Kenneth J. Arrow†

Abstract

Different forms of resource allocation—by markets, cooperative games, and by social choice—are unified by one condition, *limited arbitrage*, which is defined on the endowments and the preferences of the traders of an Arrow Debreu economy. Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium in economies with or without short sales, and with finitely or infinitely many markets. The same condition is also necessary and sufficient for the existence of the core, for resolving Arrow's paradox on choices of large utility values, and for the existence of social choice rules which are continuous, anonymous and respect unanimity, thus providing a unified perspective on standard procedures for resource allocation. When limited arbitrage does not hold, *social diversity* of various degrees is defined by the properties of a topological invariant of the economy, the cohomology rings CH of a family of cones which are naturally associated with it. CH has additional information about the resource allocation properties of subsets of traders in the economy and of the subeconomies which they span.

*Director, Program on Information and Resources, and Professor of Economics, Columbia University. 1994-5 Salinbemi Professor, University of Siena, Italy. Address: 405 Low Library, Columbia University, 116th. and Broadway, New York N.Y. 10025, email: gc9@columbia.edu, fax: 212 678 0405. The main results of this paper appeared in "Markets, Arbitrage and Social Choice", paper presented at the October 23, 1991, Conference "Columbia Celebrates Arrow's Contributions", Columbia University, New York. Invaluable editorial comments and intellectual input from Kenneth Arrow are gratefully acknowledged.

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Social diversity is central to resource allocation. People trade because they are different. Gains from trade and the scope for mutually advantageous reallocation depend naturally on the diversity of the traders' preferences and endowments. The market owes its existence to the diversity of those who make up the economy.

An excess of diversity could however stretch the ability of economic institutions to operate efficiently. This is a concern in regions experiencing extensive and rapid migration, such as Canada, the USA and the ex-USSR. Are there natural limits on the degree of social diversity with which our institutions can cope? This paper will argue that there are. I will argue that not only is a certain amount of diversity essential for the functioning of markets, but, at the other extreme, that too much diversity of a society's preferences and endowments may hinder its ability to allocate resources efficiently.

Somewhat unexpectedly, the very same level of diversity which hinders the functioning of markets also hinders the functioning of democracy, and other forms of resource allocation which are obtained through cooperative games, such as the core.[1] The main tenet of this paper is that there is a crucial level of social diversity which determines whether all these forms of resource allocation will function properly.

Social diversity has been an elusive concept until recently. I give here a precise definition, and examine its impact on the most frequently used forms of resource allocation. From this analysis a new unified perspective emerges: a well-defined connection between resource allocation by markets, games and social choices, which have been considered distinct until now. I define a limitation on social diversity which links all these forms of resource allocation. This limitation is a condition on the endowments and the preferences of the traders of an Arrow Debreu economy. In its simpler form I call this **limited arbitrage**[2]. This concept is related with that of "no-arbitrage"[3] used in finance, but it is nonetheless different from it. I show that limited arbitrage is necessary and sufficient for the existence of an equilibrium in Arrow Debreu economies, and this equivalence extends to economies with or without short sales[4] and with finitely or infinitely many markets,[5] Theorems 2 and 5. Limited arbitrage is also necessary and sufficient for the existence of the core,[6] Theorem 7, and its simplest failure is sufficient for the existence of the **supercore**, a concept which is introduced to gauge social cohesion, Theorem 8. In addition, limited arbitrage is necessary and sufficient for solving Arrow's paradox [1] on choices of

[1] The core is an allocation which no subset of players can improve upon within their own endowments.

[2] *Limited arbitrage* was introduced and named in [11], [13], [16], [14], [15].

[3] No-arbitrage is discussed in Section 2.

[4] These results were first established in [13], [15].

[5] This result was first established in [23].

[6] This result was first established in [17], [14], and it was presented and the paper circulated at the Econometric Society Meetings in Boston, January 3-5, 1994.

large utility value, i.e. for the existence of well-defined social choice rules,[7] Theorem 9. It is also necessary and sufficient for the existence social choice rules which are continuous, anonymous and respect unanimity [7], [9], Theorem 13. The success of all four forms of resource allocation, by financial and real competitive markets, by cooperative games and by social choice, hinges on precisely the same limitation on the social diversity of the economy.

Shifting the angle of inquiry slightly sheds a different light on the subject. The results predict that a society which allocates resources efficiently by markets, collective choices or cooperative games, must exhibit no more than a certain degree of social diversity. This is an implicit prediction about the characteristics of those societies which implement successfully these forms of resource allocation. Increases in social diversity beyond this threshold may call for forms of resource allocation which are different from all those which are used today.

The results of this paper are intuitively clear. New forms of resource allocation appear to be needed in order to organize effectively a diverse society. But the issue is largely avoided by thinkers and policy makers alike because the institutions required for this do not yet exist, creating an uncomfortable vacuum. This paper attempts to formalize the problem within a rigorous framework and so provide a solid basis for theory and policy.

As defined here *social diversity* comes in many "shades", of which limited arbitrage is only one. The whole concept of social diversity is subtle and complex, and it is encapsuled in algebraic objects, the cohomology rings[8] denoted CH, which are naturally associated with a family of cones defined from the endowments and preferences of the traders in the economy. Limited arbitrage simply measures whether the cones intersect or not, while the rings CH measure this and more: CH reveal the intricate topology of how these cones are situated with respect to each other. The cohomology rings CH give a **topological invariant** of the economy, in the sense that CH is invariant under continuous deformations of the measurement of commodities. It is also structurally stable, remaining invariant under small errors of measurement. This concept of diversity is therefore ideal for the social sciences where measurements are imprecise and difficult to obtain. The properties of CH predict specific properties of the economy such as which subeconomies have a competitive equilibrium and which do not, which have a social choice rule and which do not, which have a core, and which have a supercore, Theorem 10. The latter concept, the supercore, measures the extent of social cohesion, namely the extent to which a society has reasons to stay together or break apart. I prove that, somewhat paradoxically, the mildest form of social diversity predicts whether the supercore exists, even in economies where the preferences may not be convex.

[7] A result first established in [14], [16].

[8] CH is defined in Section 10.

The results presented here have two distinguishing features. One is that they provide a minimal condition which ensures that an Arrow Debreu equilibrium,[9] the core and social choice rules exist, namely a condition which is simultaneously necessary and sufficient for the existence of solutions to each of these three forms of resource allocation. The second is they extend and unify the Arrow Debreu formulation of markets to encompass economies with or without short sales[10] and with finitely or infinitely many markets.

While sufficient conditions for the existence of a competitive equilibrium have been known for about forty years, starting from the works of Von Neumann, Nash, Arrow and Debreu, the study of necessary and sufficient for resource allocation introduced in [11], [13], [12], [16], [13] had been neglected previously. A necessary and sufficient condition is a useful tool. As an illustration consider the necessary and sufficient ("first order") conditions for partial equilibrium analysis of convex problems. These are among the most widely used tools in economics: they identify and help compute solutions in the theories of the consumer and of the firm, and in optimal growth theory. Equally useful could be a necessary and sufficient condition for the existence of market clearing allocations. Furthermore, in order to prove the equivalence between different problems of resource allocation one needs "tight" characterizations: a necessary and sufficient condition for equilibrium, the core and social choice is needed to establish the equivalence of these different forms of resource allocation.

It seems useful to elaborate on a geometric interpretation of limited arbitrage because it clarifies its fundamental links with the problem of resource allocation. It was recently established that the non-empty intersection of the cones which defines limited arbitrage is equivalent to a topological condition on the spaces of preferences [8] [12]. The topological condition is **contractibility**, a form of similarity of preferences[11] [7], [31]. Contractibility is necessary and sufficient for the existence of social choice rules, see [24]. It turns out that the equivalence between non-empty intersection and contractibility is the link between markets and social choices. The contractibility of the space of preferences is necessary and sufficient for the existence of social choice rules, while non-empty intersection (limited arbitrage) is necessary and sufficient for the existence of a market equilibrium. One main result brings all this together: *a family of convex sets has a non-empty intersection if and only if every subfamily has a contractible union,* see [8] and [12].[12]

[9]It is possible to use lesser concepts of equilibrium, such as quasiequilibrium and compensated equilibrium, or equilibria where there may be excess supply in the economy. These exist under quite general conditions, but fail to provide Pareto efficient allocations and are therefore less attractive from the point of view of resource allocation, so they are not used here.

[10]I.e. whether trades are bounded below or not.

[11]Contractibility ensures that the preferences of all traders can be continuously deformed into one and is therefore a form of similarity of preferences.

[12]This theorem is also valid for certain non-convex sets, and is shown in [12] to imply Brower's fixed point theorem, the KKM theorem, Caratheodory's theorem and Leray's theorem, but it is not implied by

Using similar topological results,[13] Theorem 6 establishes a link between the number of traders and the number of commodities: the economy has limited arbitrage if and only if every subeconomy of $N+1$ traders does, where N is the number of commodities traded in the market.

As already mentioned, I consider economies with or without short sales: net trades are either bounded below, as in a standard Arrow Debreu economy, or they are not bounded at all. This is a considerable extension from the Arrow Debreu theory, as it includes financial markets in which short trades typically occur.[14] In addition, the economy could have finitely or infinitely many markets: the results obtain in either case[15], Theorem 3.

It is somewhat surprising that the same condition of limited arbitrage is necessary and sufficient for the existence of a market equilibrium with or without short sales (Theorem 2).[16] The non-existence of a competitive equilibrium is seemingly a different phenomenon in economies with short sales than it is in economies without short sales. With short sales, the problem of non-existence arises when traders with very different preferences[17] desire to take unboundedly large positions against each other, positions which cannot be accommodated within a bounded economy. Instead, without short sales, the problem arises when some traders have zero income. Yet I show that in both cases the source of the problem is indeed the same: the diversity of the traders leads to ill-defined demand behavior at the potential market clearing prices, and prevents the existence of a competitive equilibrium. Limited arbitrage ensures that none of these problems arises: with or without short sales it bounds the diversity of traders precisely as needed for a competitive equilibrium to exist. Theorem 3 links the number of markets with the number of traders in a somewhat unexpected manner.

It is also somewhat surprising that the same condition of limited arbitrage ensures the existence of an equilibrium in economies with either finitely or infinitely many markets. The problem of existence appears to be different in these two cases, and indeed they are treated quite differently in the literature. A typical problem in economies with infinitely

them.

[13]Connected with Helly's theorem.

[14]Their formalization of markets assume that the consumption sets of the individuals are bounded below, an assumption motivated by the inability of humans to provide more than a fixed number of hours of labor per day.

[15]Chichilnisky and Heal [23] proved that limited aritrage is necessary and sufficient for the existence of a competitive equilibrium in economies with infinitely many markets.

[16]I work within a standard framework where preferences are convex and uniformly non-satiated, Section 1. These include all standard convex preferences including linear of partly linear, constant elasticity of substitution (CES), Cobb Douglas, Leontief preferences, strictly convex preferences with indifference surfaces which intersect the coordinate lines or not and which contain half lines or not, see also [13], [11], [14].

[17]Or expectations.

many markets is that positive orthants have empty interior, so that the Hahn-Banach theorem cannot be used to find equilibrium prices for efficient allocations.[18] A solution to this problem was found in 1980: [21] extended the Hahn-Banach theorem by introducing a **cone condition** and proving that it is necessary and sufficient for supporting convex sets whether or not they have an interior. Thereafter the cone condition has been used extensively to prove existence in economies with infinitely many markets and is by now a standard condition on preferences defined on infinitely many markets, known also under the name of "properness" of preferences in subsequent work.[19] The fundamental new fact presented here is that limited arbitrage implies the cone condition on efficient and affordable allocations, Theorem 3.[20] Therefore by itself limited arbitrage provides a unified treatment of economies with finitely and infinitely many markets, being necessary and sufficient for the existence of equilibrium and the core in all cases.

In a nutshell: in all cases limited arbitrage bounds gains from trade, Proposition 4, and is equivalent to the compactness of the set of Pareto efficient utility allocations, Theorem 1.[21] Gains from trade and the Pareto frontier are fundamental concepts involved in most forms of resource allocation: in markets, in games and in social choice. Limited arbitrage controls them all.

1 Definitions and Examples

An Arrow Debreu market $\mathsf{E} = \{X, \Omega_h, u_h, h = 1, ..., H\}$ has $H \geq 2$ traders, indexed $h = 1, ..., H$, $N \geq 2$ commodities and consumption or trading space[22] $X = R^N_+$ or $X = R^N$; in Section 5 X is a Hilbert space of infinite dimensions. The vector $\Omega_h \in R^N_+$ denotes trader h's property rights or initial endowment and $\Omega = (\sum_{h=1}^H \Omega_h) >> 0$ is the total endowment of the economy.[23] Traders may have zero endowments of some goods. Each trader h has a continuous and convex preference represented by $u_h : X \to R$. All the assumptions and the results in this paper are ordinal;[24] therefore without loss of generality one normalizes utilities so that for all h, $u_h(0) = 0$ and $\sup_{\{x:x\in X\}} u_h(x) = \infty$. When $X = R^N_+$ the preference is increasing, i.e. $x > y \Rightarrow u_h(x) > u_h(y)$; its indifference surfaces of positive utility can either be contained in the interior of X, R^N_{++}, such as

[18] As is done in finite dimensions. The Hahn-Banach theorem requires that one of the convex sets being separated has a non-empty interior.

[19] See e.g. [19].

[20] See also Chichilnisky and Heal [23].

[21] Called the *Pareto frontier*. The connection between limited arbitrage and the compactness of the Pareto frontier is of central importance for resource allocation. This connection was first pointed out and established in [13], [23], [15], [11], [14], and [17].

[22] $R^N_+ = \{(x_1, ..., x_N) \in R^N : \forall i, x_i \geq 0\}$.

[23] If $x, y \in R^N$, $x \geq y \Leftrightarrow \forall i\ x_i \geq y_i$, $x > y \Leftrightarrow x \geq y$ and for some $i, x_i > y_i$, and $x >> y \Leftrightarrow \forall i, x_i > y_i$.

[24] Namely independent of the utility representations.

Cobb-Douglas and CES preferences, or they can intersect the boundary of the positive orthant ∂R^N_+ transversally.[25]

Definition 1 *A preference is* **uniformly non-satiated** *when it is represented by a utility with a minimum rate of increase bounded away from zero:* $\forall \varepsilon > 0, \exists \delta > 0$ *such that* $x \in X \Rightarrow \exists y \in X$ *such that* $u_h(x+y) - u_h(x) \geq \varepsilon\delta$ *and* $\|y\| \leq \varepsilon$.

Uniformly non-satiated preferences are rather common. For example, any preference which is representable by a concave utility function is uniformly non-satiated.[26] If the preference has a smooth utility representation, all that is required is that the norm of its gradient be uniformly bounded below, i.e. that for all $x \in X$, $\exists \varepsilon > 0$ such that $\|Du_h(x)\| \geq \varepsilon$.

Proposition 1 *A utility function u_h is uniformly non satiated, if and only if its indifference surfaces are within uniform distance from each other, i.e.* $\forall r, s \in R, \exists N(r,s) \in R$ *such that* $x \in u_h^{-1}(r) \Rightarrow \exists y \in u_h^{-1}(s)$ *with* $\|x - y\| \leq N(r,s)$.

Proof. This is immediate from the definition. ■

Assumption 1. The preferences in the economy E are uniformly non-satiated.

