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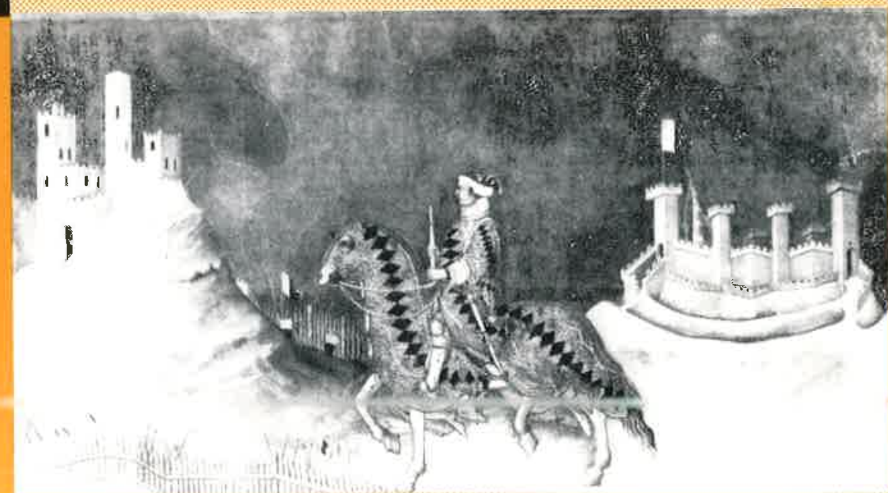
MAURO CAMINATI

BOUNDEDLY-RATIONAL LEARNING AND THE PRESERVATION OF ENDOGENOUS CYCLES



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Mauro Caminati

BOUNDEDLY-RATIONAL LEARNING AND THE PRESERVATION OF ENDOGENOUS CYCLES

Siena, Settembre 1995

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Introduction[♣]

The attention paid to chaotic dynamical systems brought to economists a new possible paradigm for the explanation of the observed persistent and aperiodic fluctuations in macroeconomic time series. A difficulty with this approach is that the statistical analysis of these series has not revealed clear evidence of low-dimensional attractors, contrary to what is expected, when time series are generated by a purely-deterministic system. Still, the widespread evidence of nonlinearities suggests that the non-linear approach to macroeconomic fluctuations should not be rejected, although it may need an integration with stochastic factors. It may be worth observing that the unavoidable presence of small measurement errors in all real systems already brings in one such stochastic factor.

The non-linear endogenous approach to macroeconomic fluctuations raises two distinct but related issues, which have been only recently considered in the literature.

A first point which has been made is that, if observable dynamic paths are complex[1], economic agents would be facing a substantial randomness, and it is an open question whether their reaction to it would preserve endogenous cycles. The contributions on this issue consider very specific examples of economic environments and investigate the existence of prediction rules which: *a*) give rise to endogenous cycles; *b*) are self-confirming, that is, are "plausible" in the face of the time series observed when forecasts are determined by these rules. This problem concerns the preservation of particular, well identified deterministic oscillations.

A second discussion is about the generality of the non-linear approach to macroeconomic fluctuations. The question is now whether it is possible to identify a sufficiently large family of prediction rules which share the properties of (*i*) being "plausible" in the face of complex macroeconomic time series of some kind, without being necessarily self-confirming and of (*ii*) meeting requirement *a*) above. Supposing that such a family could be identified, and qualified to be large in some relevant sense,

[♣] I am indebted to Nicola Dimitri for helpful comments and discussions. Financial support from the italian MURST (60%) and the EEC Human Capital and Mobility programme is gratefully acknowledged.

[1] The term complex dynamics is used here to identify not only chaotic trajectories, but also trajectories which are periodic, but their periodicity is not easy to detect.

it may well be the case that only a very small subset of prediction rules within the family is self-confirming. Still, endogenous cycles could not be regarded as a curiosum. Indeed, the persistence of any given forecasting mechanism is not a necessary condition for the preservation of the class of endogenous oscillations, as distinguished from the preservation of a particular cycle (produced by a well identified prediction rule). An example of the latter preservation issue is discussed in section 1. Sections 2 and 3 instead are mainly concerned with the former and more general preservation problem.

A preliminary difficulty arising in this context is that of providing an uncontroversial definition of a "plausible" prediction strategy.

The Bayesian learning literature insists that any such rule must be the outcome of an explicit maximization procedure and rooted on the expected utility approach to decision making, or the generalizations thereof (cf. Kiefer and Nyarko (1995)). Although the assumption is not crucially needed by the Bayesian approach, most Bayesian models postulate that the learner has a correct perception of the true structure of the world, up to some unknown parameter, that the support of the prior probability distribution on the parameter space contains the true parameter value and that the agent can compute costlessly the distribution on the state space which is induced by his prior and his choice of policy. Bayesian models incorporating these assumptions are labelled rational-learning models. Rational learners are equipped with an unrealistically high amount of information and of computing abilities.

Prediction rules considered by the boundedly-rational-learning literature are essentially "rules of thumb". As will be shown later, the ad-hocness implicit in this stand point is reduced if agents are assumed to contemplate a possibly large set of prediction rules and consequently evaluate their relative performance. Thus, a plausible prediction rule will be one which survives the ensuing learning process. A preliminary issue arising in this context is that of deciding how bounded should agents' rationality be. It turns out that the choice has direct implications on the preservation of deterministic cycles. Two main alternatives have been considered in this work.

The first alternative presents unsophisticated agents, which are unaware of the strong serial correlation present in time series. Chaotic trajectories, but also periodic

trajectories, provided they have a sufficiently high period, are interpreted as the realizations of a sequence of independent and identically distributed random variables. It may be worth observing that the above interpretation can be corroborated by the evidence only, and at most, from a very long-run perspective (asymptotically). This would be the case, in particular, if the actual law of motion $f(\cdot)$ is such that the Sinai-Ruelle-Bowen ergodic theorem (Guckenheimer and Holmes (1983), p. 282) applies. When this is the case, solution trajectories would visit any open subset A of the attractor with asymptotic frequency $\mu(A)$, where μ is a unique (ergodic, absolutely continuous, invariant) measure supported on the attractor[2]. The problem posed by Evans, Honkapohja and Sargent (1993) is: would complex deterministic cycles be preserved, when the agents perceive them as random fluctuations? In the case considered by the authors the answer is no, but, admittedly, the question is not well posed. What is unconvincing in this approach is that the agents, by ignoring serial correlation, are forced to make very poor short-term predictions. This conflicts with the fact that: *a*) serial correlation is easy to detect; *b*) chaotic trajectories (let alone periodic ones) do not forbid satisfactory short-term prediction even if the law of motion is not fully known (on this point see Kravtsov (1993)); *c*) short-term prediction is highly relevant to economic decisions. By contrast, knowing the asymptotic frequency may be of little guidance to such decisions, since the intertemporal distribution of events is crucial in economics. It turns out that the agent envisaged by this first alternative, more than boundedly rational, is unnecessarily dull.

