## 1. INTRODUCTION.

An effectivity function (EF) models the allocation of decision power among players and coalitions as induced by some underlying decision mechanism. Both the structure and the stability properties of EFs have been extensively studied in several different settings ( see e.g. Moulin,Peleg(1982), Moulin(1983), Peleg(1984 a,b), Ichiishi(1986), Abdou,Keiding(1991), Danilov,Sotskov(1993), Otten,Borm,Tijs,Storcken(1995) ) . The present paper provides a further contribution to the study of the *structure* of EFs by analysing some of their inner *lattice-theoretic* properties. Namely, the Galois lattice of an EF ( i.e. the product of its dually isomorphic complete lattices of Galois closure systems of coalitions and issues, respectively ) is introduced and used as a meaningful invariant in order to classify EFs . In that respect, the Galois lattice of an EF can serve us in several ways. We should like to emphasize the following two. First, it suggests new criteria for classifying EFs, including those which refer to the degree of power-sharing they embody. Second, it provides at least three natural measures of structural complexity for an EF ( namely, depth- or length- complexity, width complexity, and size complexity ). All that, in turn, invites the attempt at characterizing newly defined classes of EFs as well as an exploration of the connections between such Galois-latticial properties of an EF and other prominent properties of its such as (core) stability . The present paper provides some basic results on those matters.

In particular, it is shown that : i) the Galois lattice of an EF E is a chain if and only if E is representable by a pair of capacities ( monotonic real-valued measures ); ii) the Galois lattice of an EF E is made up of topological closure systems ( hence E is topology-inducing ) if and only if its Galois-closed sets are meet-irreducible whenever they are singleton- generated ; iii) convex (hence strongly stable) EFs of any possible width-complexity exist, while the length-complexity of the "longest" known convex EF on a pair (N,X) is well below the maximum length of an EF on (N,X) .

## 2. PRELIMINARIES : THE GALOIS LATTICE OF AN EFFECTIVITY FUNCTION.

Let N be the nonempty countable player set, and X the nonempty outcome set. We denote by $\mathbb{P}N$ ( $2^N$) the set of all ( nonempty ) subsets - or coalitions - of N, and by $\mathbb{P}X$ ( $2^X$ ) the set of all (nonempty ) subsets - or issues - of X . An **effectivity relation (ER)** on (N,X) is a binary relation $E \subseteq \mathbb{P}N \times \mathbb{P}X$ s.t. :

i) $(N,A) \in E$ for any $A \in 2^X$

ii) $(S,X) \in E$ for any $S \in 2^N$

iii) $(\emptyset,A) \notin E$ for any $A \in \mathbb{P}X$

iv) $(S,\emptyset) \notin E$ for any $S \in \mathbb{P}N$ .

The **effectivity function** $\mathcal{E}(E)$ attached to E is the function $\mathcal{E}(E):\mathbb{P}N \longrightarrow \mathbb{P}\mathbb{P}X$ defined by the rule $\mathcal{E}(E)(S)=\{ A \subseteq X : (S,A) \in E \}$ . Of course, i') $\mathcal{E}(E)(N)=2^X$ , ii') $X \in \bigcap_{S \in 2^N} \mathcal{E}(E)(S)$ , iii')$\mathcal{E}(E)(\emptyset)=\emptyset$ and iv') $\emptyset \notin \bigcup_{S \in 2^N} \mathcal{E}(E)$ . Conversely any **effectivity function (EF)** on (N,X), i.e. any function $\mathcal{E}:\mathbb{P}N \longrightarrow \mathbb{P}\mathbb{P}X$ satisfying properties i'-iv' induces an ER E(e) on (N,X) . Indeed, there is an obvious bijection between ERs and EFs on (N,X) , and we shall safely identify an EF with its relational counterpart. In view of this fact we shall henceforth indulge in a slight abuse of language and will denote by the same E both an ER and its corresponding EF on (N,X).

In fact, we are mainly interested in certain EFs -namely $\alpha$-EFs - arising from strategic game correspondences. A **strategic game correspondence** for (N,X) is a tuple $G=( N, (S_i)_{i \in N}, g )$ , where the outcome correspondence $g : \Pi_{i \in N} S_i \longrightarrow \mathbb{P}X$ is nonempty-valued and s.t. for any $x \in X$ a strategy profile $s \in \Pi_{i \in N} S_i$ exists : $g(s)=\{x\}$. A (strategic) game form is a single-valued (strategic) game correspondence . The **$\alpha$-EF** $E^\alpha(G)$ of a strategic game correspondence G is defined as follows : $E^\alpha(G)(\emptyset)=\emptyset$ and for any $S \in 2^N$, $E^\alpha(G)(S)=\{ A \subseteq X$ : an $s^S \in \Pi_{i \in S} S_i$ exists s.t. $G(s^S, t^{N \setminus S}) \subseteq A$ for any $t^{N \setminus S} \in \Pi_{i \in N \setminus S} S_i \}$ . It can be shown ( see Moulin(1983), Peleg(1984a) ) that an ER E on (N,X) is - modulo the equivalence defined above - the $\alpha$-EF of some strategic game correspondence on (N,X) if and only if it satisfies