This includes Cobb-Douglas and CES preferences, preferences which are strictly convex or not, preferences whose indifference surfaces of positive utility intersect the boundary or not, and preferences whose indifference surfaces are bounded below or not.

The space of **feasible allocations** is $\Upsilon = \{(x_1, ..., x_H) \in X^H : \sum_{h=1}^H x_h = \Omega\}$. The set of **supports to individually rational affordable efficient resource allocations** is:

$$S(\mathsf{E}) = \{v \in R^N : \exists (x_1 ... x_H) \in \Upsilon \text{ with } u_h(x_h) \geq u_h(\Omega_h) \forall h = 1, ... H, \\ \forall h, \langle v, x_h - \Omega_h \rangle = 0, \text{ and } \forall z_h \in X,\ u_h(z_h) \geq u_h(x_h) \text{ implies } \langle v, z_h - x_h \rangle \geq 0\}. \tag{1}$$

The set of **prices orthogonal to the endowments** is[27]

$$\mathsf{N} = \{v \in R^N_+ - \{0\} : \exists h \text{ with } \langle v, \Omega_h \rangle = 0\}. \tag{2}$$

[25] I.e. if $x \in \partial R^N_+$, and $u(x) > 0$, then $Du(x)$ is not orthogonal to ∂R^N_+. This includes strictly convex preferences, Cobb Douglas and CES preferences, Leontieff preferences $u(x,y) = \min(ax, by)$, preferences which are indifferent to one or more commodities, such as $u(x,y,z) = \sqrt{x+y}$, preferences with indifference surfaces which contain rays of ∂R^N_+ such as $u(x,y,z) = x$, and preferences defined on a neighborhood of the positive orthant or the whole space, and which are increasing along the boundaries, e.g. $u(x,y,z) = x + y + z$.

[26] This is not difficult to prove, but it is excluded for space reasons by request of the editor.

[27] N is empty when $\forall h$, $\Omega_h >> 0$.

The **utility possibility set** of the economy E is the set of feasible and individually rational utility allocations:

$$U(\mathsf{E}) = \{(V_1, ..., V_H) : \forall h, V_h = u_h(x_h) \geq u_h(\Omega_h) \geq 0, \text{ for some } (x_1, ..., x_H) \in \Upsilon \}$$

The **Pareto frontier** of the economy E is the set of feasible, individually rational and efficient utility allocations:

$$P(\mathsf{E}) = \{(V_1, ..., V_H) \in U(\mathsf{E}) :\sim \exists (W_1, ..., W_H) \in U(\mathsf{E}) : \\ \forall h = 1, ..., H, \; W_h \geq V_h \text{ and } W_h > V_h \text{ for some } h\} \subset R^H_+. \tag{3}$$

A **competitive equilibrium** of E consists of a price vector $p^* \in R^N_+$ and a feasible allocation $(x^*_1...x^*_H) \in \Upsilon$ such that x^*_h optimizes u_h over the budget set $B_h(p^*) = \{x \in X : \langle x, p^* \rangle = \langle \Omega_h, p^* \rangle\}$.

1.1 Global and Market Cones

Two cases, $X = R^N$ and $X = R^N_+$, are considered separately.

- **Consider first** $X = R^N$.

Definition 2 *For trader h define the* **global cone** *of directions along which utility increases without bound:*

$$A_h(\Omega_h) = \{x \in X : \forall y \in X, \exists \lambda > 0 : u_h(\Omega_h + \lambda x) > u_h(y)\}$$

This global cone contains global information on the economy and is new in the literature.[28] In ordinal terms, the rays of this cone intersect all indifference surfaces corresponding to bundles preferred by u_h to Ω_h.

For trader h define the **cone** $G_h(\Omega_h)$ as the set of directions along which utility is strictly increasing:

$$G_h(\Omega_h) = \{x \in X : u_h(\Omega_h + \lambda x) > u_h(\Omega_h + \mu x) \text{ if } \lambda > \mu > 0\}. \tag{4}$$

This is a convex cone because u_h is quasiconcave.

[28]The global cone $A(\rho_h, \Omega_h)$ has points in common with Debreu's [25] "asymptotic cone" corresponding to the preferred set of u_h at the initial endowment Ω_h, in that along any of the rays of $A_h(\Omega_h)$ utility always increases. Under Assumption 1, its closure $\overline{A}(\Omega_h)$, equals the "recession" cone introduced by Rockafeller as shown in Proposition 2, but not generally. However along the rays in $A(\Omega_h)$ not only does utility increase forever, but it increases beyond the utility level of any other vector in the space. This condition need not be satisfied by Debreu's asymptotic cones [25], or by Rockafeller's "recession" cones. For example, for Leontieff type preferences the recession cone through the endowment is the closure of the upper contour, which includes the indifference curve itself. By contrast, the global cone $A_h(\Omega_h)$ is the interior of the upper contour set. Related concepts appeared in Chichilnisky [6], [10], [13] and [15]; otherwise there is no precedent in the literature for global cones.

Definition 3 *The* **market cone** *of trader h is*

$$D_h(\Omega_h) = \{z \in X : \forall y \in G_h(\Omega_h), \langle z, y \rangle > 0.\} \tag{5}$$

D_h is the cone of prices assigning strictly positive value to all directions of net trades leading to ever increasing utility. This is a convex cone.

The following proposition establishes the structure of the global cones, and is used in proving the connection between limited arbitrage, equilibrium and the core:

Proposition 2 *If the function $u_h : R^N \to R$ is uniformly non-satiated: (i) the interior of the global cone is*

$$G^o_h(\Omega_h) = A_h(\Omega_h) = \{z \in G_h(\Omega_h) : \lim_{\lambda \to \infty} u_h(\Omega_h + \lambda z) = \infty\} \neq \emptyset,$$

(ii) the boundary of the cone $G_h(\Omega_h)$, denoted $\partial G_h(\Omega_h)$, consists of two parts: (a) those directions in $G_h(\Omega_h)$ along which utility increases towards a bounded value that is never reached:

$$B_h(\Omega_h) = \{z \in G_h(\Omega_h) : \lim_{\lambda \to \infty} u_h(\Omega_h + \lambda z) < \infty\} \subset G_h(\Omega_h),$$

as well as (b) those sequences in the complement of $G_h(\Omega_h)$, $G^c_h(\Omega_h)$, along which the utility eventually achieves a constant value:

$$C_h(\Omega_h) = \{z \in G^c_h : \; \exists N : \lambda, \mu > N \Rightarrow u_h(\Omega_h + \lambda z) = u_h(\Omega_h + \mu z)\}$$

(iii) starting from Ω_h the utility approaches a unique value u^o along all directions defined by vectors in the set $B_h(\Omega_h) \cup C_h(\Omega_h)$ (iv) the interior of the global cone, its boundary and its closure,[29] and the cones G_h and D_h are uniform across all vectors in the space, i.e. $\forall \Omega, \Lambda \in X$:

$$G^o_h(\Omega) = G^o_h(\Lambda) = A_h$$

$$\partial G_h(\Omega) = \partial G_h(\Lambda) = B_h \cup C_h,$$

$$G_h(\Omega) = G_h(\Lambda) \; , \; D_h(\Omega) = D_h(\Lambda)$$

and, in particular,

$$\overline{G_h}(\Omega) = \overline{G_h}(\Lambda),$$

(v) For general non-satiated preferences, $G_h(\Omega_h)$ and $D_h(\Omega_h)$ may not be uniform.

[29]Under Assumption 1, the closure of the global cone $A_h(\Omega_h)$, $\overline{A_h}(\Omega_h)$, is the "recession" cone defined by the preferred set of u_h at Ω_h, but this is not true generally.

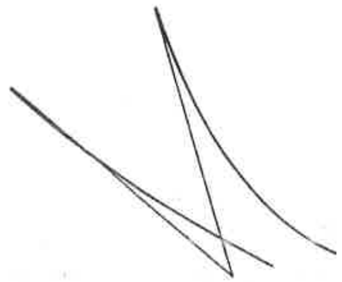

Figure 1: A preference satisfying Assumption 1 cannot have more than one utility level associated to directions in the set $B_h(\Omega_h) \cup C_h(\Omega_h)$.

Proof. Observe that the three sets $A_h(\Omega_h), B_h(\Omega_h)$ and $C_h(\Omega_h)$ are disjoint pairwise, i.e. $A_h(\Omega_h) \cap B_h(\Omega_h) = \emptyset, C_h(\Omega_h) \cap B_h(\Omega_h) = \emptyset$ and $A_h(\Omega_h) \cap C_h(\Omega_h) = \emptyset$, and that

$$A_h(\Omega_h) \cup B_h(\Omega_h) \cup C_h(\Omega_h) \cup G_h^c(\Omega_h) = R^N. \tag{6}$$

where $G_h^c(\Omega_h)$ is the complement of the set $G_h(\Omega_h)$.

The first step is to prove that there is a unique utility value associated to all vectors $z \in C_h(\Omega_h) \cap B_h(\Omega_h)$. Consider a sequence of allocations defined by a vector $z \in B_h(\Omega_h) \cup C_h(\Omega_h)$, denoted $\{\Omega_h + \lambda_j z\}_{j=1,2,...}$, where $\lambda_j \to \infty$. The sequence of utility levels corresponding to this sequence of allocations, $\{u_h(\Omega_h + \lambda_j z)\}$ approaches $u^o = \lim_{j\to\infty} u_h(\Omega_h + \lambda_j z) < \infty$. By the definition of the sets $B_h(\Omega_h)$ and $C_h(\Omega_h)$, either the utility levels converge to and reach u^o, or else $\lim_{j\to\infty} u_h(\Omega_h + \lambda_j z) = u^o$ but the sequence never reaches this limit. I will now show that under Assumption 1, there can be only *one* utility level, say $u^o = \lim_{\lambda\to\infty} u_h(\Omega_h + \lambda z)$, associated with all vectors in $B_h(\Omega_h) \cup C_h(\Omega_h)$. The proof is by contradiction. Assume not; then $\exists v : z \neq v$ satisfying: $u^1 = \lim_{\lambda_j\to\infty} u_h(\Omega_h + \lambda_j v) \neq u^o < \infty$. This implies that two indifference surfaces, of value u^o and u^1 respectively, drift away from each other without bound: $\forall M > 0, \exists y \in R^N$ such that $u_h(y) = u^1$, and if w satisfies $u_h(w) = u^o$, then $\|y - w\| > M$. By Proposition 1 this contradicts Assumption 1. See Figure 1. Therefore there can be only one limiting utility value, say u^o, associated at the limit to all vectors in the set $B_h(\Omega_h) \cup C_h(\Omega_h)$, proving (iii).

The next step is to show that if $z \in B_h(\Omega_h) \cup C_h(\Omega_h)$ and $s \in X$ satisfies $u_h(\Omega_h + s) > u_h(\Omega_h + z)$, then along the ray defined by a vector[30] s from the endowment Ω_h the utility u_h cannot achieve a maximum. The proof uses the same argument as above: if the utility u_h achieved a maximum along a direction defined by s, say at the vector $\Omega_h + \lambda s$, then the indifference surface of utility level $u_h(\Omega_h + \lambda s)$ and that of utility level $u_h(\Omega_h + z)$ would drift apart from each other without bound, i.e. $\forall M > 0$ there exists x such that

[30] The ray defined by s from a vector $\Omega \in X$ is $\{u = \Omega + \lambda s, \lambda > 0\}$.

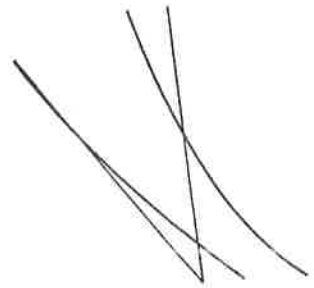

Figure 2: Under Assumption 1, if $u_h(\Omega_h + s) > u_h(\Omega_h + z)$ and $z \in B_h(\Omega_h) \cup C_h(\Omega_h)$ then $s \in A_h(\Omega_h)$.

$u_h(x) = u_h(\Omega_h + z)$ and if y satisfies $u_h(y) = u_h(\Omega_h + \lambda s)$, then $\|x - y\| > M$. This again contradicts Proposition 1, see Figure 2.

It follows that if $z \in B_h(\Omega_h) \cup C_h(\Omega_h)$, then for all $s \in R^N$,

$$u_h(\Omega_h + s) > u_h(\Omega_h + z) \Rightarrow s \in A_h(\Omega_h) \tag{7}$$

This implies, in particular, that in every neighborhood of $z \in B_h(\Omega_h) \cup C_h(\Omega_h)$ there is an element of the set $A_h(\Omega_h) \subset G_h(\Omega_h)$.

Now consider a vector $s \in X$ satisfying $\lim_{\lambda\to\infty} u_h(\Omega_h + \lambda s) < u_h(\Omega_h + z)$. By definition, the direction defined by s from the vector Ω_h is not in the set $A_h(\Omega_h)$. In addition, as proved above $s \notin B_h(\Omega_h) \cup C_h(\Omega_h)$ because $u_h(\Omega_h + v) \neq u_h(\Omega_h + z)$ and $z \in B_h(\Omega_h) \cup C_h(\Omega_h)$: we showed above that there cannot be two different utility levels associated in the limit to directions in the set $B_h(\Omega_h) \cup C_h(\Omega_h)$. Therefore, by (6), when $z \in B_h(\Omega_h) \cup C_h(\Omega_h)$ then for all $s \in R^N$,

$$u_h(\Omega_h + s) < u_h(\Omega_h + z) \Rightarrow s \in G_h^c(\Omega_h) - B_h(\Omega_h) \cup C_h(\Omega_h). \tag{8}$$

Since the utility u_h is non-satiated by Assumption 1, (7) and (8) together imply that every neighborhood of a vector $z \in B_h(\Omega_h) \cup C_h(\Omega_h) \subset \partial G_h(\Omega_h)$ contains a vector s in the set $A_h(\Omega_h)$ and another v in the set $G_h^c(\Omega_h)$. Since $A_h(\Omega_h) \subset G_h(\Omega_h)$, this implies that the set $B_h(\Omega_h) \cup C_h(\Omega_h)$ is in the boundary of the set $G_h(\Omega_h)$. This completes the proof of part (ii).

Part (i) now follows immediately from the definition of the sets $B_h(\Omega_h)$, $C_h(\Omega_h)$ and $G_h(\Omega_h)$: by definition, the interior of $G_h(\Omega_h)$, denoted $G_h^o(\Omega_h)$, equals $G_h(\Omega_h)$ minus its boundary, i.e. $G_h^o(\Omega_h) = G_h(\Omega_h) - [B_h(\Omega_h) \cup C_h(\Omega_h)] = A_h(\Omega_h)$. The proofs of parts $(i), (ii)$, and (iii) are now complete.