The agents contemplated by the second alternative referred to above are more sophisticated to the extent that they are aware of the existence of serial correlation in time series. Their view of the world reflects this awareness and is used, together with the knowledge of (current and) past realizations, for forecasting purposes. If considered in this perspective, the preservation of deterministic cycles becomes an aspect of the more general problem of boundedly rational learning in macroeconomic models. The models in this literature are so conceived that economic agents (typically,

[2] It may be worth observing in this respect that there is no ground to believe that the conditions for the applicability of the Sinai-Ruelle-Bowen theorem would frequently hold in economic examples (cf. Medio (1992)). Indeed, they do not hold , when the map f() is of the logistic type! (See Brock (1986), p. 181.)

a representative agent or generation) hold an exogenously given model of the world, or a set of such models. All these models give an incorrect specification of the actual law of motion during the learning process; still, they are mostly required to meet some weaker or stronger "rationality" requirement: for instance, prediction models may be asked to extrapolate a steady state dynamics, to extrapolate constant sequences, or to detect periodic trajectories up to some given prime period *k*. The a priori constraint placed on the agents' view of the world and on their forecasting methods determines what they can learn: as long as a stationary or periodic equilibrium of the actual dynamics with learning is perfectly predicted (in force of the above requirements), the primary issue addressed in these models is the stability of these equilibria under the learning process. It is worth stressing that the agents' forecasting models are typically inconsistent with the perfect foresight dynamics off the stationary or periodic equilibria. The possibility of convergence to an aperiodic perfect-foresight trajectory is therefore not contemplated.

The conditions for the local stability of a periodic equilibrium under learning are studied by Grandmont and Laroque (1986, 1990), Guesnerie and Woodford (1991) Evans and Honkapohja (1992*a*), among others. This work will not be reviewed here. These notes will focus instead on a line of research suggested by Grandmont and Laroque (1989) and Grandmont (1994). Some instability results concerning the actual dynamics with learning of a class of macro models are interpreted by these authors as providing a general argument in favour of the endogenous interpretation of business cycles. An implication of their findings is that predictions are never fulfilled along the endogenous (periodic or aperiodic) trajectories. The instability result of Grandmont and Laroque (1989) is likely to hold if the agent is locally prepared to extrapolate a large variety of dynamical behaviours. This will typically occur, according to the authors, if the agent is "sufficiently uncertain" about the true law of motion of the system.

As will be argued below, the above suggestion is unconvincing; it will be shown by means of examples that an increase of uncertainty can have a stabilizing influence on the actual dynamics with learning. Our counter argument postulates the existence of alternative "views of the world" (forecasting models) and of a distribution

dynamics of the degree of reliability attached to them[3]. Indeed, the forecasting model which "performs better", given the existing reliability distribution on the alternative models, can be expected to gain support. It is suggested that in large, decentralised, economies the relative predictive performance of the subjective models should be evaluated in terms of the agent's pay-off function, rather than in terms of the relative size of the prediction error or other statistical criteria.

Although uncertainty does not seem to provide a general argument in favour of the instability of learning, the idea behind Grandmont's line of argument is worth exploring. The idea is that the preservation of deterministic cycles is related to those circumstances which induce boundedly rational agents to make systematic mistakes in their forecasting activities.

The plan of this paper is as follows. Section 1 reformulates a result of Evans, Honkapohja and Sargent (1993) on the non-preservation of deterministic complex oscillations, when the agents perceive them as random fluctuations. Section 2 is mainly concerned with the suggested genericity of endogenous fluctuations, which Grandmont and Laroque (1989) and Grandmont (1994) conjecture on the ground of some relevant instability examples. The delicate connections between uncertainty and the deterministic models of boundedly-rational learning are briefly considered in section 3. Here, Grandmont's idea that the higher the degree of uncertainty, the larger the number of dynamical behaviours which the agent is locally prepared to extrapolate, is also critically considered. Section 4 presents an example in which even a very small subjective probability attached to a prediction rule which gives little weight to recent observations, can stabilize the steady state of an overlapping-generations economy, which is unstable under a least-squares learning rule. In this economy one must distinguish between the activity of learning on future states (forecasting) and that of learning on prediction rules. The simulation results of the overall learning process are presented in section 5 together with the insights offered on the up-rise and preservation of endogenous fluctuations.

[3] By contrast, learning in macroeconomic models has been mostly studied in environments in which agents hold a single, identical view of the world, which may or may not turn out to be consistent with the local stability of the given rational-expectations equilibrium.

1. Non existence of deterministic aperiodic oscillations regarded as random fluctuations: the loan-consumption overlapping-generations case.

To fix our ideas let us consider the loan-consumption overlapping-generations model (Samuelson (1958), Gale (1973), Benhabib and Day (1982), Grandmont (1985)). The choice is exclusively motivated by the fact that Evans, Honkapohja and Sargent (1993), Bullard (1994), Benassy and Blad (1989) and many other examples in the learning literature refer to some version of this model, or are consistent with it.

The agents live for two periods and are endowed with an amount of consumption good w_0 when young and an amount w_1 when old. Population is constant. Each young agent born at time t has the separable utility function,

$U(c_0(t), c_1(t+1)) = u(c_0(t)) + v(c_1(t+1))$

where c_0 and c_1 denote consumption when young and consumption when old, respectively. It is assumed that the Arrow-Pratt measure of relative risk aversion on u() is increasing in c_0. In particular, $\lim_{c_0 \to \infty} - c_0(t)u''(c_0(t)) / u'(c_0(t)) = \infty$.

Depending on the endowment vector, preferences and price expectations, the young agent may wish to save or to borrow. In the former case he sells to the old generation, in exchange for purchasing power which can be transferred to the next period. In the latter case he borrows from a central agency and buys from the old generation. The central agency makes sure that all debts will be repaid in the following period. The nominal interest rate is zero, for simplicity. As is well known, the situation in which the young is prepared to save (borrow) when one unit of consumption at time t can be exchanged for one unit of consumption at time t + 1, gives rise to the "Samuelson" ("Classical") case, as defined by Gale (1973).

The budget constraint, as perceived by the young agent at time t is:

$$c_1(t+1) = w_1 + (w_0 - c_0(t)) p_t / p^e_{t+1}, \qquad (1.1)$$

where p^e_{t+1} denotes the prediction taken at time t on the price at time t+1. Actual consumption by the old agent at time t +1, given his choice when young, will of course depend on the realized, rather than on the predicted price, at time t +1. The first order condition for an optimal consumption plan of the young agent at t is:

$$u'(c_0(t)) = v'(w_1 + (w_0 - c_0(t)) p_t / p^e_{t+1})\ p_t / p^e_{t+1} \qquad (1.2)$$

Market clearing requires:

$$w_0 - c_0(t) + w_1 - c_1(t) = 0 \qquad (1.3)$$

Combining market clearing with the budget constraint yields:

$$w_1 + w_0 - c_0(t) = c_1(t) = w_1 + (w_0 - c_0(t\text{-}1)) p_{t-1} / p_t \qquad (1.4)$$

hence:

$$p_t / p_{t-1} = (w_0 - c_0(t\text{-}1)) / (w_0 - c_0(t)) \qquad (1.5)$$

Evans Honkapohja and Sargent (1993) carry out the following thought experiment.

It is supposed that the economy has converged to a deterministic cycle of prime period k, as can be generated, for instance, by the perfect foresight dynamics of this economy. The question is: is there a deterministic k-cycle which would be preserved, should all the agents perceive this cycle as a random fluctuation? In other words, is there an expectational equilibrium, which is a deterministic *k*-cycle in prices and such that every agent makes only unconditional predictions based on the asymptotic price frequences generated by the *k*-cycle? The authors address this question for the Samuelson and the classical case, separately. A unified framework for the statement and proof of their conclusions is proposed below. Here the old agent is endowed with a non negative (rather than zero) quantity of consumption also in the Samuelson case.

Proposition 1.1:

Assume that the equilibrium-price dynamics of the overlapping-generations consumption-loan economy follows a deterministic *k*-cycle: $\{p_1, ..., p_k\}$. Assume also that predictions at time t on p_{t+1} are made using the unconditional price distribution $\Pr\{p_{t+1} = p_j\} = 1/k \quad j = 1, ..., k$. The *k*-cycle can be preserved in the Samuelson case only if $k = 1$ (stationary state) and can be preserved in the classical case only if $k = 1$ or $k = 2$.