**X-Monotonicity (X-MON)** : for any $S \subseteq N$, $A \subseteq B \subseteq X$, if $(S,A) \in E$ then $(S,B) \in E$ , and

**Superadditivity (SUPA)** : for any $S,T \subseteq N$, $A,B \subseteq X$ if $(S,A) \in E$, $(T,B) \in E$ and $S \bigcap T=\emptyset$ then $(S \bigcup T, A \bigcap B) \in E$ .

It is easily checked that SUPA entails the following basic properties :

**N-Monotonicity (N-MON)** : for any $S \subseteq T \subseteq N$ and $A \subseteq X$ if $(S,A) \in E$ then $(T,A) \in E$ , and

**Regularity (R)** : for any $S \subseteq N$ , $A \subseteq X$ , if $(S,A) \in E$ then $(N \setminus S, X \setminus A) \notin E$ .

Finally, an EF E is said to satisfy **Monotonicity (MON)** iff it enjoys both X-MON and N-MON. *Indeed, in what follows we shall mainly be focussed on such monotonic* EFs in view of the fact that both the $\alpha$-EF and the $\beta$-EF of any game correspondence satisfy MON.

The relationship between the $\alpha$-EF and the $\beta$-EF of a game correspondence can be generalized to any EF E on (N,X) by defining its **polar** $E^*$ by the following prescription : $(\emptyset,B) \in E^*$ for no $B \subseteq X$, and for any $S \subseteq N$, $S \neq \emptyset$, $B \subseteq X$, $(S,B) \in E^*$ iff $B \bigcap C \neq \emptyset$ for any $C \in E(N \setminus S)$ [ if E satisfies MON then this prescription reduces to $(S,B) \notin E^*$ if $\emptyset \in \{S,B\}$ and $(S,B) \in E^*$ iff $(N \setminus S, X \setminus B) \notin E$ otherwise .

Next, we use the classic construction due to Birkhoff ( see Birkhoff(1967) ) in order to attach to any ER E - hence to any EF - on (N,X) its own *Galois connection between* $\mathbb{P}N$ *and* $\mathbb{P}X$ . A **Galois connection** between two preordered sets ( Y, $\geq$ ) , ( Z, $\geq^*$) is a pair < f,g> of functions $f:Y\longrightarrow Z$ , $g:Z\longrightarrow Y$ such that for any $y,y' \in Y$ , $z,z' \in Z$ : GC i) $y \geq y'$ implies $f(y') \geq^* f(y)$ and $z \geq^* z'$ implies $g(z') \geq g(z)$ and GC ii) $(g \circ f)(y) \geq y$ , $(f \circ g)(z) \geq z$ .

A **closure operator** on a preordered set ( Y, $\geq$ ) is a function $F: Y \longrightarrow Y$ satisfying C i) $F(y) \geq y$ for any $y \in Y$ ; C ii) $F(x) \geq F(y)$ for any $x,y \in Y$ s.t. $x \geq y$ ; C iii) $F(y) \geq F(F(y))$ for any $y \in Y$. An element $z \in Y$ is closed w.r.t. F if $F(z)=z$ .

It is well known ( see e.g. Birkhoff(1967) , Theorem V.2 and Corollary ) that for any closure operator F on a set-inclusion ordered powerset ( $\mathbb{P}Y$ , $\supseteq$ ) the set $\mathcal{C}(F)=\{ Z \subseteq Y : F(Z)=Z\}$ of its closed elements - also called a **closure system** - is a *complete lattice* under the following natural definitions of inf and sup: $\inf\{Z_i\}_{i \in I}= \bigcap_{i \in I} Z_i$ and $\sup\{Z_i\}_{i \in I}=F( \bigcup_{i \in I} Z_i )$ ( we recall here that a **complete lattice** is a partially ordered set ( L , $\geq$ ) with L including both a $\geq$-greatest-lower-bound - or inf - and a $\geq$-lowest- upper-bound - or sup - for *any* subset $L' \subseteq L$ ; in particular, a complete lattice is *bounded* i.e. it is endowed with both a $\geq$-maximum - or *top* - and a $\geq$-minimum - or *bottom* - element. Moreover, an *atom* is a $\geq$-minimal non-bottom element, and a *co-atom* is a $\geq$-maximal non-top element ; a lattice L is *(co-) atomic* if for any non-bottom (non-top) $x \in L$ an atom ( co-atom ) a exists s.t. $x \geq a$ ( $a \geq x$ ) .