The next step is to show, as stated in (iv), that for every function u_h, $A_h(\Omega_h)$ is identical everywhere. For this it suffices to show that if two different half-lines l and m are parallel to each other, and $l \in A_h(\Omega_h)$, then $m \in A_h(\Lambda_h), \forall\ \Lambda_h \in m$. Consider first the case $N = 2$, which is the easiest to visualize. If the half-line m were not in $A_h(\Omega_h)$, then by definition u_h can only attain utility values on m which are bounded above. Let u^o be the infimum of such bounds, and let $u_h^o = \{y \in R^2 : u_h(y) > u^o\}$ be the preferred set of utility u^o. Observe that the set u_h^o must intersect l because $l \in A_h(\Omega_h)$. Now let H_m be the half-space of R^2 bounded by the half-line m and which does *not* contain l. Since by construction m supports the preferred set of utility u^o, then all vectors in H_m must have utility values lower than u^o. But *all* halflines starting at $\Omega_h \in l$ will intersect H_m, no matter how close they are to l : this implies that along any such half-line, the utility level eventually drops below u^o. This contradicts the openness of the cone $A_h(\Omega_h)$ shown in (i). Since the contradiction arises from assuming that m is not in $A_h(\Omega_h)$, m must be $A_h(\Omega_h)$ as we wished to prove. When $N > 2$ the argument is the same, by considering the function u_h restricted to the two-dimensional space defined by the two parallel lines l and m. This completes the proof that the global cone A_h is the same everywhere.

It was already shown, in the proof of (ii), that $B_h(\Omega_h) \cup C_h(\Omega_h)$ is the same everywhere. Since $A_h(\Omega_h)$ is the same everywhere, it follows that the interior of the set G_h, G_h^o, its closure $\overline{G_h}(\Omega_h)$, and its boundary $\partial G_h = B_h(\Omega_h) \cup C_h(\Omega_h)$ are the same everywhere. Therefore to complete the proof of (iv) it remains only to show that G_h and D_h are the same everywhere under Assumption 1.

Before doing so, observe that for a general convex preference represented by a utility u_h the set $G_h(\Omega_h)$ itself may in principle vary as the vector Ω_h varies, since the set $B_h(\Omega_h)$ itself may vary with Ω_h: at some Ω_h a direction $z \in \partial G_h$ may be in $B_h(\Omega_h)$ and at others $B_h(\Omega_h)$ may be empty and $z \in C_h(\Omega_h)$ instead. This occurs when along a ray defined by a vector z from one endowment the utility levels asymptote to a finite limit but do not reach their limiting value, while at other endowments, along the same direction z, they achieve this limit. In such cases, the set $\partial G_h - G_h$ is empty from some initial endowments, and it not empty at others, completing the proof of (v). I will show, however, that under Assumption 1 the set $\partial G_h - G_h$ and the cones G_h are always the same as stated in (iv). To establish this consider two parallel half-lines l and m in $\partial G_h - G_h(\Omega_h)$ and $\partial G_h - G_h(\Lambda_h)$ respectively. Assume that along l the utility levels of u_h achieve a limit u^o, which is the unique utility limit associated to vectors in $\partial G_h(\Omega_h)$ according to (iii), while the indifference surface of utility u^1 associated to $\partial G_h(\Lambda_h)$ asymptotes to, but never meets, the half-line m. In such a case $l \in \partial G_h - G_h(\Omega_h)$ but $\partial G_h - G_h(\Lambda_h)$ is an empty set. Given $\delta > 0$, let $y(x)$ denote the vector of norm $\epsilon > 0$ along which $u_h(x+y)$ increases maximally, i.e. along which the difference $\Delta u_h(x) = u_h(x+y) - u_h(x)$ is maximized. By Assumption 1, $\exists \delta \geq 0$ such that $\forall x \in X$ the set of vectors where $\Delta u_h(x) \geq \epsilon\delta > 0$ is not empty. Observe that when u_h is smooth, $y(x)$ is a multiple of the standard gradient of u_h; we therefore denote this vector the "gradient" at x, $Du_h(x)$. Observe that, by definition of l, along the indifference surface of utility level u^o the set of gradients $\{Du_h(x), x \in l\}$, form a closed set. If m and l are close enough, and $u^1 > u^o$ implies that there exists

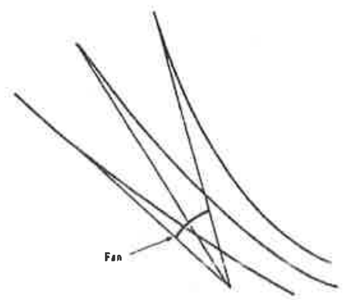

Figure 3: The preference illustrated in Figure 3 has a "fan" of different directions along which the utility values are bounded above but never reached. The existence of the "fan" implies that two indifference surfaces approaching different directions in the fan are not at a uniform distance from each other. Assumption 1 is therefore not satisfied.

a sequence of vectors $\{v^j\}_{j=1,2,...} \subset l$ along which the norm of the gradient of u_h, $\|Du_h(v^j)\|$, is a strictly increasing sequence, as it takes "increasingly shorter" distances for l (which equals the indifference of utility u^o) to reach utility level u^1. Furthermore since m and l are at a uniform distance from each other by Proposition 1, this sequence $\{\|Du_h(v^j)\|\}_{j=1,2,...}$ is bounded above. But an increasing sequence which is bounded above has a limit, and the gradients of the indifference surface of utility u^o form a closed set, so that the sequence of gradients $\{\|Du_h(v^j)\|\}_{j=1,2,...}$ eventually reaches its limit on l, and therefore the indifference surface of utility u^1 is from then on parallel to that of utility u^o, i.e. $m \in \partial G_h - G_h(\Lambda_h)$, as we wished to prove. A similar argument holds when $u^o > u^1$. Observe that this argument implies that the set of parallel half-lines along which the utility approaches and eventually reaches its limiting value, i.e. those which are asymptotes of indifferences with a closed set of gradients, is an open set. A parallel argument implies that its complement, namely the set of parallel half-lines along which the utility asymptotes to but never reaches its limiting value, is also an open set: this is because, under Assumption 1, the norms of "gradients" to an indifference surface assume a minimum non-zero value. However, the only sets which are open and closed within a connected space are the empty set and the whole space. Therefore the set of half-lines corresponding to indifferences on which the set of gradients is closed can be either the empty set, or the whole space. Therefore $\partial G_h - G_h$ is indeed uniform across the space. Since, as seen above, when Assumption 1 is satisfied G_h^o and $\overline{G_h}$ are also uniform, so is G_h. By definition, therefore, D_h is also uniform under Assumption 1. This completes the proof of (iv) and of the proposition. ■

When Assumption 1 does not hold, Proposition 1 need not hold either; for an example see Figure 3.

- **Consider next the case:** $X = R_+^N$:

Definition 4 *For trader h the cone $G_h(\Omega_h)$ is*

$$G_h(\Omega_h) = \{y \in X : u_h(\Omega_h + \lambda y) > u_h(\Omega_h + \mu y) \text{ when } \lambda > \mu > 0\} \tag{9}$$

Definition 5 *The* **market cone of trader** h *is:*

$$\begin{aligned} D_h^+(\Omega_h) &= D_h(\Omega_h) \bigcap S(\mathsf{E}) \text{ if } S(\mathsf{E}) \subset \mathsf{N}, \\ &= D_h(\Omega_h) \text{ otherwise.} \end{aligned} \tag{10}$$

where $S(\mathsf{E})$ and N are defined in (1) and (2).[31]

There is no analog to Proposition 2 (iv) when $X = R_+^N$; indeed, when $X = R_+^N$ the market cones $D_h^+(\Omega_h)$ typically vary with the initial endowments. However, when $\Omega_h \in R_{++}^N$, the interior of R_+^N, then $D_h^+(\Omega_h) = D_h\ (\Omega_h)$ and therefore $D_h^+(\Omega_h)$ is the same for all endowments in R_{++}^N.

Proposition 3 *When $X = R_+^N$ and an indifference surface of u_h corresponding to a positive consumption bundle $x > 0$ intersects a boundary ray*[32] *$r \subset \partial X$, then u_h is strictly increasing along r, i.e. $r \in G_h(0)$.*[33]

Proof. Recall that we assumed $u_h(0) = 0$, and that the preference's indifference surfaces of positive utility are either (a) contained in the interior of R_+^N, R_{++}^N, such as Cobb-Douglas or CES, or (b) they intersect the boundary of R_+^N and if so, transversally: if $\exists x \in \partial R_+^N$ such that $u_h(x) > 0$, then $Du_h(x)$ is not orthogonal to ∂R_+^N at x. In case (a) the proposition is satisfied trivially, because no indifference of strictly positive value ever intersects the boundary of R_+^N. In case (b) the proposition follows immediately from the definition of transversality. Observe that it is possible that $\sup_{x \in r}(u_h(x)) < \infty$. ■

1.2 The Core and the Supercore

Definition 6 *The* **core** *of the economy E is the set of allocations which no coalition can improve upon within its own endowments:*

$$C(\mathsf{E}) = \{(x_1, ..., x_H) \in R^{N \times H} : \sum_h (x_h - \Omega_h) = 0 \text{ and} \sim J \subset \{1, ..., H\} :$$

[31]The market cone D_h^+ is the whole consumption set $X = R^{N+}$ when $S(E)$ has a vector assigning strictly positive income to all individuals. If some trader has zero income, then this trader must have a boundary endowment.

[32]A boundary ray r in R_+^N is a set which consists of all the positive multiples of a vector $v \in \partial R_+^N$: $r = \{w \in R_+^N : \exists \lambda > 0 \text{ s.t. } w = \lambda v\}$

[33]This includes all concave preferences used on R_+^N such as: Cobb-Douglas, constant elasticity of substitution (CES), preferences with indifference surfaces of positive consumption contained in the interior of R_+^N, linear preferences, piecewise linear preferences, Leontief preferences, preferences with indifference surfaces which intersect the boundary of the positive orthant (Arrow and Hahn [2]) and smooth utilities defined on a neighborhood of X which are transversal to its boundary ∂X.

$$\begin{aligned} &\textstyle\sum_{j \in J}(y_j - \Omega_j) = 0, \forall j \in J, u_j(y_j) \geq u_j(x_j), \\ &\text{and } \exists j \in J : u_j(y_j) > u_j(x_j)\}. \end{aligned}$$

Definition 7 *The* **supercore** *of the economy is the set of allocations which no* **strict** *sub coalition can improve using only its own endowments. It is therefore a superset of the core:*

$$SC(\mathsf{E}) = \{\{(x_1, ..., x_H) \in R^{N \times H} : \sum_h (x_h - \Omega_h) = 0 \text{ and} \sim J \subset \{1, ..., \}\} :$$

$$\begin{aligned} &J \neq \{1, ..., H\} : \forall j \in J, u_j(y_j) \geq u_j(x_j), \textstyle\sum_{j \in J}(y_j - \Omega_j) = 0, \text{ and} \\ &\exists j \in J : u_j(y_j) > u_j(x_j)\}. \end{aligned}$$

and by construction, $C(\mathsf{E}) \subset SC(\mathsf{E})$. The motivation for this concept is as follows: if an allocation is in the supercore, no strict subcoalition of traders can improve upon this by itself. A non-empty supercore means that no strict subsets of individuals can do better than what they can do by joining the entire group. The benefits from joining the larger group exceed those available to any subgroup. One can say therefore that an economy with a non-empty supercore has reasons to stay together: There is no reason for such a society to break apart. If one observes an economy that has stayed together for some time, it must therefore have a non-empty supercore.

2 Limited Arbitrage: Definition and Examples

This section provides the definition of limited arbitrage. It gives an intuitive interpretation for limited arbitrage in terms of gains from trade, and contrasts limited arbitrage with the arbitrage concept used in financial markets. It provides examples of economies with and without limited arbitrage.

Definition 8 *When $X = R^N$, E satisfies* **limited arbitrage** *when*

$$(LA) \ \bigcap_{h=1}^{H} D_h \neq \emptyset.$$

Definition 9 *When $X = R_+^N$, E satisfies* **limited arbitrage** *when*

$$(LA^+) \ \bigcap_{h=1}^{H} D_h^+(\Omega_h) \neq \emptyset. \tag{11}$$

2.1 Interpretation of Limited Arbitrage as Bounded Gains From Trade

Limited arbitrage has a simple interpretation in terms of gains from trade when $X = R^N$ and the preference's indifference surfaces have a closed set of gradients.[34] Gains from trade are defined by:

$$\mathsf{G(E)} = \sup\{\sum_{h=1}^{H}(u_h(x_h) - u_h(\Omega_h)\}, \text{ where}$$

$$\sum_{h=1}^{H}(x_h - \Omega_h) = 0, \text{ and } \forall h,\ u_h(x_h) \geq u_h(\Omega_h) \geq 0.$$

Proposition 4 *When indifferences have a closed set of gradients*[35] *the economy* E *satisfies limited arbitrage if and only if gains from trade are bounded,*[36] *i.e.,*

$$\mathsf{G(E)} < \infty.$$

Proof. Assume E has limited arbitrage. If G(E) was *not* bounded there would exist a sequence of net trades $(z_1^j ... z_H^j)_{j=1,2...}$ with correspondingly positive utility levels, and a set $J \subset \{1, ..., H\}$ of traders such that:[37] (i) $\lim_j \sum_{h\in J} z_h^j = 0$, (ii) $\|z_h^j\| \to \infty \Leftrightarrow h \in J$, and (iii) for some $h = g$, $\lim_{j\to\infty}(u_g(\Omega_g + z_g^j)) \to \infty$, in particular $z_g = \lim_{j\to\infty} z_g^j/\|z_g^j\| \in A_g$. Since $\|z_h^j\| \to \infty\ \forall h \in J$, then $z_h = \lim_j z_h^j/\|z_h^j\| \in \overline{G}_h$, for otherwise, $\lim_j(u_h(\Omega_h + z_h^j) < 0$, in contradiction with the assumption that the sequence of utility levels is positive. Since $\lim_j \sum_{h\in J} z_h^j = 0$, $\forall h \in J, z_h \in \overline{G}_h$, and $z_g \in A_g$, which is an open set by Proposition 1, then $\exists\{\gamma_h\}_{h\in J}$ such that $\sum_{h\in J}\gamma_h = 0$ and $\gamma_h \in G_h\ \forall h \in J$, contradicting limited arbitrage. Since the contradiction arises from assuming that G(E) is not bounded, G(E) must be bounded. This is true for any preference satisfying Assumption 1. Therefore under Assumption 1 limited arbitrage implies bounded gains from trade. Observe that when indifference surfaces have a closed set of gradients, as assumed here, then $G_h = A_h$. In this case the reciprocal is immediate: limited arbitrage is also necessary for bounded gains from trade, completing the proof. ■

Observe that the sufficiency part is valid for all preferences satisfying Assumption 1:

Corollary 1 *For all economies with uniformly non-satiated preferences, limited arbitrage implies bounded gains from trade.*

[34] In this case, by definition, $G_h = A_h$.

[35] In this case $G_h = A_h$.

[36] The expression $G(\mathsf{E}) < \infty$ holds when $\forall h$, $\sup_{\{x:x\in X\}} u_h(x) = \infty$; it must be replaced by $G(\mathsf{E}) < \sup_{\{x:x\in X\}}\left(\sum_{h=1}^{H} u_h(x) - u_h(\Omega_h)\right) - k$, for some positive k, when $\forall h$, $\sup_{\{x:x\in X\}} u_h(x) < \infty$.

[37] Recall that without loss of generality, we normalized utilities so that $\sup_{x\in X}(u_h(x)) = \infty$

Proof. This follows immediately from the proof of Proposition 4. ■

2.2 A Financial Interpretation of Limited Arbitrage

It is useful to explain the connection between limited arbitrage and the notion of "no-arbitrage" used in finance. The concepts are generally different, but in certain cases, which are of interest in the financial literature, they coincide. In the finance literature, arbitrage appears as a central context. Financial markets equilibrium is often defined as the absence of market arbitrage. In Walrasian markets this is not the case. It may therefore appear that the two literatures use different equilibrium concepts. Yet the link provided here draws a bridge between these two literatures. As shown below limited arbitrage, while not an equilibrium concept, is necessary and sufficient for the existence of a Walrasian equilibrium. In the following I will show the close link between the two concepts and establish the bridge between the two equilibrium theories. I will provide examples where the two concepts are identical, and others where they are different.