Proof:

If the prediction rule followed by the young agent is that specified in the proposition, the first-order condition 1.2 takes the form:

$$u'(c_0(t)) = \Sigma_{j=1}^{k} 1/k \{v'(w_1 + (w_0 - c_0(t))p_t/p_j)\ p_t/p_j\}$$

Combining this with 1.5 we obtain:

$$(w_0 - c_0(t))u'(c_0(t)) = \Sigma_{j=1}^{k} 1/k \{v'(w_1 + (w_0 - c_0(t-j))\ (w_0 - c_0(t-j))\} \qquad (1.6)$$

The derivative of the left-hand-side of 1.6 with respect to $c_0(t)$ is

$$(w_0 - c_0(t))u''(c_0(t)) - u'(c_0(t)) \qquad (1.7)$$

which is negative for $0 \leq c_0(t) \leq w_0$. Thus, the left-hand-side of 1.6 is non-negative and decreasing with respect to $c_0(t)$ within this range and there can be at most a unique value of $c_0(t) < w_0$ consistent with 1.6; this is the "Samuelson" steady state.

The left-hand-side of (1.6) is negative and first decreasing with respect to $c_0(t)$ on $(w_0, \infty]$. It is increasing on the same interval if $(w_0 - c_0(t))u''(c_0(t)) / u'(c_0(t)) > 1$. The assumption on the Arrow-Pratt measure of relative risk aversion on u() implies that this will be the case for $c_0(t)$ sufficiently large. It follows that the expression – (left-hand-side of 1.6) is unimodal on $[w_0, \infty]$ and there exist on this interval at most two distinct values of $c_0(t)$ which solve 1.6. Moreover, if there exist two distinct values c_0', c_0'' solving (1.6), then c_0', $c_0'' > w_0$. Thus, the deterministic k-cycle of proposition 1.1 must be such that $k = 1$, or $k = 2$; $k = 2$ is consistent only with the "classical" case. □

The above proposition admits a straightforward extension to the case in which the equilibrium dynamics is chaotic, provided that one can still define a true stationary unconditional distribution which is used by the "boundedly-rational" agents for their predictions.

Proposition 1.2:

Assume that the price interval I is invariant under the prevailing equilibrium dynamics of the overlapping-generations consumption-loan economy. Assume also that the asymptotic frequency distribution with which the price trajectory visits subintervals of I converges to an absolutely continuous probability distribution on I (observable ergodic chaos). If predictions at time t on p_{t+1} are unconditional forecasts based on the asymptotic price distribution on I, aperiodic cycles can not be preserved and the maximum prime period k which can be preserved is $k = 1$ for the Samuelson case, $k = 2$ for the classical case.

Proof:

The proof follows the same lines of the proof of proposition 1.1.

2. Uncertainty and non-linear oscillations: Grandmont's genericity argument

Propositions 1. and 2. above rest upon the crucial assumption that agents use unconditional price distributions for their predictions. Bounded rationality of this sort appears very naive, because it ignores available information, namely the correlation between economic variables and their lagged values. The argument that the agents may choose to ignore this information because they are facing a gap between their computing ability and the complexity of their task, is unconvincing, if referred to short-term prediction. Heiner's argument on the difficulty-competence gap (cf. Heiner (1983)) would not suggest to ignore the observed correlations, but to exploit them through the adoption of a simplified prediction rule (a rule of thumb).

An alternative predictive strategy is that of using the available information to form a view of the world in which there is scope for conditional predictions. There is clear evidence that in environments in which the observed law of motion is independent of learning, the second strategy yields better results, if used for short-term predictions, even when this law of motion is a perturbed chaotic trajectory (see Kravtsov (1993) and the references quoted therein). Unfortunately, the relevant economic environment is not of this type. Considering, for instance, "large" economies with no problems of strategic interaction, we observe that the isolated agent lives in a world in which all the other agents are performing the same task of learning from experience. What is being observed and learned is the system law of motion as is affected by the activity of learning; this activity must therefore be brought into one's view of the world. The self-referential structure of the problem is a potential source of non stationarity, at least if we admit that the maintained view of the world is not given once and for all. The possibility of innovations in learning algorithms makes the structure of the environment non stationary. This issue will not be addressed here. We shall instead be concerned with the uncertainty concerning exogenously given views of the world. Before doing so we shall review some recent results which relate the local stability of the learning processes to some relevant properties of the perfect foresight dynamics and of the expectation formation. In particular, we shall present conditions, identified by Grandmont and Laroque (1989), for the local instability of a stationary state under the actual law of motion with learning.

Let us consider a competitive economy with identical agents. The state of this economy at time t is summarized by the scalar variable x_t. The true law of motion in the neighbourhood of the stationary state $\underline{x}$ is implicitly defined by

$$T(x_{t-1}, x_t, x^e_{t+1}) = 0 \tag{2.1}$$

where the super-script ^{e} denotes expectation and expectations referring to date t are taken at t - 1. The partial derivative of T ( ) at ($\underline{x}$, $\underline{x}$, $\underline{x}$), with respect to its third, second and first argument are a, b and d respectively. The assumption $a \neq 0$ means that expectations matter (at least locally); the assumption $b \neq 0$ implies that x_t is locally a well defined function of x_{t-1} and x^e_{t+1}.

It is worth observing that the condition $d = 0$ is allowed; this means that the following analysis may well apply also if, locally, the pre-determined variable is irrelevant (in particular, if there is not a predetermined variable). Indeed, the temporary-equilibrium map T( ) will typically arise in some standard formulations of the overlapping generations model with or without production, or in infinite-horizon models with cash in advance constraints. Nonetheless, it must be pointed out that those environments which give rise to a temporary-equilibrium map like T( ) remain quite specific. Even simple formulations of the consumption-loan overlapping-generations model, as long as consistent with both the Samuelson and the classical case (see section 1), give rise to a different temporary-equilibrium map (with the form $P(x_t, x^e_t, x^e_{t+1}) = 0$).

The substitution of x_{t+1} for x^e_{t+1} in T( ) yields the local (forward) perfect-foresight dynamics, which can be written in explicit form as $(x_{t+1}, x_t)' = F(x_t, x_{t-1})'$, where the superscript ' is the transpose operator. The Jacobian matrix of F( ) at ($\underline{x}$, $\underline{x}$) is easily seen to be

$$\begin{bmatrix} -a^{-1}b & -a^{-1}d \\ 1 & 0 \end{bmatrix}$$

with characteristic polynomial $P_F(\lambda) = \lambda^2 + a^{-1}b\lambda + a^{-1}d = 0$.

Agents in this economy are boundedly rational and form predictions on the base of current and past observations:

$$x^e_{t+1} = \Psi(x_t, x_{t-1}, ..., x_{t-L}) \tag{2.2}$$

The assumption that x_t is an argument of Ψ( ) implies a simultaneous determination of x^e_{t+1} and x_t.

The expectation function Ψ( ) is said to detect period k if for any sequence (x_t, x_{t-1}, ..., x_{t-L}) of prime period k, $\Psi(x_t, x_{t-1}, ..., x_{t-L}) = x_{t-k+1}$. The following restriction, which amounts to a minimum rationality requirement, is made throughout: Ψ( ) detects period 1. Ψ( ) is also assumed to be continuous and differentiable; its partial derivative with respect to x_{t-j} evaluated at ($\underline{x}$, $\underline{x}$, ..., $\underline{x}$) is c_j j = 0, 1, ..., L. Since Ψ( ) detects period 1, $\Sigma_0^L c_j = 1$. One can also write:

$(x^e_{t+1}, x_t, x_{t-1}, ..., x_{t-L+1})' = Q((x_t, x_{t-1}, ..., x_{t-L})')$.