It is easily checked that if <f,g> is a Galois connection between $\mathbb{P}Y$ and $\mathbb{P}Z$ - ordered by set-inclusion - then $g \circ f : \mathbb{P}Y \longrightarrow \mathbb{P}Y$ and $f \circ g : \mathbb{P}Z \longrightarrow \mathbb{P}Z$ are both *closure operators* such that $f \circ g \circ f = f$ , and $g \circ f \circ g = g$ .

Also, it is well known from Birkhoff's classic work ( see Birkhoff(1967) , chpt. V) that a Galois connection $<f(\rho),g(\rho)>$ between $\mathbb{P}Y$ and $\mathbb{P}Z$ ( ordered by set-inclusion ) can be attached in a natural way to any binary relation $\rho \subseteq Y \times Z$ by the following rules : for any $B \subseteq Y$ , $C \subseteq Z$

$f(\rho)(B)=\{ z \in Z : (b,z) \in \rho$ for all $b \in B \}$ and $g(\rho)(C) = \{ y \in Y : (y,c) \in \rho$ for all $c \in C \}$ .

Hence, a pair of *closure operators* $K(\rho)= g(\rho) \circ f(\rho)$ , $K^*(\rho)=f(\rho) \circ g(\rho)$ ( on ( $\mathbb{P}Y, \supseteq$ ) and ($\mathbb{P}Z, \supseteq$ ) , respectively ) can be attached in a canonical way to any binary relation $\rho \subseteq Y \times Z$ . Clearly, the corresponding sets of closed sets $\mathcal{C}(K(\rho))$ and $\mathcal{C}(K^*(\rho))$ - or closure systems of $\rho$- are *complete lattices* under the definitions mentioned above of the *inf* and *sup* operations . Moreover, such lattices are dually isomorphic ( i.e. a latticial isomorphism between ( $\mathcal{C}(K(\rho))$ , $\subseteq$ ) and ( $\mathcal{C}(K^*(\rho))$ , $\supseteq$ ) can be defined ) : see again Birkhoff(1967) . The complete lattices of closed sets thus defined can be "merged" to obtain the **Galois lattice of** $\rho$ i.e. a complete lattice on $\mathcal{C}(K(\rho)) \times \mathcal{C}(K^*(\rho))$ as defined by the following rules :

for any $\{A_i\}_{i \in I} \subseteq \mathcal{C}(K(\rho))$ , $\{A'_i\}_{i \in I} \subseteq \mathcal{C}(K^*(\rho))$ s.t. $A'_i=f(A_i)$ , for any $i \in I$

$$\sup(\{(A_i,A'_i)\}_{i\in I}) = (\bigcap_{i\in I}A_i,\ K^*(\rho)(\bigcup_{i\in I}A'_i))$$
$$\inf(\{A_i,A'_i)\}_{i\in I}) = (K(\rho)(\bigcup_{i\in I}A_i),\ \bigcap_{i\in I}A'_i)\ .$$

In view of the previous observation concerning the equivalence between EFs and ERs ( on a fixed pair (N,X) ) , taking an ER as the basic relation $\rho$, and noticing that

$\alpha$) $\#(\mathcal{C}(K(\rho))\times\mathcal{C}(K^*(\rho)))\leq\min\{\#\mathbb{P}N,\ \#\mathbb{P}X\}$ and

$\beta$) ( $K(\rho)(\{N\})$ , $2^X$ ) is the unique co-atom - and ( $2^N$, $K^*(\rho)(\{X\})$ ) the unique atom - of the Galois lattice of $\rho$ thus defined, we obtain the following :

**PROPOSITION 1.** Let E be an effectivity function on (N,X) . Then a *complete lattice L(E) with a unique atom and a unique co-atom - the Galois lattice of E,* uniquely defined up to isomorphisms - can be canonically attached to E . Moreover, if either N or X is finite then L(E) is also finite.

**Remark 1.** It follows from Proposition 1 that for any EF E, $L(E)=\mathbf{1}\oplus L\oplus\mathbf{1}$ , for some lattice L, where **1** is the degenerate 1-element lattice, and $\oplus$ denotes the linear (or ordinal ) sum operation ( see e.g. Birkhoff(1967) chpt. VIII.10 ). L will also be called the **bulk** of L(E), and denoted by B(L(E)) .