In financial markets an arbitrage opportunity exists when unbounded gains can be made at no cost, or, equivalently, by taking no risks. Consider, for example, buying an asset in a market where its price is low while simultaneously selling it at another where its price is high: this can lead to unbounded gains at no risk to the trader. No-arbitrage means that such opportunities do not exist, and it provides a standard framework for pricing a financial asset: precisely so that no arbitrage opportunities should arise between this and other related assets. Since trading does not cease until all arbitrage opportunities are extinguished, at a market clearing equilibrium there must be no-arbitrage.

The simplest illustration of the link between limited arbitrage and no-arbitrage is an economy E where the traders' initial endowments are zero, $\Omega_h = 0$ for $h = 1, 2$, and the set of gradients to indifference surfaces are closed. Here no-arbitrage at the initial endowments means that there are no trades which could increase the traders' utilities at zero cost: gains from trade in E must be zero. By contrast, E has limited arbitrage when no trader can increase utility beyond a given bound at zero cost; as seen above, gains from trade are bounded.

In brief: *no-arbitrage* requires that there should be no gains from trade at zero cost while *limited arbitrage* requires that there should be only bounded utility arbitrage or limited gains from trade.

Now consider a particular case of the same example: when the traders' utilities are defined by linear real valued functions. Then the two concepts coincide: there is limited arbitrage if and only if there is no-arbitrage as defined in finance. In brief: in linear economies, limited arbitrage "collapses" into no-arbitrage.

In general, the two concepts are related but nonetheless quite different: no-arbitrage is a market clearing condition used to describe an allocation at which there is no further reason to trade. It can be applied at the initial allocations, but then it means that there is no reason for trade in the economy as a whole: the economy is autarchic and therefore not very interesting. By contrast, limited arbitrage is applied only to the economy's initial data, the traders' endowments and preferences. Limited arbitrage does not imply that the economy is autarchic; quite to the contrary, it is valuable in predicting whether the economy can ever reach a competitive equilibrium. It allows to do so by examining the economy's initial conditions.

2.3 Examples of markets with and without limited arbitrage

Example 1 *Figures 4 and 5 illustrate an economy with two traders trading in $X = R^2$; in Figure 4 the market cones intersect and the economy has limited arbitrage. In Figure 5 the market cones do not intersect and the economy does not have limited arbitrage. Figure 6 illustrates three traders trading in $X = R^3$; each two market cones intersect, but the three market cones do not intersect, and the economy violates limited arbitrage. This figure illustrates the fact that the union of the market cones may fail to be contractible: indeed, this failure corresponds to the failure of the market cones to intersect, as proven in Chichilnisky [12]*

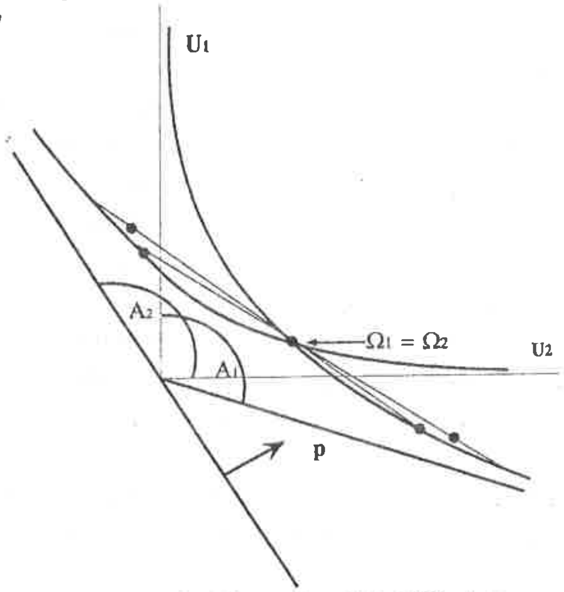

Figure 4: limited arbitrage is satisfied. The two global cones lie in the halfspace defined by P. There are no feasible trades that increase utilities without limit: these would consist of pairs of points symmetrically placed about the common initial endowment, and such pairs of points lead to utility values below those of the endowments at a bounded distance from the initial endowments.

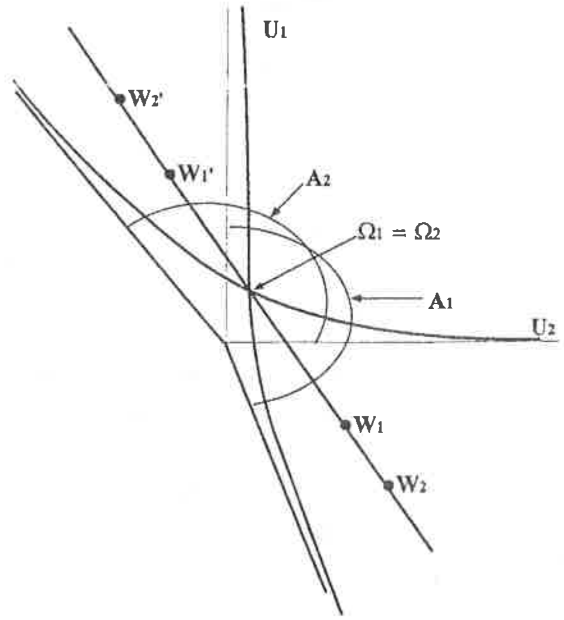

Figure 5: Limited arbitrage does not hold. The global cones are not contained in a half space, and there are sequences of feasible allocations such as (W1,W1'), (W2,W2') which produce unbounded utilities.

Figure 6: Every two cones intersect, but all three do not. The economy does not have limited arbitrage.

Example 2 *When the consumption set is $X = R^N_+$, limited arbitrage is always satisfied if all indifference surfaces through positive consumption bundles are contained in the interior of X, R^N_{++} (as in Debreu [25]). Examples of such preferences are those given by Cobb-Douglas utilities, or by utilities with constant elasticity of substitution (CES) with elasticity of substitution $\sigma < 1$. This is because all such preferences have the same global cone, namely the positive orthant, and therefore their market cones always intersect. Since their global cones are identical, these preferences are very similar to each other on choices involving large utility levels: this is a form of similarity of preferences. Economies where the individuals' initial endowments are strictly interior to the consumption set X always satisfy the limited arbitrage condition in the case $X = R^N_+$, since in this case $\forall h$, $R^N_{++} \subset D^+_h(\Omega_h)$ for all $h = 1, ..., H$.*

Example 3 *When $X = R^N_+$ the limited arbitrage condition may fail to be satisfied when some trader's endowment vector Ω_h is in the boundary of the consumption space, ∂R^N_+, and at all supporting prices in $S(\mathsf{E})$ some trader has zero income, i.e. when $\forall p \in S(\mathsf{E})\ \exists h$ such that $\langle p, \Omega_h \rangle = 0$. In this case, $S(\mathsf{E}) \subset \mathsf{N}$. This case is illustrated in Figure 7; it is a rather general case which may occur in economies with many individuals and with many commodities. When all individuals have positive income at some price $p \in S(\mathsf{E})$; then limited arbitrage is always satisfied since by definition in this case $\forall h$, $= R^N_{++} \subset D^+_h(\Omega_i)$ for all $h = 1, ..., H$.*

Example 4 *A competitive equilibrium may exist even when some traders have zero income, showing that Arrow's "resource relatedness" condition is sufficient but not necessary for existence of an equilibrium. Figure 8 illustrates an economy where at all supporting prices some trader has zero income: $\forall p \in S(\mathsf{E})\ \exists h$ such that $\langle p, \Omega_h \rangle = 0$, i.e. $S(\mathsf{E}) \subset \mathsf{N}$; in this economy, however, limited arbitrage is satisfied so that a competitive equilibrium exists. The initial allocation and a price vector assigning value zero to the second good is such an equilibrium.*

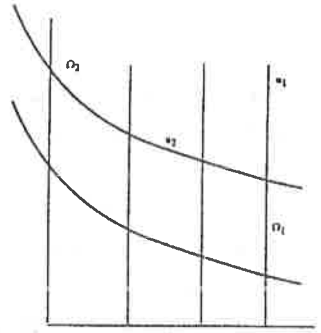

Figure 7: Limited arbitrage fails: one traders' endowment in the boundary of the consumption set.

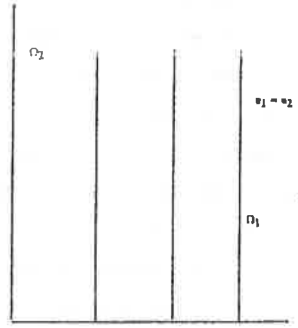

Figure 8: Equilibrium exists even when some trader has zero income: "resource relatedness" is not necessary for existence of an equilibrium in this example.

3 Limited Arbitrage and the Compactness of the Pareto Frontier

The Pareto frontier $P(\mathsf{E})$ is the set of feasible, efficient and individually rational utility allocations. With H traders it is a subset of R^H_+, cf. Section 1.2. Proving the boundedness and closedness of the Pareto frontier is a crucial step in establishing the existence of a competitive equilibrium and the non-emptiness of the core. The main theorem of this section shows that limited arbitrage is necessary and sufficient for this.

There is a novel feature of the results which are presented here, a feature which is shared which those that were previously established in [13], [15], [11], [14], [16] and [23]. It starts from the observation that the compactness of the Pareto frontier need not imply the compactness of the set of feasible commodity allocations. The Pareto frontier is defined in *utility* space, R^H_+ while the commodity allocations are in the product of the *commodity space* with itself, X^H. When $X = R^N$, the commodity allocations are in $R^{H \times N}$. This observation is useful to distinguish the results presented here and in [13], [15], [11], [14], [16] and [23] from all others in the literature. All other conditions used in the literature which are sufficient for the existence of an equilibrium and the core ensure that—with or without short sales—the set of individually rational and feasible commodity allocations is compact, see e.g. Chichilnisky and Heal [22], Werner [41] and Koutsougeras [36] among others; the latter [36] proves in detail that Werner's proof of existence of an equilibrium [41] relies on the compactness of the set of feasible allocations. But as already observed, and as is shown below, the boundedness of the set of feasible commodity allocations is *not* needed for existence. Indeed, such boundedness is not used in this paper, nor was it used in the results of [13] [15], [11], [14], [16] and [23]: these are therefore the first results in the literature proving the existence of equilibrium and the non-emptiness of the core in economies where the set of feasible allocations is unbounded. In addition, of course, these results establish conditions which are simultaneously necessary and sufficient for the existence of equilibrium and the core, another novel feature. As a result, here the set of all possible efficient allocations, the contract curve, and the set of possible equilibria and the set of all possible core allocations, may be unbounded sets. In brief: The proofs of existence of an equilibrium given below, for economies with finitely or infinitely many markets, and also that of [13] [15], [11], [14], [16] and [23], are original in that they apply to economies where the set of feasible and individually rational allocations of commodities may be unbounded. The same is true for the proof of non-emptiness of the core given below: it applies to economies with possibly unbounded commodity allocations.

First we review some examples to illustrate and better appreciate the nature of the problems that can arise.

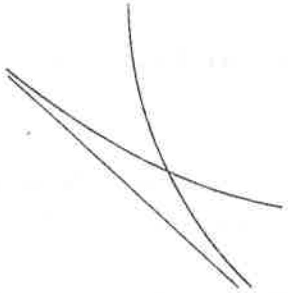

Figure 9: The Pareto frontier may fail to be closed even in finite dimensions.

Example 5 *Figure 9 shows that the Pareto frontier may fail to be closed even in finite dimensional models, provided the consumption set is the whole Euclidean space. It shows agents with indifference curves having the line $y = -x$ as asymptote. Consumption sets are the whole space and feasible allocations are those which sum to zero. Utility functions are $u_i = x_i + y_i \pm e^{-(x_i - y_i)}, i = 1, 2$. Limited arbitrage rules out such cases.*

Example 6 *Another example is a two-agent economy where both agents have linear preferences: if the preferences are different the set of feasible utility allocations is unbounded. Of course, limited arbitrage rules out such situations.*

Example 7 *Even when the consumption set is bounded below, but the commodity space is infinite dimensional, examples can be provided where the Pareto frontier is not closed.*[38]

Theorem 1 *Consider an economy* E *as defined in Section 1. Then* **limited arbitrage** *is necessary and sufficient for the compactness of the Pareto frontier.*[39]

Proof. This result always holds when the consumption set is bounded by some vector in the space,[40] and in that case it is proved using standard arguments, see e.g. Arrow and Hahn [2]. Therefore in the following I concentrate in the case where X is unbounded.

Sufficiency first. Recall that by definition $P(\mathsf{E}) \subset U(\mathsf{E}) \subset R^N_+$. Proposition 4 proves that $U(\mathsf{E})$ is bounded when limited arbitrage is satisfied, so that $P(\mathsf{E})$ is bounded also.

The first step is to prove that $P(\mathsf{E})$ is closed when limited arbitrage is satisfied. Consider a sequence of allocations $\{z^j_h\}_{j=1,2...}$, $h = 1, 2, ..., H$, satisfying $\forall j$, $\sum_{h=1}^H z^j_h \leq \Omega$, and $\lim_{j\to\infty} \sum_{h=1}^H z^j_h = \Omega$. Assume that $(u_1(z^j_1), ..., u_H(z^j_H)) \subset R^N_+$ converges to a utility allocation $v = (v_1, ..., v_H) \in R^H_+$, which is efficient, i.e. v is not dominated by the utility allocation of any other feasible allocation. Observe that the vector v may or not be the utility vector of a feasible allocation: when limited arbitrage is satisfied, I will prove that it is. The result is immediate if the set of feasible allocations is bounded; therefore I concentrate in the case where the set of feasible allocations is not bounded. Let M be the set of all traders $h \in \{1, 2.,,, H\}$, for whom the corresponding sequence of allocations $\{z^j_h\}_{j=1,2...}$ is bounded, i.e. $h \in M \Leftrightarrow \exists K_h : \|z^j_h\| < K_h$; let J be its complement, $J = \{1, 2, ..., H\} - M$, which I assume to be non empty. There exists a subsequence of the original sequence of allocations, which for simplicity is denoted also $\{z^j_h\}_{j=1,2...}$, $h = 1, 2, ..., H$, along which $\forall h \in M$, the $\lim_j \{z^j_h\}_{j=1,2...} = z_h$ exists, and $\sum_{h\in M} z_h + \lim_{j\to\infty} \sum_{h\in J} z^j_h = \Omega$. Recall that by Proposition 2 the cones G_h are uniform, so that we may translate the origin of the space without loss of generality. Therefore we may assume without loss that $\sum_{h\in M} z_h = \Omega$, i.e. that $\lim_{j\to\infty} \sum_{h\in J} z^j_h = 0$. For each $h \in J$, the normalized sequence $\{\frac{z^j_h}{\|z^j_h\|}\}_{j=1,2...}$, is contained in a compact space, the unit sphere; consider a convergent subsequence of this, denoted also $\{\frac{z^j_h}{\|z^j_h\|}\}_{j=1,2...}$. Let $z_h = \lim_j \{\frac{z^j_h}{\|z^j_h\|}\}$, $h \in J$. The complement of $\overline{G_h}$, denoted $\overline{G_h}^c$, is an open cone so that if $z_h \in \overline{G_h}^c$, then $z^j_h \in \overline{G_h}^c$ from some j on. However, by construction, $u_h(z^j_h) \geq 0$, while $\lim_j \|z^j_h\| = \infty$ so that if $z^j_h \in \overline{G_h}^c$, eventually $u_h(z^j_h) < 0$, a contradiction. Therefore $\forall h \in J$, $z_h \notin \overline{G_h}^c$, or,

[38] A standard example of this phenomenon is in $L_\infty = \{f : R \to R : \sup_{x\in R} \|f(x)\| < \infty\}$. Society's endowment is $\Omega = (1,1,...,1,...)$, trader one has a preference $u_1(x) = \sup(x_i)$, and trader two has a preference $u_2(x) = \sum_i u(x_i)\lambda^{-i}$, $0 < \lambda < 1$. Then giving one more unit of the ith good to trader two always increases trader two's utility without decreasing that of trader one, and the Pareto frontier cannot be closed.