The Jacobian matrix **DQ** of Q( ) evaluated at ($\underline{x}$, $\underline{x}$, ..., $\underline{x}$) and here shown for the case L = 3 is:

$$\mathbf{DQ} \equiv \begin{bmatrix} c_0 & c_1 & c_{L-1} & c_L \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.3}$$

with characteristic polynomial $P_Q(\mu) = \mu^{L+1} - \Sigma_0^L c_j \mu^{L-j}$.

The actual dynamics with learning is defined implicitly by inserting (2.2) into (2.1):

$T(x_{t-1}, x_t, \Psi(x_t, x_{t-1}, ..., x_{t-L})) = 0$

which can be written equivalently as

$Z(x_t, x_{t-1}, ..., x_{t-L}) = 0$.

The partial derivative of Z( ) with respect to its first argument and evaluated at ($\underline{x}$, $\underline{x}$, ..., $\underline{x}$) is $ac_0 + b$, with $b \neq 0$ (see above) and $ac_0 \neq - b$. The last assumption implies that x_t is locally a function of (x_{t-1}, ..., x_{t-L}) so that the actual dynamics with learning is well defined in a neighbourhood of the stationary state.

As a consequence of the simultaneity induced by the assumption that x_t is observed at the time of forecasting x^e_{t+1}, the actual dynamics with learning is not well defined when the agents are perfectly rational, in which case $c_{t-j} = 0$ for $2 \leq j \leq L$ and $c_0 = -a^{-1}b$, $c_1 = -a^{-1}d$. Conversely, the local perfect-foresight dynamics is not the particular local actual dynamics with learning which is obtained by imposing that the expectation function yields correct predictions, that is $Q((x_t, x_{t-1}, \ldots, x_{t-L})') = F(x_t, x_{t-1})'$, with L = 1. The difficulty disappears if, as is required by a non-instantaneous forecasting activity, x_t is not an argument of $\Psi()$.

The actual dynamics with learning can be written in explicit form as:

$$(x_t, x_{t-1}, \ldots, x_{t-L+1})' = W(x_{t-1}, \ldots, x_{t-L})' \tag{2.4}$$

The Jacobian matrix of (3) evaluated at ($\underline{x}, \underline{x}, \ldots, \underline{x}$) has the form of the matrix shown below, which corresponds to the case L = 4:

$$\mathbf{DW} = \begin{bmatrix} \frac{-(ac_1+d)}{ac_0+b} & \frac{-ac_2}{ac_0+b} & \frac{-ac_{L-1}}{ac_0+b} & \frac{-ac_L}{ac_0+b} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.5}$$

The associated characteristic polynomial is $P_W(z)$.

To the sake of fixing our ideas we begin by making a number of quite straightforward remarks.

Remark 2.1: If expectations do not matter ($a = 0$) the actual dynamics with learning and the true law of motion coincide locally around the stationary state. They boil down to $x_t = f(x_{t-1})$, with $f' = -b/d$. An equivalent, but less transparent statement is (cf. Grandmont and Laroque 1989) that for $a = 0$ the roots of $P_F(\lambda) = 0$ are $\lambda_1 = -b/d$ and $\lambda_2 = +\infty$, or $-\infty$, while $P_W(z) = 0$ has one root $z_1 = -b/d$ and the other L - 1 roots are zero.

Remark 2.2: The first row of **DW** can be rewritten as:

$$-\left[\frac{c_1+(d/a)}{c_0+(b/a)} \quad \frac{c_2}{c_0+(b/a)} \quad \frac{c_{L-1}}{c_0+(b/a)} \quad \frac{c_L}{c_0+(b/a)}\right]$$

If the true law of motion is dominated by expectations ($a = \infty$) the local stability of the actual dynamics with learning depends only on expectations. In particular, if $-1 < c_j/c_0 < 0$, i = 1, ... , L, and $-\Sigma_1^L c_j / c_0 < 1$, then all the roots of $P_W(z) = 0$ are less than 1 in absolute value, hence the actual dynamics is locally stable.

The last sentence of remark 2.2 above draws upon the following lemma, which is worth stating explicitly for the sake of later reference.

Lemma: Let **C** be a square matrix of the form (2.3) above, with $c_j \geq 0$ and $\Sigma_0^L c_j = 1$. $\mu = 1$ is an eigenvalue of **C** and no other eigenvalue of **C** has modulus greater than 1. The Frobenius root of **C** is therefore $\mu_F = 1$. If $\Sigma_0^L c_j < 1$, then $\mu_F < 1$.

Proof: **C** is a non negative square matrix; its Frobenius root μ_F is therefore real. The characteristic polynomial of **C**, namely $P_C(\mu)$, has the same form of $P_Q(\mu)$. It follows that, if the sum of the elements in the first row of **C** equals (is less than) 1, then $P_C(1) = (>) 0$. The term $|\mu^{L+1}|$ grows faster than $|\Sigma_0^L c_j \mu^{L-j}|$ as $|\mu|$ increases starting from $|\mu| = 1$. This proves the lemma.

Remark 2.3: Assume that the true law of motion is such that expectations matter ($a \neq 0$). If the last-period observation and/or the predetermined variable matter (c_1 and/or $d \neq 0$), but more distant observations have locally no influence on expectations around the stationary state (i. e. $c_j = 0$ j = 2, ..., L), then the actual dynamics with learning is locally stable, provided that expectations are highly sensitive to current results ($|c_0|$ sufficiently large).

The above remark shows how highly misspecified expectations functions can be stabilizing; it can also provide a rationale for expectational rules of thumb, which place so large a weight on current observations.

Remark 2. 4: If the perfect-foresight dynamics does not admit cycles of period k, the actual dynamics with learning can converge to a k cycle only if the expectation function Ψ does not detect period k. In particular, if the perfect-foresight dynamics does not admit cycles of period $k > 1$ and the stationary state(s) of the actual dynamics with learning is (are) unstable, expectations would be never fulfilled on the actual dynamics.

We are now ready to state the main result of Grandmont and Laroque (1989) (see also Grandmont (1994)).

Proposition 2.5: Assume $a \neq 0$ (expectations matter) and $ac_0 \neq -b$ (the actual dynamics with learning is locally well defined). Assume that the characteristic polynomial $P_Q(\mu)$ associated to the expectation function (2.2) has two real roots $\mu_1 \geq 1$, $\mu_2 \leq -1$. Assume that the real roots of the characteristic polynomial $P_F(\lambda)$ associated with the perfect-foresight dynamics belong to the open interval (μ_2, μ_1). The actual dynamics with learning is unstable near the given stationary state.
Proof: Grandmont and Laroque (1989), Grandmont (1994).