A converse of Proposition 1 is also true .

**PROPOSITION 2.** Let L be a countable complete lattice with a unique atom and a unique co-atom. Then for some ( indeed, infinitely many ) pairs (N,X) an effectivity function $\mathcal{E}$ on (N,X) exists such that its Galois lattice L(E) is isomorphic to L .

**Remark 2.** The foregoing proof implicitly shows that some - and only some - EFs are entirely determined by their own Galois lattices, i.e. E(L(E))=E ( modulo bijections ) holds true for them. .

The construction underlying Proposition 1 above makes it clear that Galois-closure systems provide a significant invariant for EFs, though clearly not a complete or characteristic invariant, in view of Proposition 2 . This opens up the possibility that certain relevant properties of an EF be amenable to a natural Galois lattice-theoretical formulation ( as we shall see below).

## 3. CLASSIFYING EFFECTIVITY FUNCTIONS BY MEANS OF THEIR GALOIS LATTICES .

The Galois lattices of an EF E and of its polar $E^*$ - or their bulks ( see Remark 1 ) - embody a considerable amount of information on E, thereby constituting a most useful classificatory tool for EFs. Indeed, the length and the width of L(E) provide two useful structural complexity measures of the underlying EF ( the size or cardinality of L(E) could also be considered in that respect ) . Moreover, the latticial structure of L(E) - and of $L(E^*)$ - can also course be used in order to describe those properties of an EF E that can be inferred by inspection of latticial properties of L(E) . Finally, the concrete structure of L(E) as a product of *closure systems* - i.e. of lattices *of sets* - can be considered to the effect of identifying more specialized properties of E .

To begin with we define some useful ( order-theoretic ) parameters of the Galois lattice of an EF . We recall that the *width* $w(L)$ of a ( finite ) lattice L is the size of the largest antichain ( i.e. set of pairwise incomparable elements ) in L, the *length* $\ell(\mathcal{C})$ of a chain $\mathcal{C}$ of k+1 distinct elements is k, that the *height* h(x) of an element x of a lattice L is the least upper bound to the length of chains in L having x as their maximum, and the *length* $\ell(L)$ *of the lattice* itself is the least upper bound to the length of chains in L. Similarly, we define the *height* $h_E(S)$ *of a coalition* $S$ under a fixed EF E as the height of the highest closed set of coalitions $\mathcal{C} \in L_1(E)$ s.t. $S \in \mathcal{C}$.

A few interesting properties of an EF can be readily expressed in terms of height. Indeed, let $\mathcal{E}$ be an EF on (N,X) . Then E is

**Simple (S)** if a set $\mathcal{W} \subseteq \mathbb{P}N$ exists s.t. for any $S \subseteq N$, $B \subseteq X$ , $B \in \mathcal{E}(S)$ if and only if $S \neq \emptyset$ , $B \neq \emptyset$, and either $S \in \mathcal{W}$ or B=X, or equivalently for any $S \subseteq N$ either $h_{\mathcal{E}}(S)=1$ or $h_{\mathcal{E}}(S)=0$ .

**Consensual (CO)** iff $A \in \mathcal{E}(S)$ for any $A \neq \emptyset$ implies S=N ( i.e. the grand coalition is the only coalition endowed with full decision power ) or equivalently iff $h_{\mathcal{E}}(N) > h_{\mathcal{E}}(S)$ for any $S \subseteq N$ , $S \neq N$ .

**Fully Distributed (FD)** iff for any $i \in N$ an $A \subseteq X$ , $A \neq \emptyset$ exists s.t. $A \in \mathcal{E}(\{i\})$ ( i.e. any single player is endowed with some non-trivial decision power ) or equivalently $h_{\mathcal{E}}(\{i\}) > 1$ for any $i \in N$ .

**Unspecialized (US)** iff for any $S,T \subseteq N$, $h_{\mathcal{E}}(S)=h_{\mathcal{E}}(T)$ entails $\mathcal{E}(S)=\mathcal{E}(T)$ ( i.e. two coalitions having the same height are endowed with the same decision power ) .

**Strictly Hierarchical (SH)** iff for any $S,T \subseteq N$ , $S \neq T$ entails $h_{\mathcal{E}}(S) \neq h_{\mathcal{E}}(T)$ ( i.e. the coalitions can be linearly ordered w.r.t. their decision power and their height ).