[39] Recall that the Pareto frontier is defined as the set of individually rational, feasible and efficient utility allocations, see Section 1.2.

[40] A set $X \subset H$ is bounded by v when there exists $v \in H : \forall x \in H, x > v$ or $\forall x \in H, v < x$.

equivalently, $\forall h \in J$, $z_h \in \overline{G_h}$. Furthermore if $\forall h \in J$, $z_h \in \partial G_h$ and $z_h \notin G_h$ then by Proposition 2, eventually the utility values of the traders attain their limit for all h; in this case the utility vector v is achieved by a feasible allocation and the proof is complete. It remains therefore to consider only the case where for some trader $g \in J$, $z_g \in G_g$. Define now the convex cone C of all strictly positive linear combinations of the vectors $\{z_h\}_{h \in J}$, $C = \{w = \sum_{h \in J} \mu_h z_h, \mu_h > 0\}$. There are two mutually exclusive and exhaustive cases: either (a) the cone C is contained *strictly* in a half-space of R^N, or else (b) the cone C is a subspace of R^N. By construction $\lim_{j \to \infty} \sum_{h \in J} z_h^j = 0$, which eliminates case (a) . Therefore case (b) must hold: in particular the vector $\{0\}$ belongs to C. This means that $\forall h \in J, \exists \lambda_h > 0$, such that

$$\sum_{h \in J} \lambda_h z_h = 0. \tag{12}$$

The final step is to show that (12) contradicts limited arbitrage. By limited arbitrage $\exists p \in R^N$ such that $\forall h \in J$, $\langle p, z_h \rangle > 0$ for all $z_h \in G_h$. It follows that $\forall h \in J, \langle p, z_h \rangle \geq 0$ $\forall z_h \in \overline{G_h}$. Since in our remaining case $\forall h \in J$, $z_h \in \overline{G_h}$ and $z_g \in G_g$, the inner product $\langle p, \sum_{h \in J} \lambda_h z_h \rangle$ must be strictly positive, contradicting (12). Since the contradiction arises from assuming that the Pareto frontier $P(\mathsf{E})$ is not closed, $P(\mathsf{E})$ must be closed. Therefore limited arbitrage implies a compact Pareto frontier.

Necessity is established next. If limited arbitrage fails, there is no vector $y \in H$ such that $\langle y, z_h \rangle > 0$ for all $\{z_h\} \in G_h$. Equivalently, there exist a set J consisting of at least two traders and, for each $h \in J$, a vector $z_h \in G_h$ such that $\sum_{h \in J} z_h = 0$. Then either for some h, $z_h \in G_h^o$ so that the Pareto frontier is unbounded and therefore not compact, or else for some h, $z_h \in \partial G_h \cap G_h$ and therefore the Pareto frontier is not closed, and therefore not compact either. In either case, the Pareto frontier is not compact when limited arbitrage fails. Therefore compactness is necessary for limited arbitrage. ■

Proposition 5 *When $X = R^N$, limited arbitrage implies that the Pareto frontier $P(\mathsf{E})$ is homeomorphic to a simplex.*[41]

Proof. This follows from Theorem 1 and by the convexity of utilities, cf. Arrow and Hahn [2]. ■

4 Competitive Equilibrium and Limited Arbitrage

This section establishes the main result linking the existence of a competitive equilibrium with the condition of limited arbitrage.[42] The result is that limited arbitrage is simul-

[41] A topological space X is homeomorphic to another Y when there exists an onto map $f : X \to Y$ which is continuous and has a continuous inverse.

[42] The results on equilibrium in this paper originated from a theorem in Chichilnisky and Heal [24], as paper which was submitted for publication in 1984, nine years before it appeared in print: these dates

taneously necessary and sufficient for the existence of a competitive equilibrium,[43] and it was established first in [13], [11], [14]. Other noteworthy features are: the equivalence between limited arbitrage and equilibrium applies equally to economies with or without short sales, and with or without strictly convex preferences. It therefore includes the Arrow Debreu market which has no short sales, a classic case which was neglected previously in the literature on no-arbitrage conditions. In addition, the equivalence applies to economies where the set of feasible allocations may be unbounded; this is also new in the literature.[44] Finally, the equivalence between limited arbitrage and equilibrium extends to economies with infinitely many markets, see the next Section.

The result presented below was established first in [13], [11], [14] for uniformly non-satiated convex preferences which are either strictly convex, or which have indifference surfaces with a closed set of gradients. The result presented here extends these earlier results in that it applies to all uniformly non-satiated convex preferences, and in particular to all convex preferences which admit a representation by a concave utility, see Section 2: this extension was proposed by Kenneth Arrow in October 5, 1994:

are recorded in the printed version. This paper [24] provided a no-arbitrage condition and proved it is sufficient for the existence of a competitive equilibrium with or without short sales, with infinitely or finitely many markets. See also the following footnote.

[43] Chichilnisky and Heal [24], Hart [30] Hammond [29] and Werner [41] among others, have defined various no-arbitrage conditions which they prove, under certain conditions on preferences, to be sufficient for existence of an equilibrium. None of these no-arbitrage conditions is generally necessary for existence. Within economies with short sales (which exclude Arrow Debreu's markets), and where preferences have no halflines in the indifference surfaces (which exclude "flats"), Werner [41] remarks (p. 1410, last para.) that another related condition (p. 1410, line -3) is necessary for existence, without however providing a complete proof of the equivalence between the condition which is necessary and that which is sufficient. The two conditions in [41] are defined on different sets of cones: the sufficient condition is defined on cones S_i (p. 1410, line -14) while the necessary condition is defined on other cones, D_i (p. 1410, -3). The equivalence between the two cones depends on properties of yet another family of cones W_i (see p. 1410, lines 13-4). The definition of W_i on page 1408, line -15 shows that W_i is different from the recession cone R_i, (which are uniform by assumption) and therefore the cone W_i need not be uniform even when the recession cones are, as needed in Werner's Proposition 2. Therefore [13], [11], [14] and the results presented here provide the first complete proof of a condition (limited arbitrage) which is simultaneously necessary and sufficient for the existence of a competitive equilibrium, even for the special case of economies with short sales and with strictly convex preferences.

[44] The no-arbitrage conditions in Chichilnisky and Heal [24], Hart [30], Hammond [29] and Werner [41] do not extend to, and do not apply to, all the economies considered in this paper: all prior results (except for those in [13], [11], [14]) depend crucially on the fact that the set of feasible allocations is compact. By contrast, the boundedness of feasible allocations is neither required, nor it is generally satisfied, in the economies considered in this paper, because although the feasible allocations may be unbounded, there exists a bounded set of allocations which reach all possible feasible utility levels. This paper extends all prior results to economies where all convex preferences are allowed, as long as Assumption 1 is satisfied, as proposed by K. Arrow. As already mentioned in Section 2, Assumption 1 is automatically satisfied by preferences admitting a representation by a concave utility function, whether or not this function is strictly concave.

Theorem 2 *Consider an economy* $\mathsf{E} = \{X, u_h, \Omega_h, h = 1, ..., H\}$, *where* $H \geq 2$, *with* $X = R^N$ *or* $X = R^N_+$ *and* $N \geq 1$. *Then the following two properties are equivalent:*

(i) The economy E *has limited arbitrage*

(ii) The economy E *has a competitive equilibrium*

Proof. Necessity first. Consider first the case $X = R^N$ and assume without loss of generality that $\Omega_h = 0$ for all h. The proof is by contradiction. Let p^* be an equilibrium price and let $x^* = (x^*_1, ...x^*_H)$ be the corresponding equilibrium allocation. Then if limited arbitrage does not hold, $\exists h$ and $v \in G_h$ such that $\langle p^*, v\rangle \leq 0$, so that $\forall \lambda > 0$, λv is affordable at prices p^*. However, G_h is the same at any endowment by Proposition 2. It follows that $\exists \lambda > 0 : u_h(x^*_h + \lambda v) > u_h(x^*_h)$, which contradicts the fact that x^*_h is an equilibrium allocation. This completes the proof of necessity when $X = R^N$.

Consider next $X = R^N_+$. Assume that $\forall q \in S(\mathsf{E})\ \exists\, h \in \{1, ..., H\}$ such that $\langle q, \Omega_h\rangle = 0$. Then if limited arbitrage is not satisfied $\bigcap_{h=1}^H D^+_h(\Omega_h) = \emptyset$, which implies that $\forall p \in R^N$, $\exists h$ and $v(p) \in G_h(\Omega_h)$:

$$\langle p, \lambda v(p)\rangle \leq 0,\ \forall \lambda > 0 \tag{13}$$

I will now show that this implies that a competitive equilibrium price cannot exist. By contradiction. Let p^* be an equilibrium price and $x^* \in X^H$ be the corresponding equilibrium allocation. Consider $v(p^*) \in G_h(\Omega_h)$ satisfying (13). If $\lim_{\lambda\to\infty} u_h(\Omega_h + \lambda v(p)) = \infty$ this leads directly to a contradiction, because $\langle p^*, \lambda v(p^*)\rangle \leq 0$, so that for all λ, $\lambda v(p^*)$ is affordable, and therefore there is no affordable allocation which maximizes h's utility at the equilibrium price p^*. Consider next the case where $v(p^*) \in G_h(\Omega_h) - A_h(\Omega_h)$. By definition, $u_h(\Omega_h + \lambda v(p^*))$ is strictly increasing in λ, and $\lim_{\lambda\to\infty} u_h(\Omega_h + \lambda v(p^*)) < \infty$. If $u_h(x^*_h) > \lim_{\lambda\to\infty} u_h(\Omega_h + \lambda v(p^*))$ then there exists a vector, namely x^*_h, which has utility strictly larger than $v(p^*) \in \partial G_h(\Omega_h)$ so that, as shown in Proposition 2, the direction defined by the vector $x^* - \Omega_h$ must be contained in $A_h(\Omega_h)$. But this is a contradiction with the assumption that x^*_h is an equilibrium allocation, because if $x^* - \Omega_h \in A_h(\Omega_h)$, $\lim_{\lambda\to\infty} u_h(\Omega_h + \lambda(x^*_h - \Omega_h)) = \infty$, while $\langle p^*, \lambda(x^*_h - \Omega_h)\rangle \leq 0$ so that x^*_h cannot be an equilibrium allocation. Therefore limited arbitrage is also necessary for the existence of a competitive equilibrium in this case.

It remains to consider the case where $\exists p \in S(\mathsf{E})$ such that $\forall h \in \{1, ..., H\}, \langle p, \Omega_h\rangle \neq 0$. But in this case by definition $\bigcap_{h=1}^H D^+_h(\Omega_h) \neq \emptyset$ since $\forall h \in \{1...H\}$, $R^N_{++} \subset D^+_h(\Omega_h)$, so that limited arbitrage is always satisfied when an equilibrium exists. This completes the proof of necessity.

Sufficiency next. The proof uses the fact that the Pareto frontier is homeomorphic to a simplex. When $X = R^N_+$ the Pareto frontier of the economy $P(\mathsf{E})$ is always homeomorphic to a simplex, see Arrow and Hahn [2]. In the case $X = R^N$ this may fail. However, by Theorem 1 above, if the economy satisfies limited arbitrage then the Pareto frontier is

compact; under the assumptions on preferences, it is then also homeomorphic to a simplex [2]. Therefore in both cases, $P(\mathsf{E})$ is homeomorphic to a simplex and one can apply the by now standard Negishi method of using a fixed point argument on the Pareto frontier to establish the existence of a **pseudoequilibrium.**[45] It remains however to prove that the pseudoequilibrium is also a competitive equilibrium.

To complete the proof of existence of a competitive equilibrium consider first $X = R^N$. Then $\forall h = 1, ..., H$ there exists an allocation in X of strictly lower value than the pseudoequilibrium x^*_h at the price p^*. Therefore by Lemma 3, Chapter 4, page 81 of Arrow and Hahn [2], the quasi-equilibrium (p^*, x^*) is also a competitive equilibrium, completing the proof of existence when $X = R^N$.

Next consider $X = R^N_+$, and a quasi-equilibrium (p^*, x^*) whose existence was already established. If every individual has a positive income at p^*, i.e. $\forall h$, $\langle p^*, \Omega_h\rangle > 0$, then by Lemma 3, Chapter 4 of Arrow and Hahn [2] the quasi-equilibrium (p^*, x^*) is also a competitive equilibrium, completing the proof. Furthermore, observe that in any case the pseudoequilibrium price $p^* \in S(\mathsf{E})$, so that $S(\mathsf{E})$ is not empty. To prove existence we consider therefore two cases: first the case where $\exists q^* \in S(\mathsf{E}) : \forall h$, $\langle q^*, \Omega_h\rangle > 0$. In this case, by the above remarks from Arrow and Hahn [2], (q^*, x^*) is a competitive equilibrium. The second case is when $\forall q \in S(\mathsf{E}), \exists h \in \{1, ..., H\}$ such that $\langle q, \Omega_h\rangle = 0$. Limited arbitrage then implies:

$$\exists q^* \in S(\mathsf{E}) : \forall h,\ \langle q^*, v\rangle > 0 \text{ for all } v \in G_h(\Omega_h). \tag{14}$$

Let $x^* = x^*_1, ..., x^*_H \in X^H$ be a feasible allocation in Υ supported by the vector q^* defined in (14): by definition, $\forall h$, $u_h(x^*_h) \geq u_h(\Omega_h)$ and q^* supports x^*. Note that any h minimizes costs at x^*_h because q^* is a support. Furthermore x^*_h is affordable under q^*. Therefore, (q^*, x^*) can fail to be a competitive equilibrium only when for some h, $\langle q^*, x^*_h\rangle = 0$, for otherwise the cost minimizing allocation is always also utility maximizing in the budget set $B_h(q^*) = \{w \in X : \langle q^*, w\rangle = \langle q^*, \Omega_h\rangle\}$.