The above result is important because it gives sufficient conditions for the instability of the actual dynamics with learning in terms of the local properties of the perfect-foresight dynamics and of the characteristic roots associated to the expectation function (2.2). In particular, the distribution of the roots of $P_Q(\mu) = 0$ in the complex plane plays a crucial role; indeed, the information we can derive from Proposition 2.3 is conditional on the knowledge of circumstances affecting this distribution. These are so described by Grandmont and Laroque:

> ...the L + 1 eigenvalues of Ψ describe the local regularities that the traders are able to recognize in, and extrapolate from, past history near the stationary state. (Grandmont and Laroque (1989), p. 11)

For instance, if the agents are prepared to extrapolate sequences of period k (i. e. Ψ detects period k), then if μ is a complex number such that $\mu^k = 1$, μ is a root of $P_Q(\mu) = 0$ (see Grandmont and Laroque (1986), Lemma 3.1). If the agents are prepared locally to extrapolate growth paths which, if expressed in terms of deviations from the stationary state, have constant growth rate μ, then μ is a real root of $P_Q(\mu) = 0$.
The conjecture advanced by the authors is that the stronger the uncertainty concerning the future behaviour of the economy, the larger the patterns of dynamical behaviour that the agents must be prepared to track down and reflected in their expectation function. It then follows that

> if the traders assign a positive, but perhaps very small, relative weight to the possibility that the system might experience high inflation or deflation rates, one may expect that for a large L, the forecasting rule Ψ would have two real roots $\mu_1 \geq 1$, $\mu_2 \leq -1$ such that [the conditions of Proposition 2.3 are fulfilled]... (cf. Grandmont and Laroque (1989), pp. 15-6).

> [Thus,] if the agents are rather uncertain about the local dynamics of the system and are thus ready, on average, to extrapolate a wide range of local regularities (trends) out of past deviations from the equilibrium, and if the influence of expectations on the dynamics of the system is significant, the actual dynamics with learning should be locally unstable (cf. Grandmont (1994), p. 11).

An immediate corollary to this argument is that if the (representative) agent is sufficiently uncertain about the future, and explosive trajectories are ruled out by the non-linear feed-backs which are present in all real systems, the actual dynamics with learning must generically display non-linear oscillations (on which expectations may or may nor be fulfilled; see remark 2.4).

3. Uncertainty, expectations and stability of learning

Grandmont's line of argument in suggesting that the instability condition of Proposition (2.3) is likely to be fulfilled in real-world economies is faced with two difficulties.

The first one is related to the limits of the approach which is taken by this strand of literature on learning. It is not so much that there are obvious shortcomings in any representative-agent framework. One may indeed observe, in passing, that Grandmont (1994) moves some steps towards extending his arguments to an environment with heterogeneous agents. The first difficulty is, instead, that the true law of motion T( ) is so conceived that it appears to be inconsistent with any subjective degree of uncertainty, except under highly restrictive assumptions. It is worth expanding briefly on this point.

The expectation function ψ reflects the perceived law of motion and the agent's best guess on the value of the parameters which are built into this law. It is a prominent feature of the derivation of the actual law of motion W( ) that x^e_{t+1} is subjectively perceived as true, or at least that the agent is prepared to take decisions as if x^e_{t+1} were so perceived. This must be the case because the true law of motion T( ), which is used to obtain W( ), is defined as the implicit dynamics which holds when expectations are perceived as true. For instance, substitution of x_{t+1} for x^e_{t+1} in T( ) yields the perfect-foresight dynamics.

In so far as x^e_{t+1} is conceived as a deterministic variable, and decisions are taken as if the prediction x^e_{t+1} were subjectively perceived as true, it should be obvious that the above framework is not well suited to the analysis of the relation between uncertainty and the stability of learning. To illustrate the difficulty, we can refer to those simple cases of uncertain predictions in which x^e_{t+1} given $(x_t, ..., x_{t-L})$, that is Ψ, is defined by a conditional (subjective) probability distribution on {X}. As is well known, it is only in particular circumstances[4] that the conditional expectation $E(x_{t+1} \mid (x_t, ..., x_{t-L}))$ can be substituted for the random variable x^e_{t+1} in the agent's optimization problem, hence in T( ). In such cases one can hold to the definitions of section 2 and think of Ψ as the deterministic function which computes the (subjective) conditional expectation of x_{t+1}. Grandmont and Laroque are of course well aware of these facts; indeed, Grandmont (1994) is explicit about the assumption that, either expectations are subjectively perceived as true, or the utility function is linear. With the former assumption we are confronted with an irrational agent who, though uncertain about the future, is prepared to take his predictions as true; the latter assumption is prohibitively restrictive, but (to the end of holding to the framework of section 2) can be relaxed in applications with the weaker assumption of quasi-linearity. As will be shown in section 4, a quasi-linear utility function is sufficient to discuss examples raising the issue of the preservation of deterministic cycles.

Using this particular case as an illustrative example, we move to consider how Grandmont's line of argument is faced with a second difficulty, which is most relevant in the present context. The effect of a higher degree of uncertainty, so the argument goes, is that of increasing the number of local regularities, which the agent is prepared to extrapolate near the stationary state. According to this suggestion, the number of multiplicities within the roots of $P_Q(\mu) = 0$ is bound to decrease with uncertainty if L is fixed, or else L is bound to increase with uncertainty, or both. However, a closer examination reveals that the argument is unconvincing.

[4] The standard condition is that, either the objective function must be quadratic and the constraint linear, or *vice-versa*.

No matter how high is the uncertainty on the local regularities to be extrapolated, we are not entitled to draw the conclusion which are envisaged by Grandmont and Laroque. To see this, it must be observed that in a state of bounded rationality any single perceived law of motion and/or prediction rule, which is adopted by our representative agent, will generally reflect the fact that (s)he is uncertain between *different* laws of motion, and/or prediction rules (see, for instance, Blume and Easley (1982)). In terms of our simplified example (which elicits the use of expectation functions, rather than distributions), the agent contemplates n alternative expectations functions ψ_i $i = 1, ..., n$.

Each ψ_i has in the eye of the agent the relative strength θ_i. This can be generally interpreted as an index of the reliability of the corresponding expectation function, or more specifically as a second-order probability.

It is only in the latter case that a unique subjective conditional probability distribution for x^e_{t+1} on {X} can still be defined. If the functions ψ_i have the same domain, it is also possible to define the unique expectation function $\Psi(x_t, ..., x_{t-L}) = \sum_{i=1}^{n} \theta_i \Psi_i(x_t, ..., x_{t-L})$. This shows that the representative agent holds in fact a unique belief concerning the law of motion.

If the θ_i $i = 1, ..., n$ do not represent second-order probabilities, a unique perceived law of motion can not be defined and the agent is faced with truly ambiguous beliefs. The axiomatic foundations of the decision framework which is appropriate in such a context are considered in a recent and fast growing literature[5].

Grandmont and Laroque refer to situations in which the agent holds a unique and fixed view of the world; still, their expectation function Ψ can be thought of as resulting from the averaging of a set of beliefs: $\Psi(x_t, ..., x_{t-L}) = \sum_{i=1}^{n} \theta_i \Psi_i(x_t, ..., x_{t-L})$.

Let us now consider two alternative distributions θ^1_i, θ^2_i $i = 1, ..., N$ and let us call them distribution 1 and 2, respectively. If $\theta^2_i \leq \theta^1_i$ whenever $\theta^1_i \neq 0$ and there exists at least one j such that $\theta^2_j \neq 0$ and $\theta^1_j = 0$, then distribution 2 reflects a higher degree of uncertainty than distribution 1. Still it may well be the case that the real parts of all the eigenvalues which are associated to the average expectation function Ψ^2 are

[5] An informative survey is offered by Camerer and Weber (1992).

strictly inside the unit circle, while the same does not hold for the eigenvalues which are associated to the average expectation function Ψ^1.