**Remark 3 .** Clearly enough, the foregoing properties are not independent from each other . Indeed , both **S** *and* **SH** entail **US** ; moreover, both [ **CO** plus **FD** ] and **SH** alone entail **not S** ( whenever #N > 1 ). It should also be remarked that, as it is easily checked, *E is simple if and only if* $E^*$ *is simple* *too.*

The following elementary result is easily proved :

**PROPOSITION 3.** i) B(L(E))=**1** ( i.e. the degenerate one-element lattice ) if and only if $E^*$ is simple and consensual ; ii) B(L(E))=**2** ( the simple two-element Boolean lattice ) if and only if $E^*$ is simple and *not* consensual .

**Remark 4.** Since the (only) simple and consensual EF is the EF of the unanimity rule it is somehow remarkable that the EF of such extreme and most unpractical of rules turns out to be uniquely connected to the degenerate lattice **1** , while the rest of simple EFs is uniquely connected to the *simple* Boolean lattice **2** ( we recall here that a lattice is simple iff its only congruences are the trivial ones i.e. the identity congruence and the universal congruence ).

We proceed now to the characterization of another class of EFs, namely the class of monotonic EFs whose Galois lattice is a **chain** i.e. a totally ordered set . In order to do that, two more definitions are needed.

**Definition.** Let A be a nonempty set. A **capacity** on $\mathbb{P}A$ is a nonnegative monotonic function $\mu:\mathbb{P}A\longrightarrow\mathbb{R}$ i.e. for any $B,C\subseteq A$ $f(B)\geq 0$ and $B\subseteq C$ entails $f(B)\leq f(C)$ .

**Definition.** An EF E on (N,X) is **capacity-representable (CR)** if a pair of capacities $\mu:\mathbb{P}N\longrightarrow\mathbb{R}$ , $\nu:\mathbb{P}X\longrightarrow\mathbb{R}$ exist such that for any $\emptyset\neq S\subseteq N$, $\emptyset\neq B\subseteq X$ : $(S,B)\in E$ iff $\mu(S)>\nu(X\backslash B)$ .

**PROPOSITION 4.** Let E be a monotonic EF on (N,X) with N finite. Then L(E) is a chain iff E is CR .

**Remark 5.** The class of EFs having a chain as their own Galois lattice comprises most of the EFs which have been extensively studied in the literature, including *simple* EFs ( usually under the "simple games" sobriquet ) and *additive* EFs ( see Moulin,Peleg(1982), Moulin(1983), Peleg(1984) ).

Next, we move to the Galois-latticial characterization of a still wider class of EFs . This time, however, we shall have to rely on the concrete structure of the relevant Galois lattices as *lattices of sets* ( as opposed to their abstract latticial structure ) . Indeed, if we recall the fact that the Galois lattice of an EF is made up of two ( dually isomorphic ) closure systems that are in turn induced by two corresponding *closure operators.* Therefore, a natural question immediately arises : *under what circumstances are such closure operators topological ?*

The following definition is to be recalled here:

**Definition.** A closure operator K on a set-inclusion ordered powerset ($\mathbb{P}Z$ , $\subseteq$ ) is **topological** if F is i)*normal* - i.e. $K(\emptyset)=\emptyset$ - and ii) $\bigcup$-*additive* i.e. for any $A,B \subseteq Z$ $K(A \bigcup B)= K(A) \bigcup K(B)$ .

Indeed, whenever an EF E happens to have topological closure operators, the resulting closure system(s) define a topology on the player set ( and/or on the outcome set ) and the underlying EF is therefore **topology-inducing** ( a notion that is not to be confused with the usual notion of topological EF which simply refers to the case of EF whose outcome space is endowed with a given topological structure : see e.g. Abdou,Keiding (1991) ) .

**Definition.** An EF E on (N,X) is said to be **N-topology-inducing** ( **X-topology-inducing** ) whenever K(E) ( $K^*(E)$ , respectively ) is a topological closure operator .

**Remark 6.** It is easily checked that for any EF E, both $K(E)(\emptyset)=\emptyset$ and $K^*(E)(\emptyset)=\emptyset$ . Thus, the topological nature of K(E) and $K^*(E)$ depends solely on their $\bigcup$-additivity ( or lack of it ).

A few more notions are now to be introduced .