It remains therefore to prove existence when $\langle q^*, x^*_h\rangle = 0$ for some h. Since by the definition of $S(\mathsf{E})$, x^* is individually rational, i.e. $\forall h$, $u_h(x^*_h) \geq u_h(\Omega_h)$, then $\langle q^*, x^*_h\rangle = 0$ implies $\langle q^*, \Omega_h\rangle = 0$, because by definition q^* is a supporting price for the equilibrium

[45] A pseudoequilibrium, also called quasiequilibrium, is an allocation which clears the market and a price vector at which traders minimize cost within the utility levels achieved at their respective allocations. For the proof of existence of a quasiequilibrium cf. Negishi [37], who studied the case where the economy has no short sales. For cases where short sales are allowed, and therefore feasible allocations may be unbounded, a method similar to Negishi's can be used, see e.g. Chichilnisky and Heal [24], [23] and Chichilnisky [13]. With strictly convex preferences, limited arbitrage implies that feasible allocations form a bounded set; otherwise, when indifferences have "flats", the set of feasible allocations may be unbounded. However in this latter case there exists a bounded set of feasible allocations which achieves all feasible utility levels, and this suffices for a Negishi-type proof of existence to go through.

allocation x^*. If $\forall h$, $u_h(x^*_h) = 0$ then $x^*_h \in \partial R^N_+$, and by the monotonicity and quasi-concavity of u_h, any vector y in the budget set defined by the price p^*, $B_h(q^*)$, must also satisfy $u_h(y) = 0$, so that x^*_h maximizes utility in $B_h(q^*)$, which implies that (q^*, x^*) is a competitive equilibrium. Therefore (q^*, x^*) is a competitive equilibrium unless for some h, $u_h(x^*_h) > 0$.

Assume therefore that the quasiequilibrium (q^*, x^*) is not a competitive equilibrium, and that for some h with $\langle q^*, \Omega_h\rangle = 0, u_h(x^*_h) > 0$. Since $u_h(x^*_h) > 0$ and $x^*_h \in \partial R^N_+$ then an indifference surface of a commodity bundle of positive utility $u_h(x^*_h)$ intersects ∂R^N_+ at $x^*_h \in \partial R^N_+$. Let r be the ray in ∂R^N_+containing x^*_h. If $w \in r$ then $\langle q^*, w\rangle = 0$, because $\langle q^*, x^*_h\rangle = 0$. Since $u_h(x^*_h) > 0$, by Proposition 3 u_h strictly increases along r, so that $w \in G_h(x^*_h)$. But this contradicts the choice of q^* as a supporting price satisfying limited arbitrage (14) since

$$\exists h \text{ and } w \in G_h(\Omega_h) \text{ such that } \langle q^*, w\rangle = 0. \tag{15}$$

The contradiction between (15) and (14) arose from the assumption that (q^*, x^*) is not a competitive equilibrium, so that (q^*, x^*) must be a competitive equilibrium, and the proof is complete. ■

5 Economies with Infinitely Many Markets

The results of Theorem 2 are also valid for infinitely many markets. As already seen, the existence of inner products is useful in defining limited arbitrage. For this reason and because of the natural structure of prices in Hilbert spaces, I work on a Hilbert space of commodities in which inner products are defined.

5.1 Hilbert Spaces and the Cone Condition

All Hilbert spaces have positive orthants with empty interior. This can make things difficult when seeking to prove the existence of an equilibrium, which depends on finding supporting prices for efficient allocations. Supporting prices are usually found by applying the Hahn-Banach theorem, and without such prices a competitive equilibrium does not exist. Therefore the Hahn Banach theorem is crucial for proving existence of an equilibrium. However this theorem requires that the convex set being supported has a non-empty interior, a condition which is never satisfied within the positive orthant of a Hilbert space. This problem, which is typical in infinite dimensional spaces, was solved in 1980 by Chichilnisky and Kalman [21] who introduced a condition on preferences, the **cone condition** (**C-K**,) and proved that it is necessary and sufficient for separating convex sets with or without non-empty interior, thus extending Hahn Banach's theorem to

encompass all convex sets, whether or not they have an empty interior. Since its introduction the **C-K cone condition** has been used extensively to prove the existence of a market equilibrium and in game theory; it is now a standard condition of economies with infinitely many markets and is known also under the name of "properness", cf. [19].

In addition to the cone condition, one more result is needed to extend directly the proof of Theorem 2 to economies with infinitely many markets: the compactness of the Pareto frontier. Recall that this frontier is always a finite dimensional object when there are a finite number of traders: it is contained in R^H_+, where H is the number of traders.

5.2 Limited Arbitrage and the Cone Condition

A somewhat unexpected result is that limited arbitrage implies the **C-K cone condition** on efficient allocations [21], cf. [23]. Because of this, limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium and the core, with or without short sales, in the infinite dimensional space H. Limited arbitrage therefore unifies the treatment of finitely and infinitely many markets.

Consider an economy E as defined in Section 2 except that here $X =$ H or $X =$ H$^+$; more general convex sets can be considered as well, see [24]. The global cones and the market cones, and the limited arbitrage condition, are the same as defined in the finite dimensional cases when $X = R^N$ and $X = R^N_+$ respectively. To shorten the presentation, here the market cones are assumed to be uniform across initial endowments, a condition which is automatically satisfied under Assumption 1 when $X =$ H, and which is not needed for the main results, cf. [24]. Therefore here either limited is satisfied at every endowment or not at all. The results on existence of an equilibrium presented below are due to Heal and Chichilnisky, see [24].

Definition 10 *The cone defined by a convex set $D \subset X$ at a point $x \in D$ is $C(D, x) = \{z \in X : z = x + \lambda(y - x)$, where $\lambda > 0$ and $y \in D\}$.*

Definition 11 *A convex set $D \subset$ X satisfies the* **C-K cone condition** *of [21] at $x \in$D when there exists a vector $v \in$X which is at positive distance $\varepsilon(D, x)$ from the cone with vertex x defined by the set $D, C(D, x)$.*

Definition 12 *A* **preference** *$u_h : X \to R$ satisfies the* **C-K cone condition** *of [21] (also known in subsequent work as "properness") when for every $x \in$X, the preferred set $u^x_h = \{y : u_h(y) \geq u_h(x)\} \subset X$ of u_h at x satisfies the C-K condition, and $\varepsilon(P_x, x)$ is independent of x.*

Theorem 3 *(Chichilnisky-Heal) Consider an economy E as defined in Section 2, where the trading space is either $X =$ H$^+$, or $X =$ H, and where H is a Hilbert space of finite* *or infinite dimensions. Then limited arbitrage implies the* **C-K** *[21] cone condition. In particular, the second welfare theorem applies under limited arbitrage: a Pareto efficient allocation is also a competitive equilibrium.*

Proof. See also [23]. When $X = \mathsf{H}$, limited arbitrage means that there exists a non-zero price $p \in \mathsf{H}$ which supports every cone G_h i.e. $\forall h = 1, 2..., H$:

$$\langle p, x_h \rangle > 0 \; \forall x_h \in G_h; \tag{16}$$

when $X = H^+$, limited arbitrage means that there exists a price p and an efficient affordable allocation $x = x_1, ..., x_h \in S(\mathsf{E})$ such that either

$$\langle p, \Omega_h \rangle > 0 \text{ for all } h, \text{ or}$$

$$\langle p, x_h \rangle > 0 \; \forall x_h \in G_h,$$

see Section 2. To prove that the cone condition is satisfied for a given trader h, it suffices to establish that the preferred set of u_h at x_h, $u_h^x = \{y \in X \colon u_h(y) \geq u_h(x)\}$, can be supported by a non-zero price p, see Chichilnisky and Kalman, [21] p. 25, Theorem 2.1, and [19] p. 180.

First consider $X = \mathsf{H}$. Recall that by Proposition 2, $B_i \cup C_i = \partial G_i \subset \overline{G_i}$, which, by (16) implies that p supports the preferred set $u_h^x = \{y \in H^+ : u_h(y) \geq u_h(x)\}$ corresponding to the single utility value $u_h(x)$ which is associated with all the directions in the set $B_i \cup C_i$: there is such a utility value and it is unique by Proposition 2 part (iii). Therefore p supports at least one preferred set of trader i and therefore, for that preferred set, the vector $p = p_h$ is at a positive distance, say ε_h, from the corresponding cone $C(u_h^x, x)$, as we wished to prove.

Next consider $X = \mathsf{H}^+$: we will show first that limited arbitrage, as defined in Section 2, implies that there exists a vector $p \neq 0$ in $S(\mathsf{E})$. The proof is by contradiction. If $\sim \exists p \neq 0$ in $S(\mathsf{E})$, then the intersection of the dual cones in Definition 6 must be empty, i.e. $\cap_{h=1}^{H} D_h^+ = \emptyset$: this occurs either because for some h, the set $D_h^+ = D_h \cap S(\mathsf{E})$ is empty, or aalternaitvely because the set $S(\mathsf{E})$ itself is empty. In either case this leads to a contradiction with limited arbitrage which requires that $\cap_{h=1}^{H} D_h^+ \neq \emptyset$. Since the contradiction arises from assuming that $\sim \exists p \neq 0$ in $S(\mathsf{E})$, it follows that $\exists p \in S(\mathsf{E}), p \neq 0$, i.e. the preferred set of u_h can be supported by a non-zero price$p$ at some x_h which is part of a feasible affordable efficient and individually rational allocation, $x = x_1, ..., x_H$. Furthermore, since in this section limited arbitrage is satisfied simultaneously at every initial endowment, this completes the proof that the cone condition is always satisfied when $X = \mathsf{H}^+$, as we wished to prove.

The next step is to show that for all $x \in \mathsf{H}$, i.e. the support p is at a positive distance from $C(u_h^x, x)$. By Proposition 1, any two indifference surfaces of u_h are at a uniform

distance from each other. Since p supports u_h^x then p supports any translation of u_h^x. But any other preferred set u_h^y, $\forall y \in \mathsf{H}$, is contained in a translation of u_h^x. Therefore p necessarily supports any other preferred set u_h^y ,$\forall y \in \mathsf{H}$. It follows that all preferred sets of trader h admit the same support, of course, each at different points. Therefore $p = p_h$ is at a positive distance from $C(u_h^x, x)$, for all $x \in \mathsf{H}$, as we wished to show.

The last step is to show that there exists one vector v, the same for all traders, which is at a positive distance ε from $C(u_h^x, x)$ *for every trader* h as well as for every $x \in X$. Consider now the vector $v = \sum_{h=1}^{H} p_h$, where p_h is the support whose existence was established above, and let $\varepsilon = \min_{i=1,2,...,H}\{\varepsilon_i\}$. By (16) the vector v satisfies the definition of the cone condition C-K.[46] ■

Theorem 4 *(Chichilnisky-Heal) Consider an economy* E *as defined in Section 2, where* $X = \mathsf{H}$, *or* $X = \mathsf{H}^+$, *where* H *is a Hilbert space of finite or infinite dimensions. Then limited arbitrage is necessary and sufficient for the compactness of the Pareto frontier.*

Proof. Since the cone condition holds, the proof is a straightforward extension of Theorem 1 which holds for the finite dimensional case. See also [23]. ■

Theorem 5 *(Chichilnisky-Heal) Consider an economy* E *as defined in Section 2, where* $X = \mathsf{H}^+$ *or* $X = \mathsf{H}$, *a Hilbert space of finite or infinite dimensions. Then limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium.*

Proof. The proof is the same as in the finite dimensional case, see also [23]. ■

5.3 Subeconomies with Competitive Equilibria

The condition of limited arbitrage need not be tested on all traders simultaneously: in the case of R^N, it needs only be satisfied on subeconomies with no more traders than the number of commodities in the economy,[47] N, plus one.

Definition 13 *A* $k-$*trader sub-economy of* E *is an economy* F *consisting of a subset of* $k \leq H$ *traders in* E, *each with the endowments and preferences as in* E: $\mathsf{F} = \{X, u_h, \Omega_h, h \in J \subset \{1, ..., H\}$, *cardinality* $(J) = k \leq H\}$.

Theorem 6 *The following four properties of an economy* E *with trading space* R^N *are equivalent:*

[46] Obviously the supports can be achieved at different points for each trader, and these points may or not define feasible allocations. Therefore the fact that all indifferences of all traders admit the same support does not by itself imply the existence of an equilibrium.

[47] A similar result can be obtained for R_+^N, but requires a slightly longer statement and proof, since in R_+^N the market cone of each trader may vary as the number of traders changes.

(i) E *has a competitive equilibrium*

(ii) Every sub economy of E *with at most* $N+1$ *traders has a competitive equilibrium*

(iii) E *has limited arbitrage*

(iv) E *has limited arbitrage for any subset of traders with no more that* $N+1$ *members.*

Proof. Theorem 1 implies (i)$\Leftrightarrow$(iii) and (ii)$\Leftrightarrow$(iv). That (iii)$\Leftrightarrow$(iv) follows from the following theorem derives from Helly's theorem, which is a corollary in Chichilnisky [12]: Consider a family $\{U_i\}_{i=1...H}$ of convex sets in R^N, $H, N \geq 1$. Then

$$\bigcap_{i=1}^{H} U_i \neq \emptyset \text{ if and only if} \bigcap_{j \in J} U_i \neq \emptyset$$

for any subset of indices $J \subset \{1...H\}$ having at most $N+1$ elements.

In particular, an economy E as defined in Section 2 satisfies limited arbitrage, if and only if it satisfies limited arbitrage for any subset of $k = N+1$ traders, where N is the number of commodities in the economy E. ■

6 Limited Arbitrage Equilibrium and The Core with Finitely or Infinitely Many Markets

The following result proves that limited arbitrage is necessary and sufficient for the nonemptiness of the core[48] and shows the equivalence between equilibrium and the core.

Theorem 7 *Consider an economy* $E = \{X, u_h, \Omega_h, h = 1, ..., H\}$, *where* $H \geq 2$, $X = R^N$ *and* $N \geq 1$, *or* X *is a Hilbert space* H. *Then the following four properties are equivalent:*

(i) The economy E *has limited arbitrage*

(ii)The economy E *has a core*

(iii) The economy E *has a competitive equilibrium*

(iv) When $X = R^N$, *every subeconomy of* E *with at most* $N+1$ *trades has a core*

Proof. Necessity first. If a core allocation exists, then it is Pareto efficient, and, by the second welfare theorem, this allocation must be a competitive equilibrium for some initial endowments—for the infinite dimensional case, this is Theorem 3 above. Since by Proposition 2 limited arbitrage is satisfied simultaneously for all initial endowments, and

[48] I proved this result in the finite dimensional case while at Stanford University in the Spring and Summer of 1993, stimulated by conversations with Curtis Eaves; it was written in the Fall of 1993 in [17]. I presented this result and its proof, and circulated the paper with its proof [17] at the January 3-5, 1994 Meetings of the Econometric Society in Boston.

limited arbitrage is necessary for the existence of an equilibrium by Theorems 2 and 5, then limited arbitrage must be satisfied in E.

Sufficiency is immediate from Theorem 2, because a competitive equilibrium is always in the core. Therefore I have established (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii).

Finally I establish (iv)$\Leftrightarrow$(i): this follows directly from the fact that an economy has a competitive equilibrium if and only if every subeconomy of at most $N+1$ traders has limited arbitrage, Theorem 6. ■

7 Social Diversity and the Supercore

The supercore was defined and motivated in Section 1.2. It measures the extent to which a society has reasons to stay together.

Up to now I have considered convex economies. In this section I consider instead the case where the economy E may have either convex or non-convex preferences; otherwise the economy is as in Section 2. Short sales are allowed, so that the trading space is $X = R^N$. To simplify notation I assume without loss of generality that all endowments are zero, $\forall h$, $\Omega_h = 0$.