4. Updating the reliability of beliefs

In the above reasoning the relative strengths attached to the perceived laws of motion were purely exogenous. Adaptive revision of these relative strengths adds a new dimension to the learning processes considered so far. An early attempt of studying this type of learning is the model of Blume and Easley (1982) (further results on a simplified version of the model are shown in Foster and Frierman (1990)), which considers an economy with random shocks and informed and uninformed agents. In these papers the uninformed agent contemplates a set of models, puts weights on them and makes predictions using the resulting average model. The prior weight on a model is then increased if and only if the model predicts the outcome better than does the average model.

The simultaneous updating of the prior relative strengths of the given models, based on a direct comparison of their respective performance, is justified in the context of a "large" economy in which the observed outcome is perceived as independent of the agent's action. The different learning rule presented in Brown (1993) (see also Sacco (1994)) is such that a single perceived model is selected in each run and the adaptation of the relative strengths takes place only with reference to the performance of the selected model. Brown's learning algorithm is appropriate in environments in which the agent is aware that his predictions, like those of the others, affect the observed outcome; this provides an incentive to experiment with prediction rules[6].

A shortcoming of the learning rule adopted in Blume and Easly (1982) is that the justification for evaluating a model performance in terms of its relative predictive power is not self evident. Indeed, learning is not an economic goal per-se[7]. It may

[6] In this context the models are randomly selected for prediction according to a frequency distribution defined by the set of relative strengths. The possibly optimal performance of a model with a relatively low initial strength would not be spotted, if the model is not given a chance. An incentive to experiment arises also if the agent does not know his pay-off function, but can only observe the realized pay-off associated to every chosen act.

[7] This point is insisted upon also by consistent Bayesians; see in particular Kiefer and Nyarko 1995.

well be the case, for instance, that a specific model, though leading to a prediction error which is larger than the prediction error associated to the average model, would have yielded a larger pay-off. This can be the case in so far as the pay-off function is in general sensitive to the sign, as well as to the absolute size, of the prediction error. In what follows, the relative performance of the given models is evaluated according to the pay-offs which the agent perceives they would have yielded, if adopted for prediction. Obviously enough, the framing of the learning process in the context of a large economy plays a crucial role also in this respect.

The learning rule summarised above is applied to the overlapping-generations consumption-loan economy presented in section 1, as modified by the introduction of government spending, which is financed by a positive flow of money supply. The stock of money supply is assumed to grow at the gross rate $m \geq 1$ (cf. Marcet and Sargent (1989b), Bullard (1994)). For the reasons explained in section 3 the utility function $U(c_0, c_1)$ is chosen to be linear in the level of consumption-when-old. In particular, for the sake of the present discussion the following convenient specification will suffice:

$$U(c_0, c_1) = log\,(c_0) + c_1 \qquad (4.1)$$

With the above utility function, saving is carried out by the young generation in the monetary equilibria of this economy, for a large set of endowments (w_0, w_1). In these monetary equilibria the money stock M_t is composed by the stock $(1/m)M_t = M_{t-1}$ in the hands of the old generation and by the quantity of money newly created by the government.

Each (identical) young agent born at time t is taught by the old family members a given set of prediction rules $\{\psi_{1,t}, \ldots, \psi_{N,t}\}$and the current distribution of the prior subjective probabilities θ_i (t) i = 1, ..., N. For the sake of simplicity, we develop the case N = 2. In the attempt of connecting different strands of literature, the agents contemplated here, unlike those of section 2, have a potentially infinite memory. On the assumption that the rules $\psi_{1,t}$ and $\psi_{2,t}$ are concerned with the prediction of the gross rate of inflation $x_t \equiv p_{t+1}/p_t$, the first-order condition for the young-agent optimal consumption plan at t is:

$$c_0(t) = \sum_{i=1,2} \theta_{i,t}\, \psi_{i,t} \equiv \psi_t \qquad (4.2)$$

where $\psi_{i,t}$ is the gross rate of inflation between time t and t + 1, as predicted by the i-th rule. The above first-order condition makes it clear that the quasi-linear form of the utility function (4.1) yields an optimal decision which is an average of the predictions.

Market clearing at time t requires:

$$w_0 - c_0(t) = M_t/p_t \equiv mM_{t-1}/p_t \tag{4.3}$$

Market clearing at t and t-1, together with (4.2) yield the true law of motion:

$$p_t = p_{t-1}\, m\, (w_0 - \psi_{t-1})/(w_0 - \psi_t) \tag{4.4}$$

The actual gross rate of inflation x is then determined by:

$$x_{t-1} = m\, (w_0 - \psi_{t-1})/(w_0 - \psi_t) \tag{4.5}$$

The prediction $\psi_{i,t-1}$ is associated to the perceived law of motion

$$p_t = p_{t-1}\, \psi_{i,t-1} \qquad i = 1, 2 \tag{4.6}$$

which is an incorrect specification of the true law (4.4), except for the case $\psi_{i,t} = \psi_{i,t-1} = m$.

The first prediction rule (i = 1) is a least-squares learning algorithm:

$$\psi_{1,t} = \psi_{1,t}(p_{t-1}, p_{t-2}, \ldots, p_0) \equiv \left[\sum_{s=1}^{t-1} p_{s-1}^2\right]^{-1} \left[\sum_{s=1}^{t-1} p_{s-1} p_s\right] \tag{4.7}$$

It is worth observing that, unlike the case considered in section 1, the prediction on x_t is made before the agent can observe p_t. Following Bullard (1994), $\psi_{1,t}$ can be written recursively as:

$$\psi_{1,t} = \psi_{1,t-1} + C_{t-1}^{-1}\, p_{t-2}\, (p_{t-1} - p_{t-2}\psi_{1,t-1}) \tag{4.8a}$$

$$C_{t-1} = \left[\sum_{s=1}^{t-1} p_{s-1}^2\right] \tag{4.8b}$$

Substituting p_{t-1} in (4.8a) with its expression derived from the actual law of motion (4.4) yields:

$$\psi_{1,t} = \psi_{1,t-1} + g_{t-1}\, [m\, ((w_0 - \psi_{t-2})/(w_0 - \psi_{t-1})) - \psi_{1,t-1}] \tag{4.9a}$$

$$g_t = [(1/g_{t-1})\, (m\, (w_0 - \psi_{t-2})/(w_0 - \psi_{t-1}))^{-2} + 1]^{-1} \tag{4.9b}$$

$$g_{t-1} = p_{t-2}^2 \left[\sum_{s=1}^{t-1} p_{s-1}^2\right]^{-1} \tag{4.9c}$$

Since 4.6 involves a specification error, the OLS estimator $\psi_{1,t}$ does not have the standard optimality properties (is not a best linear estimator). Indeed, our boundedly-rational agent is not subjectively certain of (4.6) and considers a second prediction rule (i = 2), which is most simply specified as:

$$\psi_{2,t} = \psi_{t-1} \tag{4.10}$$

The second prediction rule extrapolates the last forecast on x; it is well specified for an appropriate initial condition ψ_0 which is the prior guess at time 0 on the value of x_0. The prominent feature of this rule is that the newly available observation is not exploited for forecasting purposes. If the rule is assigned a positive probability, the average forecast will be affected accordingly: the larger $\theta_{2,t}$, the lower the influence of p_{t-1} on ψ_t.