**Definition.** Let (f,g) be a Galois connection between $\mathbb{P}Y$ and $\mathbb{P}Z$ and $K=(g \circ f)$ be the closure operator on $\mathbb{P}Y$ induced by (f,g) . Then, a K-closed set $\mathfrak{C}$ is :

i) **singleton- generated** iff $\mathfrak{C}=g(\{B\})$ for some $B \in \mathbb{P}Z$

ii) **prime** iff for any pair $\mathcal{A},\mathfrak{B}$ of K-closed sets either $\mathcal{A} \subseteq \mathfrak{C}$ or $\mathfrak{B} \subseteq \mathfrak{C}$ whenever $\mathcal{A} \bigcap \mathfrak{B} \subseteq \mathfrak{C}$ ;

iii) **meet-irreducible** iff for any pair $\mathcal{A},\mathfrak{B}$ of K-closed sets $\mathfrak{C}=\mathcal{A} \bigcap \mathfrak{B}$ entails either $\mathfrak{C}=\mathcal{A}$ or $\mathfrak{C}=\mathfrak{B}$ .

**Remark 7.** It is immediately checked that a prime closed set is also meet-irreducible while the converse does not necessarily hold.

**Definition.** A lattice L is **distributive** iff $\inf(x, \sup(y,z)) = \sup(\inf(x,y), \inf(x,z))$ for any $x,y,z \in L$.

**PROPOSITION 5.** Let E be a monotonic EF on (N,X) . Then, i) E is N-topology-inducing iff any singleton-generated $K^*(E)$-closed set is meet-irreducible and dually ii) E is X-topology-inducing iff any singleton-generated K(E)-closed set is meet-irreducible .

**Example . ( An EF with a topological closure system that is not a chain ).** Since any set-inclusion-ordered chain of sets is obviously closed w.r.t. $\bigcup$, the EF-class characterized through Proposition 5 is certainly *not smaller* than that characterized through Proposition 4 . To check that it is indeed *larger,* one may consider the following example

Let E be an ER on $(N=\{1,2,3,4,5,6\}, X=\{x_0,x_1,x_2,x_4,x_5,x_6\})$ as defined by the prescription below : for any $S \subseteq N$, $B \subseteq X$ , $(S,B) \in E$ iff one of the following clauses obtains :

a) ( either $S \supseteq \{1,3\}$ or $S \supseteq \{1,5\}$ ) and ( $B \supseteq X\setminus\{x_1,x_3\}$, or $B \supseteq X\setminus\{x_1,x_5\}$, or $B \supseteq X\setminus\{x_0,x_1\}$, or $B \supseteq X\setminus\{x_0,x_2\}$ )

b) ( either $S \supseteq \{2,4\}$ or $S \supseteq \{2,6\}$ ) and ( $B \supseteq X\setminus\{x_2,x_4\}$, or $B \supseteq X\setminus\{x_2,x_6\}$, or $B \supseteq X\setminus\{x_0,x_1\}$, or $B \supseteq X\setminus\{x_0, x_2\}$ )

c) $S=N$ and $B \neq \emptyset$

d) $S \neq \emptyset$ and $B=X$ .

It is easily checked that $L(E)=2 \oplus 2^2 \oplus 2$ , hence L(E) is definitely not a chain. However, the only meet-reducible K(E)-closed set is $\mathcal{S}^*=\{ S \subseteq N : S \supseteq \{1,3\}$, or $S \supseteq \{1,5\}$, or $S \supseteq \{2,4\}$, or $S \supseteq \{2,6\}\}$ which is not singleton-generated . Therefore, $L^*(E)$ is indeed a topological closure system.

## 4. CONVEXITY AND GALOIS-LATTICIAL COMPLEXITY .

The present section will be devoted to a preliminary analysis of the following issue : how do "structural" Galois-latticial features of an EF combine with its (core) stability properties, which are of course the main focus of current literature on EFs ?

This aim will be pursued here by studying *convex* EFs .

**Definition .** Let E be an EF on (N,X) . Then, E is **convex** iff for any $S,T \subseteq N$ , $A,B \subseteq X$ : if $A \in E(S)$ and $B \in E(T)$ then either $A \cap B \in E(S \cup T)$ or $A \cup B \in E(S \cap T)$ .

Convex EFs are very nice, from a coalitional game-theoretic point of view : it is well-known that convex EFs are core-stable, i.e. $Core(E, \succ) \neq \emptyset$ for any profile $\succ$ of -say- acyclic preferences on the outcome set ( see e.g. Peleg (1984a,b) ). Also, they are known to enjoy a more stringent **strong stability** property that ensure that any "dominated"-outcome is dominated through a subset including a core outcome, thereby preventing some kinds of obvious manipulation activities ( see e.g. Abdou,Keiding(1991) ) . Furthermore, convex EFs are *the only* (core-) stable EFs on standard "large" domains among *maximal* EFs ( see again Abdou,Keiding(1991) ) .