Definition 14 *An economy* E *is socially diverse when it does not satisfy limited arbitrage. When* $X = R^N$, *this means:*

$$\bigcap_{h=1}^{H} D_h = \emptyset.$$

When $X = R^N_+$:

$$\bigcap_{h=1}^{H} D^+_h(\Omega_h) = \emptyset.$$

Social diversity comes in many shades, one of which, the mildest possible, will be used to establish the existence of a supercore. Assume now that the gradients of the indifference surfaces of preferences are closed sets so that[49] $G_h = A_h$.

Definition 15 E *has social diversity of type 1, or* SD1, *when all sub economies with at most* $H-1$ *traders have limited arbitrage, but* E *does not.*

Theorem 8 *Consider an economy* E *with at least three traders, where the traders' preferences may or not be convex. Then if* E *has social diversity of type 1,* SD1, *its supercore is not empty.*

[49] The definitions of A_h and D_h require no convexity of preferences; if the preferences are not convex the global cones may not be convex either.

Proof. To simplify notation, assume without loss of generality that $\forall h$, $\Omega_h = 0$. Since the economy has social diversity of type 1, every subeconomy of $H-1$ traders satisfies limited arbitrage, which by Proposition 4 implies that gains from trade G(E) are bounded in every $H-1$ trader subeconomy. In particular, there is a maximum level of utility which each trader can obtain by him or herself, and the same is true for any subgroup consisting of at most $H-1$ traders.

However, by Proposition 4, gains from trade cannot be bounded in E for the set of all H traders, since E does not satisfy limited arbitrage. ■

8 Limited Arbitrage and Social Choice

Limited arbitrage is also crucial for achieving resource allocation via social choice. Two main approaches to social choice are studied here. One is Arrow's [1]: his axioms of social choice require that the social choice rule Φ be non-dictatorial, independent of irrelevant alternatives, and satisfy a Pareto condition. A second approach requires, instead, that the rule Φ be continuous, anonymous, and respect unanimity, Chichilnisky [7] [9]. Both approaches have led to corresponding impossibility results [1], and [7] [9]. Though the two sets of axioms are quite different, it has been shown recently that the impossibility results which emerge from them are equivalent, see Baryshnikov [4]. Furthermore, as is shown below, limited arbitrage is closely connected with both sets of axioms. Economies which satisfy limited arbitrage admit social choice rules with either set of axioms. Therefore, in a well defined sense, the social choice problem can only be solved in those economies which satisfy limited arbitrage.

How do we allocate resources by social choice? Social choice rules assign a social preference $\Phi(u_1...u_H)$ to each list $(u_1...u_H)$ of individual preferences of an economy E.[50] The social preference ranks allocations in $R^{N\times H}$, and allows to select an optimal feasible allocation. This is the resource allocation obtained via social choice.

The procedure requires, of course, that a social choice rule Φ exists: the role of limited arbitrage is important because it ensures existence. This will be established below. I prove here that limited arbitrage is necessary and sufficient for resolving Arrow's paradox when the domain of individual preferences are those in the economy, and the choices are those feasible allocations which give large utility value.[51]

Limited arbitrage provides a restriction on the relationship between individual preferences under which social choice rules exist. A brief background on the matter of preference diversity follows.

[50] In the economy E the traders' preferences are defined over private consumption $u_i : R^N \to R$, but they define automatically preferences over allocations in $R^{N\times H}$: $u_i(x_1...x_H) \geq u_i(y_1...y_H) \Leftrightarrow u_i(x_i) \geq u_i(y_i)$.

[51] See also [11].

Arrow's impossibility theorem established that in general a social choice rule Φ does not exist: the problem of social choice has no solution unless individual preferences are restricted. Duncan Black [5] established that the "single peakedness" of preferences is a sufficient restriction to obtain majority rules. Using different axioms, Chichilnisky [7] [9] established also that a social choice rule Φ does not generally exist; subsequently Chichilnisky and Heal [22] established the first necessary and sufficient restriction for the resolution of the social choice paradox: the *contractibility* of the space of preferences.[52] Contractibility can be interpreted as a limitation on preference diversity, Heal [31]. In all cases, therefore, the problem of social choice is resolved by restricting the diversity of individual preferences. The main result in this section is that the restriction on individual preferences required to solve the problem is precisely limited arbitrage. The connection between limited arbitrage and contractibility is discussed below.

The section is organized as follows. First I show in Proposition 6 that the economy E satisfies limited arbitrage if and only if it contains no *Condorcet cycles* on choices of large utility values.[53] Condorcet cycles are the building blocks of Arrow's impossibility theorem, and are at the root of the social choice problem. On the basis of Proposition 6, I prove in Theorem 9 that limited arbitrage is necessary and sufficient for resolving Arrow's paradox [1] on allocations of large utility values.

Definition 16 *A Condorcet cycle is a collection of three preferences over a choice set X, represented by three utilities $u_i : X \to R$, $i = 1, 2, 3$, and three choices α, β, γ within a feasible set $Y \subset X$ such that $u_1(\alpha) > u_1(\beta) > u_1(\gamma)$, $u_2(\gamma) > u_2(\alpha) > u_2(\beta)$ and $u_3(\beta) > u_3(\gamma) > u_3(\alpha)$.*

Within an economy with finite resources $\Omega >> 0$, the social choice problem is about the choice of allocations of these resources. Choices are in $X = R^{N\times H}$. An allocation $(x_1...x_H) \in R^{N\times H}$ is *feasible* if $\sum_i x_i - \Omega = 0$. Consider an economy E as defined in Section 2. Preferences over private consumption are increasing, $u_h(x) > u_h(y)$ if $x > y \in R^N$, utilities are uniformly non-satiated (Assumption 1), and indifference surfaces which are not bounded below have a closed set of gradients,[54] so that $G_h = A_h$. While the preferences in E are defined over private consumption, they naturally define preference over allocations, as follows: define $u_h(x_1...x_H) \geq u_h(y_i...y_H) \Leftrightarrow u_h(x_h) \geq u_h(y_h)$. Thus the preferences in the economy E induce naturally preferences over the feasible allocations in E.

[52] A space X is contractible when there exists a continuous map $f : X \times [0,1] \to X$ and $x_o \in X$ such that $\forall x$, $f(x, 0) = x$ and $f(x, 1) = x_o$.

[53] The concept of "large utility values" is purely ordinal; it is defined relative to the maximum utility value achieved by a utility representation.

[54] If indifferences are bounded below, nothing is required of the sets of gradients. These conditions can be removed, but at the cost of more notation.

Definition 17 *The family of preferences $\{u_1...u_H\}$, $u_h : R^N \to R$ of an economy* E *has a Condorcet cycle of size k if for every three preferences $u_1^k, u_2^k, u_3^k \in \{u_1...u_H\}$ there exists three feasible allocations $\alpha^k = (\alpha_1^k, \alpha_2^k, \alpha_3^k) \in X^{3\times H} \subset R^{3\times N\times H}; \beta^k = (\beta_1^k, \beta_2^k, \beta_3^k)$ and $\gamma^k = (\gamma_1^k, \gamma_2^k, \gamma_3^k)$ which define a Condorcet cycle, and such that each trader $h = 1,...,H$, achieves at least a utility level k at each choice:*

$$\min_{h=1,...,H}\{[u_h^k(\alpha_h^k), u_h^k(\beta_h^k), u_h(\gamma_h^k)]\} > k.$$

The following shows that limited arbitrage eliminates Condorcet cycles on matters of great importance, namely on those with utility level approaching the supremum of the utilities, which for simplicity and without loss of generality we have assumed to be ∞ :

Proposition 6 *Let* E *be a market economy with short sales ($X = R^N$) and $H \geq 3$ traders. Then* E *has social diversity if and only if its traders' preferences have Condorcet cycles of every size. Equivalently,* E *has limited arbitrage if and only for some $k > 0$, the traders' preferences have no Condorcet cycles of size larger than k.*

Proof. Consider an economy with Condorcet cycles of all sizes. For each $k > 0$, there exists three allocations denoted $(\alpha^k, \beta^k, \gamma^k) \in R^{3\times N\times H}$ and three traders $u_{k_1}^k, u_{k_2}^k, u_{k_3}^k \subset \{u_1...u_H\}$ which define a Condorcet triple of size k. By definition, for every k, each of the three allocations is feasible, for example, $\alpha^k = (\alpha_1^k, ..., \alpha_H^k) \in R^{N\times H}$, and $\sum_{i=1}^H(\alpha_i^k) = 0$. Furthermore $\min_{h=1,...,H}\{[u_h^k(\alpha_h^k), u_h^k(\beta_h^k), u_h(\gamma_h^k)]\} > k$, so that e.g. $\lim_{k\to\infty}(u_h(\alpha_h^k)) = \infty$. There exist therefore a sequence of allocations $(\theta^k)_{k=1,2...} = (\theta_1^k, ..., \theta_H^k)_{k=1,2...}$ such that $\forall k$, $\sum_{h=1}^H \theta_h^k = 0$ and $\forall h$ $\sup_{k\to\infty}(\inf_{i=1,2,3}(u_h(\theta_h^k)) = \infty$. This implies that E has unbounded gains from trade, which contradicts Proposition 3. Therefore E cannot have Condorcet cycles of every size.

Conversely, if E has no limited arbitrage, for any $k > 0$, there exist a feasible allocation $(a_1^k, a_2^k, ..., a_H^k)$, such that $\sum_{h=1}^H a_h^k \leq 0$, and $\forall h$, $u_h(a_h^k) \geq k$. For each integer $k > 0$, and for a small enough $\epsilon > 0$ define now the vector $\Delta = (\epsilon, ..., \epsilon) \in R_+^N$ and the following three allocations: $\alpha^k = (ka_1^k, ka_2^k - 2\Delta, ka_3^k + 2\Delta, ka_4^k, ..., ka_H^k)$, $\beta^k = (ka_1^k - \Delta, ka_2^k, ka_3^k + \Delta, ka_4^k, ..., ka_H^k)$ and $\gamma^k = (ka_1^k - 2\Delta, ka_2^k - \Delta, ka_3^k + 3\Delta, ka_4^k, ..ka_H^k)$. Each allocation is feasible, e.g. $ka_1^k + ka_2^k - 2\Delta + ka_3^k + 2\Delta + ka_4^k + ... + ka_H^k = k(\sum_{h=1}^H a_h^k) \leq 0$. Furthermore for each $k > 0$ sufficiently large, the three allocations $\alpha^k, \beta^k, \gamma^k$ and the traders $h = 1, 2, 3$, define a Condorcet cycle of size k : all traders except for $1, 2, 3$, are indifferent between the three allocations and they reach a utility value at least k, while trader 1 prefers α^k to β^k to γ^k, trader 3 prefers γ^k to α^k to β^k, and trader 2 prefers β^k to γ^k to α^k. Observe that this construction can be made for any three traders within the set $\{1, 2, ..., H\}$. This completes the proof. ■

The next result uses Proposition 6 to establish the connection between limited arbitrage and Arrow's theorem. Consider Arrow's three axioms: Pareto, independence of

irrelevant alternatives, and non-dictatorship. The social choice problem is to find a social choice rule $\Phi : P^j \to P$ from individual to social preferences satisfying Arrow's three axioms; the domain for the rule Φ are profiles of individual preferences over allocations of the economy E: $\Phi : P^j \to P$. Recall that each preference in the economy E defines a preference over feasible allocations in E.

Definition 18 *The economy* E *admits a resolution of Arrow's paradox if for any number of voters $j \geq 3$ there exists a social choice function from the space $P = \{u_1, ..., u_H\}$ of preferences of the economy* E *into the space Q of complete transitive preference defined on the space of feasible allocations of* E, *$\Phi : P^j \to Q$, satisfying Arrow's three axioms.*

Definition 19 *A feasible allocation $(\alpha_1^k, ..., \alpha_H^k) \in R^{N\times H}$ has utility value k, or simply value k, if each trader achieves at least a utility level k :*

$$\min_{h\in H}\{[u_1^k(\alpha_1^k), ..., u_H(\alpha_H^k)]\} > k.$$

Definition 20 *Arrow's paradox is said to be resolved on choices of large utility value in the economy* E *when for all $j \geq 3$ there exists social choice function $\Phi : P^j \to Q$ and a $k > 0$ such that Φ is defined on all profiles of j preferences in* E, *and it satisfies Arrow's three axioms when restricted to allocations of utility value exceeding k.*[55]

Theorem 9 *Limited arbitrage is necessary and sufficient for a resolution of Arrow's paradox on choices of large utility value in the economy* E.

Proof. Necessity follows from Proposition 6, since by Arrow's axiom of independence of irrelevant alternatives, the existence of one Condorcet triple of size k suffices to produce Arrow's impossibility theorem on feasible choices of value k in our domain of preferences, see [1]. Sufficiency is immediate: limited arbitrage eliminates feasible allocation of large utility value by Proposition 1, because it bounds gains from trade. Therefore it resolves Arrow's paradox, because this is automatically resolved in an empty domain of choices. ■

8.1 Social choice rules which are continuous, anonymous and respect unanimity

Consider now to the second approach to social choice, Chichilnisky [7] [9], which seeks continuous anonymous social choice rules which respect unanimity. The link connecting arbitrage with social choices is still very close but it takes a different form. In this case the connection is between the contractibility of the space of preferences, which is necessary

[55] Recall that we have assumed, without loss, that $\sup_{x\in X} u_h(x) = \infty$. Otherwise the same statement holds by replacing "$> k$" by "$> \sup_{x\in X} u_h(x) - k$."

and sufficient for the existence of continuous, anonymous rules which respect unanimity [22], and limited arbitrage.

Continuity is defined in a standard manner; anonymity means that the social preference does not depend on the order of voting. Respect of unanimity means that if all individuals have identical preferences overall, so does the social preference; it is a very weak version of the Pareto condition. It was shown in [7] [9] that, for general spaces of preferences, there exist no social choice rules satisfying these three axioms. The following result [22] established that contractibility is exactly what is needed for the existence of social choice rules. It is worth observing that *the following result is valid for any topology on the space of preferences T*. In this sense this result is analogous to a fixed point theorem or to a maximization theorem: whatever the topology, a continuous function from a compact convex space to itself has a fixed point. A continuous function of a compact set has a maximum. All these statements, and the one below, apply independently of the topology chosen:

Theorem 10 *Let T be a connected space of preferences endowed with any topology.*[56] *Then T admits a continuous anonymous map Φ respecting unanimity*

$$\Phi : T^k \to T$$

for every $k \geq 2$, if and only if T is contractible.

Proof. See Chichilnisky and Heal [22]. ■

The close relation between contractibility and non-empty intersection (which is limited arbitrage) follows from the following theorem:

Theorem 11 *Let $\{U_i\}_{i=1...I}$ be a family of convex sets in R^N. The family has a non-empty intersection if and only if every subfamily has a contractible union:*

$$\bigcap_{i=1}^{I} U_i \neq \phi \Leftrightarrow \bigcup_{i \in J} U_i \text{ is contractible } \forall J \subset \{1...I\}.$$

Proof. See Chichilnisky [12]. ■

This theorem holds for more general families of sets, such as acyclic families, and even for *simple families* which consist of sets which need not be convex, acyclic open or even connected. This theorem was shown to imply the Knaster Kuratowski Marzukiewicz theorem and Brouwer's fixed point theorem [12], but it is not implied by them. Theorem 11 establishes a close link between contractibility and non-empty intersection and is used to show that limited arbitrage, or equivalently the lack of social diversity, is necessary and sufficient for resource allocation via social choice rules. Formally:

[56] T could be the space of linear preferences on R^N or the space of strictly convex preferences on R^N, or the space of all smooth preferences. T could be endowed with the closed convergence topology, or the smooth topology, or the order topology, etc. T must satisfy a minimal regularity condition, for example to be locally convex (every point has a convex neighborhood) or, more generally, to be a parafinite CW complex. This is a very general specification, and includes all the spaces used routinely in economics, finite or infinite dimensional, such as all euclidean spaces, Banach and Hilbert spaces, manifolds, all piece-wise linear spaces, polyhedrons, simplicial complexes, or finite or infinite dimensional CW spaces.