The final element which determines the dynamic properties of the actual law of motion is the learning algorithm which is used to revise the subjective probabilities $\theta_{i,t}$. This is specified as follows:

$$\theta_{i,t} = \theta_{i,t-1}\, [1 + U(c_0(\psi_{i,t-2}), c_1(\psi_{i,t-2}, x_{t-2}))]\; 1/q_{t-1} \qquad i = 1, 2 \tag{4.11}$$

where q_{t-1} is a weighing factor, which is so defined that $\theta_{1,t} + \theta_{2,t} = 1$. $U(c_0(\psi_{i,t-2}), c_1(\psi_{i,t-2}, x_{t-2}))$ is the hypothetical utility obtained by the agent which is old at t -1 (and lives up to the end of the interval (t-1, t)), if he had used, when young, the i-th prediction rule only. Thus, the adjustment of subjective probabilities is not based on a direct comparison of the forecast errors $(x_{t-2} - \psi_{i,t-2})$, i = 1, 2, but on the utility consequences of these errors. It is also worth recalling that the observation x_{t-1} is not yet available when $\theta_{i,t}$ is used in forecasting.

Taking into account (4.1), (4.2) and (4.5), and using the fact that $\theta_{1,t} + \theta_{2,t} = 1$, (4.11) can be equivalently expressed as:

$$\theta_{1,t} = \theta_{1,t-1}\, [1 + \log(\psi_{1,t-2}) + w_1 + (w_0 - \psi_{1,t-2})(w_0 - \psi_{t-1})/(m(w_0 - \psi_{t-2}))]\; 1/q_{t-1} \tag{4.12a}$$

$$\begin{aligned} q_{t-1} \equiv\ & \theta_{1,t-1}\, [1 + \log(\psi_{1,t-2}) + w_1 + (w_0 - \psi_{1,t-2})(w_0 - \psi_{t-1})/(m(w_0 - \psi_{t-2}))] + \\ & (1 - \theta_{1,t-1})[1 + \log(\psi_{2,t-2}) + w_1 + (w_0 - \psi_{2,t-2})(w_0 - \psi_{t-1})/(m(w_0 - \psi_{t-2}))] \end{aligned} \tag{4.12b}$$

Expressions (4.9a), (4.9b), (4.10), (4.12a) together with the definitions (4.12b) and $\sum_{i=1,2} \theta_{i,t}\, \psi_{i,t} \equiv \psi_t$ t = 0, 1, ..., determine the actual law of motion with learning.

There are clearly two types of adaptive learning which are taking place simultaneously; one is directly concerned with the prediction of the inflation rate, the other with the revision of the subjective probabilities $\theta_{i,t}$.

For the sake of later reference, the actual law of motion with learning is re-written as follows:

$$\psi_{1,t} = \psi_{1,t-1} + g_{t-1}\, [m\, ((w_0 - \psi_{t-2})/(w_0 - \psi_{t-1})) - \psi_{1,t-1}] \tag{4.13a}$$

$$y_t = \psi_{1,t-1} \quad (4.13b)$$

$$g_t = [(1/g_{t-1}) (m (w_0 - \psi_{t-2})/(w_0 - \psi_{t-1}))^{-2} + 1]^{-1} \quad (4.13c)$$

$$\psi_{2,t} = \theta_{1,t-1} \psi_{1,t-1} + (1 - \theta_{1,t-1})\psi_{2,t-1} \quad (4.13d)$$

$$h_t = \psi_{2,t-1} \quad (4.13e)$$

$$\theta_{1,t} = \theta_{1,t-1} [1 + \log y_{t-1} + w_1 + (w_0 - y_{t-1})(w_0 - \psi_{t-1})/(m(w_0 - \psi_{t-2}))] \, 1/q_{t-1} \quad (4.13f)$$

where ψ_{t-1}, ψ_{t-2}, and q_{t-1} are defined as above.

As the reader can easily verify, an important feature of this system is that an ordered six-tuple $(m, m, 1 - m^{-2}, m, m, \theta)$ is a stationary state of (4.13) (with phase vector $(\psi_{1,t}, \psi_{1,t-1}, g_t, \psi_{2,t}, \psi_{2,t-1}, \theta_{1,t}))$ irrespective of the value of θ between zero and one. In such a state both ψ_1 and ψ_2 give correct predictions. Thus, there is a continuum of monetary stationary states of the actual dynamics with learning which are identical in all respects, except for the relative strength θ_1 attached to ψ_1. All these stationary states are typically non hyperbolic: the Jacobian matrix **DW** locally associated to the actual dynamics with learning has one eigenvalue z = 1. It follows that the local stability properties of the learning process can not be checked through a simple inspection of the spectrum of **DW**. This feature, and the relatively high dimensionality, suggest to take a simulation approach to the analysis of the dynamic properties of (4.13abcdef).

Before doing so, it is worth observing that the definition (4.12b) implies that if $\theta_{1,t-1} = 1$, so that $\psi_{1,t-1} = \psi_{t-1}$, then $\theta_{1,t} = 1$ and $\psi_{1,t} = \psi_t$. It follows that for the initial condition $\theta_{1,0} = 1$ the dynamics of $\theta_{1,t}$ is trivial and the dynamics of the other variables is completely determined by (4.13abc). This is in fact the subsystem studied in Bullard (1994).

Proposition 4.1

$(m, m, 1 - m^{-2})$ is the unique monetary stationary state of (4.13abc), with $\psi_{1,t} = \psi_t$ all t and phase vector $(\psi_t, \psi_{t-1}, g_t)$; the eigenvalues of the Jacobian matrix of (4.13abc) at $(m, m, 1 - m^{-2})$ lie strictly within the unit circle provided that the gross rate of money growth m is sufficiently close to one (cf. Bullard (1994), p. 474).

Proposition 4.2

There exists a critical value m^0 of the money growth, such that the monetary stationary state is locally stable if m < m^0 and unstable if m > m^0. The monetary stationary state undergoes a subcritical Hopf bifurcation when m goes through the critical value m^0. An unstable invariant closed curve exists near the stationary state for m less than but close to m^0, and coalesces to the stationary state at $m = m^0$(cf. Bullard (1994), p. 476).

The above results show that the adoption of a least-squares prediction rule can give rise to persistent deterministic oscillations on which predictions are never fulfilled. Although the periodic orbit of proposition 4.2 is repelling, our simulation results show that the rate of divergence can be very slow; a diverging trajectory can remain in a relatively small neighbourhood of the orbit for more than 80000 iterations! In the architecture of the model one iteration covers a time interval corresponding to one half of a representative life length; thus, oscillations of the type described above can be interpreted as relatively persistent. Indeed, stability and instability results require careful qualifications when their relevance for real-world economic issues is on the agenda. In most cases, even the economic structure which a model is meant to "capture" can be said to be persistent only in a weak and relative sense. This is the sense in which the term will be used in what follows. In any case, system 4.13abcdef (and the sub-system corresponding to Bullard's model) can be thought of as a local representation of a larger system in which a feed-back of the inflation rate on the growth rate of government expenditure is allowed for. The feed-back is such that the dynamics of the larger system is bounded.

Bullard's agents are far less naive than the "econometricians" of Evans Honkapohja and Sargent (1993), encountered in section 1. Still, oscillating trajectories like those implied by proposition 4.2, although possibly (indeed, generically!) non-periodic, would be regular. In particular, they would imply systematic, that is, non-random-looking forecast errors. Systematic errors may induce boundedly-rational, but smart, agents to revise their prediction rules. To answer this objection Bullard shows that a different specification of the saving function (in particular, a non-monotonic saving function) can generate complicated forecast errors within his model. These forecast errors would look in all respects random to the eye of a boundedly-rational agent.