Therefore, it is worth stressing - and perhaps a little surprising, in view of such a list of preciously rare properties they enjoy- that convex EFs comprise a class that is very rich in "structural" diversity, as the following example helps to emphasize :

**Example . ( A convex EF with a non-modular Galois-lattice )**

Let us consider the monotonic EF E on ( N= {1,2,3,4} , X={x,y,z,w} ) as defined by the following prescription:

for any $S \subseteq N$, $B \subseteq X$ , $(S,B) \in E$ if and only if one of the following clauses applies

a) $S \supseteq \{1,3,4\}$ and $B \supseteq \{x,w\}$ ;

b) $S \supseteq \{1,3\}$ and $B \supseteq \{x,z,w\}$ ;

c) $S \supseteq \{1,2\}$ and $B \supseteq \{y,w\}$ ;

d) S=N and $B \neq \emptyset$ ;

e) $S \neq \emptyset$ and B=X .

It can be easily checked that E is convex, and $L(E)=1 \oplus N_5 \oplus 1$ , where $N_5$ denotes the *pentagon* lattice, that is the well-known "archetypal" non-modular lattice ( see e.g. Birkhoff (1967) I.7 , p.17 ) . Two further observations are in order here. First, it should be stressed that in such a non-modular lattice the lengths of two different chains connecting the same pair need not be the same ( namely, the so called Jordan-Dedekind chain condition is violated ) . Thus, the height of an element ( hence of a coalition according to the definition proposed previously in Section 3), while well-defined, may lose some of its significance when applied to such an EF. Second, this example confirms - in view of Proposition 4 above - that while *some* convex ( namely, *additive* ) EFs are CR, this is not necessarily the case.

We proceed now to address the following issue : what is the possible range of Galois-latticial length- and width-complexity levels for a convex EF ? *Or, to put it in the simplest terms, what are if any the (Galois-latticial) complexity thresholds of a convex EF ?*

The following proposition provides a partial answer to that question .

**PROPOSITION 6.** Let (N,X) be a pair of nonempty finite sets, and s=min { #N , #X}. Then
i) for any $m \leq \sum^{s-1}_{h=1}(s-h)$ a monotonic convex EF E on (N,X) exists such that $\ell(L(E))=m$ ;
ii) whenever s is odd, for any $m \leq \max\{ w(L(E)) :$ E is an EF on (N,X)$\}$ a monotonic convex EF E on (N,X) exists such that $w(L(E)) = m$ .

In order to appreciate the significance of the foregoing result, a couple of observations have to be made. First, it should be stressed that Proposition 6 i) only provides a lower bound on the maximum Galois-latticial length-complexity of monotonic convex EFs . This is done in the proof by providing an example of a monotonic convex which is considerably "longer" ( in Galois-latticial terms ) of the "longest" monotonic convex EF explicitly studied in the previous literature ( to the best of the author's knowledge), namely the anonymous ( or neutral ) additive EFs ( see e.g. Moulin,Peleg(1982), Moulin(1983), Peleg(1984a), Abdou,Keiding(1991) ) . The author is unable at the moment to establish whether this lower bound can be ameliorated. However, it must be noticed that such a lower bound is well below the maximum Galois-latticial length-complexity of a monotonic EF . Indeed, it is easily seen that this maximum length for monotonic EFs amounts to the number $\alpha^*$ of antichains of the powerset of an s(N,X)-element set ( just define a chain of closed sets relying on any chain of antichains which extends the $\succeq^*$ partial order as defined above in the proof of Proposition 6i) ) . A good upper bound for $\alpha^*$ is $3^{n_{[n/2]}}$ ( see e.g. Anderson (1987), theorem 3.4.1 ) .

On the other hand, Proposition 6 ii) establishes that convexity does not force any additional limitation on the possible Galois-latticial width-complexity of a monotonic EF.

Indulging in a tentative attempt at interpretation, one might observe that Proposition 6 licenses the following conclusions : i) *augmenting the set of specialized decision tasks can be done without compromising core-stability of a decision procedure* , while *ii) a similar "stability-consistency" clause need not apply to adding hierarchical layers of decision power.*

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**APPENDIX . Proofs.**

**Proof of Proposition 2.** We just provide an outline of the relevant ( canonical ) construction. First, we may regard L as the lattice of $K^*(E)$-closed sets : hence, the unique atom $\alpha$ of L may be identified with {X} . Next, consider the set $L^1$ of elements of L that cover $\alpha$ i.e.

$L^1$={ $x \in L$ : $x \gg \alpha$ , namely $x > \alpha$ and $x > y > \alpha$ for no $y \in L$ } ,

and define the function f mapping $x \in L^1$ to { X, X\{x}} .