Definition 21 *Let P consist of all those preferences which are* similar *to those of the market economy* E, *in the sense that their gradients are in the intersection of the market cones of the traders, see [20],[12], [11].*

Intuitively, a preference is similar to that of trader h when it prefers those allocations which assign h a consumption vector which u_h prefers. In mathematical terms this means a space of preferences which have gradients within unions of subfamilies of the market cones of the economy. Formally, let the space of choices be R^N and consider a connected[57] space of preferences

$$P_J = \{u : u \text{ defines a preference on } R^N \text{ satisfying Assumption 1, and}$$

$$\exists J \subset \{1, ..., H\} : \forall x \in R^N, Du(x) \in \cup_{h \in J} D_h\}.$$

Theorem 12 *The economy E satisfies limited arbitrage if and only if for any subset of traders $J \subset \{1, 2, ..., H\}$ the union of all market cones $\cup_{h \in J} D_h$ is contractible.*

Proof. This follows directly from Theorem 11. ■

Theorem 13 *A continuous anonymous social choice rule $\Phi : P_J^k \to P_J$ which respects unanimity exists which for every $k \geq 2$ and every $J \subset \{1, ..., H\}$ if and only if the economy E has limited arbitrage, i.e. if and only if the economy has a competitive equilibrium and a non-empty core.*

Proof. See [11], Theorems 2,7, 10 and 12. ■

9 Social Diversity and Limited Arbitrage

If the economy does not have limited arbitrage, it is called **socially diverse**:

Definition 22 *The economy* E *is* **socially diverse** *when $\bigcap_{h=1}^{H} D_h = \phi$.*

[57] Since we apply Theorem 11, we require that the space of preferences P_J be connected. In a market economy, this requires that every two traders have a reason to trade, but says nothing about sets of three or more traders, nor does it imply limited arbitrage.

This concept is robust under small errors in measurement and is independent of the units of measurement or choice of numeraire. If E is not socially diverse, all economies sufficiently close in endowments and preferences have the same property: the concept is **structurally stable**. Social diversity admits different "shades"; these can be measured, for example, by the smallest number of market cones which do not intersect:

Definition 23 *The economy* E *has* **index of diversity** $I(\mathsf{E}) = H - K$ *if* $K+1$ *is the smallest number such that* $\exists J \subset \{1...H\}$ *with cardinality of* $J = K+1$, *and* $\bigcap_{h \in J} D_h = \phi$. *The index* $I(\mathsf{E})$ *ranges between* 0 *and* $H-1$: *the larger the index, the larger the social diversity. The index is smallest when all the market cones intersect: then all social diversity disappears, and the economy has limited arbitrage.*

Theorem 14 *The index of social diversity is* $I(E)$ *if and only if* $H - I(E)$ *is the maximum number of traders for which every subeconomy has a competitive equilibrium, a non-empty core, admits social choice rules which satisfy Arrow's axioms on choices giving large utility values, and admits social choice rules which are continuous, anonymous and respect unanimity on preferences similar to those of the subeconomy.*

10 A topological invariant for the market E

This section shows that the resource allocation properties of the economy E can be described simply in terms of the properties of a family of cohomology rings denoted $CH(\mathsf{E})$.

A ring is a set Q endowed with two operations, denoted $+$ and $\times$; the operation $+$ must define a group structure for Q (every element has an inverse under $+$) and the operation $\times$ defined a semi group structure for Q; both operations together satisfy an associative relation. A typical example of a ring is the set of the integers, as well as the rational numbers, both with addition and multiplication.

The *cohomology ring* of a space Y contains information about the space's topological structure, namely those properties of the space which remain invariant when the space is deformed as if it was made of rubber. For a formal definition see [40]. An intuitive explanation is as follows. The cohomology ring consists of maps defined on homology groups. Intuitively a homology group consists of "holes", defined as cycles which do not bound any region in the space Y. The homology groups are indexed by dimension. For example the circle S^1 has the simplest possible "hole": its first homology group measures that. The "torus" $S_1 \times S_1$ has two types of holes: therefore it has a non-zero first homology group as well as a non-zero second homology group. Any convex, or contractible, space has no "holes" so that its cohomology groups are all zero. In addition to the standard group structure of each cohomology group, there is another operation, called a "cup product",

which consists, intuitively, of "patching up" elements across cohomology groups. The set of all cohomology groups with these two operations defines the cohomology ring.

The rings $CH(\mathsf{E})$ are the cohomology rings corresponding to subfamilies of market cones $\{D_h\}$ of the economy E. define a topological invariant of the economy E in the sense that they are the same for any continuous deformation of the space of commodities on which the economy is defined, i.e. they are preserved under any continuous transformation in the units of measurement of the commodities. They are also preserved under small perturbations of, or measurement errors on, the traders' preferences.

Definition 24 *The* **nerve** *of a family of subsets* $\{V_i\}_{i=1,...,L}$, *in* R^M.[58] *denoted*

$$nerve\{V_i\}_{i=1,...,L}$$

is a simplicial complex defined as follows: each subfamily of $k+1$ *sets in* $\{V_i\}_{i=1,...,L}$ *with non-empty intersection is a* $k-$ *simplex of the* $nerve\{V_i\}_{i=1,...,L}$.

The **topological invariant** $CH(\mathsf{E})$ of the economy E is the family of cohomology rings[59] of the simplicial complexes defined by all subfamilies $\{F_h\}$of the family $\{D_h\}_{h=1,2...H}$, i.e. the cohomology rings of $\{nerve\{F_h\}_{h=1...H}$ where $\{F_h\} \in \{D_h\}\}$:

$$\{H^*(nerve\{F_h\}, \forall \{F_h\} \in \{D_h\}\}$$

For the following result I consider continuous deformations of the economy which preserve its convexity and Assumption 1.

Theorem 15 *The economy* E *with* H *traders has limited arbitrage, and therefore a competitive equilibrium, a non-empty core and social choice rules if and only if:*

$$CH(\mathsf{E}) = 0$$

$$i.e. \forall \{F_h\} \in \{D_h\} \ \mathsf{H}^*(nerve\{F_h\}) = 0$$

Furthermore, the economy E has social diversity index $I(\mathsf{E})$ if and only if $I(E) = H - K$, where K satisfies the following conditions: (i) for every $\{F_h\} \in \{D_h\}$ of cardinality at most K

$$\mathsf{H}^*(nerve\{F_h\}) = 0,$$

and there exists $\{J_h\} \in \{D_h\}$ with cardinality$\{J_h\} = K+1$ and

$$\mathsf{H}^*(nerve\{J_h\}) \neq 0.$$

[58] $\forall i, V_i \subset R^M$.

[59] With integer coefficients.

Proof. This follows directly from Chichilnisky [12]. ■

11 Related Literature on Market Equilibrium

The literature on the existence of a competitive (Walrasian) market equilibrium is about forty years old, starting with the classic works of Von-Neumann, Nash, Arrow and Debreu and others. This literature has focused on sufficient conditions for existence rather than on necessary and sufficient conditions as studied here. It can be reviewed in two parts: markets without short sales, such as those studied by Arrow and Debreu, and markets with short sales which appear in the literature on financial markets.

11.1 Related Literature on Equilibrium with Bounds on Short Sales

Two well known conditions are sufficient for the existence of an equilibrium[60] when $X = R^N_+$. They are Arrow's "resource relatedness" condition [2], and McKenzie's "irreducibility" condition [33] [34] [35]; both are sufficient but neither is necessary for existence. Both imply limited arbitrage, which is simultaneously necessary and sufficient for the existence of a competitive equilibrium. Resource relatedness and irreducibility ensure existence by requiring that the endowments of any trader are desired, directly or indirectly, by others, so that the treaders' incomes cannot fall to zero. Under these conditions it is easy to check that limited arbitrage is always satisfied, and a competitive equilibrium always exists. Yet traders with zero or minimum income do not by themselves rule out the existence of a competitive equilibrium. Limited arbitrage could be satisfied even when some traders have zero income. This reflects a real situation: some individuals are considered economically worthless, in that they have nothing to offer that others want in a market context. Such a situation could be a competitive equilibrium. Figure 8 provides an example. It seems realistic that markets could lead to such allocations: one observes them all the time in city ghettos. Limited arbitrage does not attempt to rule out individuals with minimum (or zero) income; instead, it seeks to determine if society's evaluation of their worthlessness is shared. Individuals are diverse in the sense of not satisfying limited arbitrage, when someone has minimal (or zero) income, and, in addition, when there is no agreement about the value of those who have minimal income. In such cases there is no competitive equilibrium.

Another condition which is sufficient but not necessary for existence of a competitive

[60] Not all Arrow-Debreu exchange economies have a competitive equilibrium, even when all individual preferences are smooth, concave and increasing, and the consumption sets are positive orthants, $X = R^N_+$, see for example Arrow and Hahn [2], Chapter 4, p. 80.

equilibrium is that the indifference surfaces of preferences of positive consumption bundles should be in the interior of the positive orthant, Debreu [25]: this implies that the set of directions along which the utilities increase without bound from initial endowments is the same for all traders. Therefore all individuals agree on choices with large utility values, again a form of similarity of preferences. It is immediate to see that such economies satisfy limited arbitrage.

11.2 Related Literature on Equilibrium in Markets With Short Sales

The literature of general equilibrium with short sales has concentrated so far on sufficient conditions for existence, for example Hart [30], Kreps [32] Hammond [29], Chichilnisky and Heal [24], and not on the question of conditions which are simultaneously necessary and sufficient for existence as studied here. In addition, the literature has neglected so far to consider economies where the feasible allocations do not form a compact set. All previous sufficient results for the existence of an equilibrium in economies with short sales rely on the fact that the set of feasible allocations is compact. Except for the results in Chichilnisky [13] [15], [17], [14], [16], and [23], there are no results on existence of equilibrium in economies with unbounded feasible allocations, not even sufficient conditions for existence. Theorem 2 above (see also Chichilnisky [13] [15], [17], [14], [16], and [23]) is original in that it provides conditions which are necessary and sufficient for existence in economies where feasible allocations may be bounded or unbounded; in addition these results are novel in that they result apply in economies with or without short sales, and for finitely or infinitely many markets.

In the context of temporary equilibrium models, which are different from Arrow Debreu models because forward markets are missing, Green [28] established early on interesting necessary and sufficient conditions on "overlapping expectations" for the existence of a temporary equilibrium; similar conditions appear in Grandmont [27] also in the context of temporary equilibrium and Green's model. Sufficient conditions for existence in economies with short sales, i.e. when $X = R^N$, include those of Debreu [26], which requires the "irreversibility" of the total consumption set $X = \sum_{h=1}^{H} X_h$: this contrasts with limited arbitrage in that it applies to the whole consumption set X rather than to global or market cones, in any case it is only a sufficient condition for existence. Other "no arbitrage" conditions have been used, for example in the finance literature. The connection between the standard notion of no-arbitrage and limited arbitrage, was discussed in Section 3.2. The no-arbitrage Condition C of Chichilnisky and Heal [24] is a 1984 antecedent for limited arbitrage: it is a no-arbitrage condition which is sufficient but not necessary for the existence of a competitive equilibrium; it requires that along a sequence of feasible allocations where the utility of one trader increases beyond bound, there exists another trader whose utility eventually decreases below this trader's utility at the initial endowment. This result is based on a bounded sets of feasible allocations, a conditions that is not generally satisfied in this paper.

Another condition of no-arbitrage appears in Werner [41] and in Nielsen [38], who provide sufficient conditions for the existence of an equilibrium in finite dimensions. The results of [41] and [38] are posterior and a subset of those in those in[61] [24]; they are restricted to finite dimensional economies with short sales and with strictly convex preferences, and are based on bounded sets of feasible allocations. The same no-arbitrage condition was used subsequently by Page [39] for special models of asset prices with strictly convex preferences, which do not have the generality of the Arrow Debreu model, and which exclude also the Arrow Debreu treatment of short sales. The no-arbitrage condition mentioned above is sufficient but not necessary in general economies for the existence of an equilibrium. Under certain conditions which exclude the Arrow Debreu treatment of short sales, and which exclude also the case of preferences which are not strictly convex and which may have different recession cones at different endowments, conditions which are not required here, Werner [41] provides two conditions, one which is proved to be sufficient for existence of equilibrium and another which is mentioned without formal statement or proof to be necessary for existence. The two conditions involve different cones and there is no complete proof that these cones, and therefore these two conditions, are the same; the details are in Section 4 above. No-arbitrage as defined in [41] is not defined on initial parameters of the economy: it must be verified in principle at all allocations, thus eliminating cases where limited arbitrage is satisfied (with the same preferences) for some initial endowments and not for others, cases which are included in the analysis of this paper, see Definition 5.

12 Conclusions

One limitation on social diversity, limited arbitrage, is necessary and sufficient for the existence of a competitive equilibrium, the core and social choice rules in Arrow Debreu economies [3]. Social diversity is however more subtle and complex: it comes in many shades. Social diversity is zero when limited arbitrage is satisfied, and it is defined generally in terms of the properties of the cohomology rings CH of the nerve of a family of

[61]The results on existence of an equilibrium in [24] (which are valid in finite or infinite dimensional economies) contain as a special case the results on existence of equilibrium in [41]. The no-arbitrage Condition C introduced in 1984 by Chichilnisky and Heal, [24] is weaker that the no-arbitrage condition defined by Werner [41]. As recorded in its printed version, Chichilnisky and Heal [24] was submitted for publication in February 1984. As recorded in its printed version, Werner's paper [7] was submitted for publication subsequently, in July 1985.

cones which are naturally associated with the economy. The cohomology rings of these nerves contain information about which subeconomies have competitive equilibria and a core, and which have social choice rules; the mildest form of social diversity is sufficient for the existence of a supercore, which consists of all those allocations which no strict subcoalition has a reason to block.

From these results an implicit prediction emerges about the characteristics of economies which have evolved mechanisms to allocate resources efficiently according to markets, cooperative game solutions, or social choice: they will exhibit only a limited amount of social diversity. Economies which do not succeed in allocating resource efficiently are not likely to be observed in practice, so that existing economies are likely to exhibit limited social diversity.

Other forms of diversity come to mind—for example, the genetic diversity of a population: this is generally believed to be favorable for the species' survival. In the biological context, therefore, diversity appears as a positive feature. This may appear to run counter to what is said here. Not so. Some diversity is desirable in economics as well: as mentioned at the beginning of this paper, without diversity there would be no gains from trade. Indeed without diversity the market would have no reason to exist. The matter is subtle: in the end, it is a question of degrees, of how much diversity is desirable or acceptable.

The tenet of this paper is that the economic organizations which prevail today require a well-defined amount of diversity, and no more, to function properly. One is led to consider the following, somewhat unsettling, question: is it possible that existing forms of economic organization restrict diversity beyond what would be desirable for the survival of our species? Or, more generally: are the forms of social and economic organization which prevail in our society sustainable?

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