The simulation exercise which is reported below is motivated by a different, but related problem. Why do Bullard's agents use a least-squares learning algorithm, in the first place? Since their model of the world is mis-specified, the OLS estimator is not optimal even within the subset of linear estimators. In other words, the ex-ante motivation for using a least-squares learning algorithm is more subjective (for instance, it may rest on the belief that the perceived law of motion is the true one) than objective. This is a problem faced by models of boundedly-rational learning in general and, indeed, almost by definition. In so far as learning is boundedly rational, the perceived law of motion is mis-specified, at least during the learning process, and the ex-ante choice of a given prediction rule, if motivated (as it should be) by the perceived law, rests on an incorrect belief. Dissatisfaction with this ad-hocness is highly mitigated if one can show that even in a larger learning space, the choice may turn out to be self-sustaining ex-post. This occurs when the prediction model survives a selection among a set of models which are plausibly in the tool-box of a boundedly-rational agent, each model being credited with a non-zero prior subjective probability. As will be shown below, the issue has direct bearings on the problem of the preservation of deterministic cycles.

5. Simulation results and discussion

Since the young-agent optimal consumption $c_0(t)$ is equal to the predicted gross rate of inflation ψ_t and predictions are correct in the monetary stationary state (given the predictions rule ψ_1 and ψ_2), it follows that the actual gross rate of money growth $m > 1$ is consistent with a monetary stationary state only if $m < w_0$. There are only three parameters in (4.13abcdef), namely w_0, w_1 and m. In what follows $w_0 = 2$, $w_1 = 1.8$ and m is allowed to vary between $\underline{m}$ and 2, where $\underline{m}$ is a point in a left-neighbourhood of m^0. (m^0 was defined in proposition 4.2; the choice $w_0 = 2$ implies $m^0 = 1.36603$.). In this range money growth is relatively high, but consistent with a monetary stationary state.

Fact 1.

Fix $g_0 = 0.2$ and a given m, such that $2 > m \geq \underline{m}$. Call $\Psi_0 \in \Re^4$ the vector of initial conditions on ψ_1, ψ_2 and define $\mathbf{A} \equiv \{(1, 2)\times(1, 2)\times(1, 2)\times(1, 2) - (m, m, m, m)\} \subset \Re^4_+$. For a given Ψ_0 in $\mathbf{A}$ let $\theta_{1,0}$ increase in the interval (0, 1). If $\theta_{1,0}$ is sufficiently low, the trajectories $\{\psi_{1,t}\}$ and $\{\psi_{2,t}\}$converge to the same correct prediction m so that the economy converges to the monetary stationary state. Convergence is fast and $\Omega(\theta_1) \cong \theta_{1,0} \cong \theta_{1,t}$, $0 < t < \infty$, where $\Omega(\theta_1) \equiv \lim_{t\to\infty} \theta_{1,t}$. The initial belief $\theta_{1,0}$ is therefore self-sustaining if $\theta_{1,0}$ is sufficiently low. For the sake of later reference, the critical value of $\theta_{1,0}$ below which Fact 1. is observed is called $\theta_1^*(g_0, \Psi_0, m)$. It is worth stressing that this critical value is close to 1 if m is close to $\underline{m}$.

Fact 2.

Hold g_0, Ψ_0 and m as in Fact 1. As $\theta_{1,0}$ is increased just above $\theta_1^*(g_0, \Psi_0, m)$ the trajectories $\{\psi_{1,t}\}$ and $\{\psi_{2,t}\}$ display regular oscillations of decreasing amplitude converging to the correct prediction m. Unlike Fact 1, $1 > \Omega(\theta_1) > \theta_{1,0}$. Since convergence to the monetary stationary state is now substantially slower, learning on θ_1 has time to take place. The trajectory $\{\theta_{1,t}\}$ is monotonically increasing, thus indicating that the prediction rule ψ_1 performs better than ψ_2; the trajectories $\{\psi_{1,t}\}$ and $\{\psi_{2,t}\}$ converge to m before the increase of θ_1 can reverse their qualitative dynamics.

Fact 3.

If the initial condition $\theta_{1,0}$ is larger than a critical value $\theta_1^{**}(g_0, \Psi_0, m) > \theta_1^*(g_0, \Psi_0, m)$, a reversal in the qualitative dynamics of $\{\psi_{1,t}\}$ and $\{\psi_{2,t}\}$ is observed as θ_1 moves along its monotonically increasing trajectory. The ongoing learning on θ_1 changes the amplitude of the oscillations of $\{\psi_{1,t}\}$ and $\{\psi_{2,t}\}$ from decreasing to increasing. If the model does not explode before $\Omega(\theta_1)$ is reached, then $\Omega(\theta_1) = 1$. The belief '$\theta_{1,0} = 1$ or sufficiently close to 1' is therefore, basically, self fulfilling in the parameter range of these simulations

Fact 4.

If $\theta_{1,0}$ is close to the critical value $\theta_1{}^{**}(g_0, \Psi_0, m)$, $\{\psi_{1,t}\}$ and $\{\psi_{2,t}\}$ remain in a small neighbourhood of a closed orbit for several thousands of iterations.

If the given m is close to $\underline{m}$, then $\theta_1{}^{**}(g_0, \Psi_0, m)$ is very close to 1. The results obtained support the conjecture that $\theta_1{}^{**}(g_0, \Psi_0, m)$ and $\theta_1{}^{*}(g_0, \Psi_0, m)$ decrease as m is increased in the interval $(\underline{m}, 2)$.

In synthesis, the belief "$\theta_{1,0}$ sufficiently low" and the belief "$\theta_{1,0}$ sufficiently close to 1" are both self-fulfilling in the parameter range of this simulation exercise. With the former belief the monetary stationary state is strongly stable and predictions are eventually correct. With the latter belief, predictions are always incorrect and the model economy gives rise to relatively persistent oscillations if m is in a left-neighbourhood of m^0. Oscillations of the same kind can also be brought about by the very process of learning on the subjective probability θ_1. This activity of learning greatly enlarges the range of values of the parameter m for which "persistent" oscillations are observed.

These findings illustrate the working of the proposed learning algorithm on a set of prediction rules. In particular, Facts 1 and 3 show that there is a potential conflict between the short-run and the long-run reward associated to a forecasting rule. Increasing the relative strength of a given prediction algorithm because of the relatively higher returns it yielded in the recent past, may lead to asymptotically incorrect predictions and to lower long-run returns. Facts 1 and 2 provide also an example of the possibly stabilizing influence of the uncertainty on the true law of motion. For this reason, the endogenous interpretation of macroeconomic fluctuations does not have the generality implied by Grandmont's argument considered in section 2. Still, the idea that local instabilities induced by plausible processes of expectation formation play a role in the up-rise and preservation of macroeconomic fluctuations receives a qualified support. Qualifications are supplied by the following remarks.

For a given exogenous rate of money growth, for instance m = 1.36029, even a minor perturbation of the prior belief, say from $\theta_{1,0}$ = 1 to $\theta_{1,0}$ = 0.999 can strongly stabilize the monetary stationary state. Conversely, for a given set of initial conditions and parameters w_0, w_1, the range of m-values which give rise to "persistent" oscillations is narrow.

The possibility of small random perturbations of m is interesting, for the following reasons. In the perturbed system, ψ_1 would *systematically* perform better than ψ_2, thus implying that the likely value of θ_1 is θ_1 = 1. If, in addition, these perturbations take place in an interval just below the critical value m^0, their influence on the actual rate of inflation would be amplified by the endogenous oscillations of $\psi_{1,t}$. In so far as the moments of ψ_1 may rapidly explode in the perturbed system (4.13abcdef), a feed-back of the actual rate of inflation on the growth of government expenditure should be introduced. This analysis provides a direction for further work. Indeed, the idea that the complexity and amplitude of the oscillations in time-series have much to do with the effects of small random perturbations within a non-linear system in which expectations matter, appears to provide a plausible paradigm for business-cycle analysis.

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