Then, consider $L^2$= { $y \in L$ : $y \gg x$ , for some $x \in L^1$ } .

For any $y \in L^2$ , we posit

$f(y)= \bigcup (f(x) : x \in L^1 , y \gg x )$ if y=supA for some $A \subseteq L^1$, and

$f(y)= \bigcup(f(x): x \in L^1, y \gg x) \bigcup\{ X\backslash\{y\}\}$ otherwise .

Working by induction, a countable subset $L^* \subseteq L$ is singled out . Repeating the argument with reference to the lattice of K(E)-closed sets it can be shown that L is isomorphic to L(E) where E=E(L) is an effectivity function on (N,X) with both N and X bijective to $L^*$ . The construction can be obviously replicated - for the same L - with suitably enlarged N and/or X . □

**Proof of Proposition 3.** i) Let B(L(E))=1 . Then $E=2^N \times 2^X$ . Now, $(S,B) \in E^*$ iff $S \neq \emptyset$, $B \neq \emptyset$ and $B \bigcap C \neq \emptyset$ for any C s.t. $(N\backslash S, C) \in E$. Of course, $(S,X) \in E^*$ for any S s.t. $\emptyset \neq S \subseteq N$ (by definition), and $(N,B) \in E^*$ for any $\emptyset \neq B \subseteq X$ since $(\emptyset, C) \in E^*$ for no $C \subseteq X$. However, $S \neq N$ and $B \neq X$ entail $(N\backslash S, X\backslash B) \in E$ whence $(S,B) \notin E^*$. Therefore $E^*$={ (N,B) : $\emptyset \neq B \subseteq X$ }$\bigcup${ (S,X) : $\emptyset \neq S \neq N$ } .

Conversely, let $E^*$ be simple and consensual i.e. $E^*$={ (N,B) : $\emptyset \neq B \subseteq X$ } $\cup$ { (S,X) : $\emptyset \neq S \neq N$}. Then, for any S,B s.t. $\emptyset \neq S$, $\emptyset \neq B$ , $(S,B) \in E$ ( since $(N\backslash S, C) \in E$ only if C=X ).

ii) It follows immediately from i) and from the fact that - as it is easily checked- E is simple if and only if $E^*$ is. □

**Proof of Proposition 4.** Let L(E) be a chain with bulk ( $(\mathfrak{C}_0,\mathfrak{D}_0)$, $(\mathfrak{C}_1,\mathfrak{D}_1)$,....., $(\mathfrak{C}_h,\mathfrak{D}_h)$ ) where $(\mathfrak{C}_{i-1},\mathfrak{D}_{i-1}) \geq (\mathfrak{C}_i,\mathfrak{D}_i)$ ( i.e. $\mathfrak{C}_{i-1} \subseteq \mathfrak{C}_i$ and $\mathfrak{D}_i \supseteq \mathfrak{D}_{i-1}$ for any i=1,..,h. . Then $\mu$ and $\nu$ are defined by the following rule. Take the unique co-atom $(\mathfrak{C}_0, \mathfrak{D}_0=2^X)$ of L(E), and its ( unique ) lower cover $(\mathfrak{C}_1,\mathfrak{D}_1)$, and posit $\mu(S)=m$ for any $S \in \mathfrak{C}_0$, and $\nu(X\backslash B) = m-k$ , with $m > h \cdot k > 0$ for any $B \in \mathfrak{D}_0\backslash\mathfrak{D}_1$ , $B \neq X$ ; $\mu(S)= m-2k$ for any $S \in \mathfrak{C}_1\backslash\mathfrak{C}_0$ , and $\nu(X\backslash B')= m-3k$ , for any $B' \in \mathfrak{D}_1\backslash\mathfrak{D}_2$ , $B' \neq X$ ; similarly, $\mu(S)= m-2i \cdot k$ for any $S \in \mathfrak{C}_i\backslash\mathfrak{C}_{i-1}$ , i=1,..,h , $\nu(X\backslash B)= m-(2i+1) \cdot k$ for any $B \in \mathfrak{D}_i\backslash\mathfrak{D}_{i+1}$, i=1,...,h − 1 , and $\nu(X\backslash B)=\nu(X)= m-2h \cdot k$ for any $B \in \mathfrak{D}_h$ .

Both $\mu$ and $\nu$ are well defined functions, since $\bigcup_i \mathfrak{C}_i=2^N$ and $\bigcup_i \mathfrak{D}_i=2^X$. Next, recall that monotonicity of E entails that both $\mathcal{C}_i$ and $\mathcal{D}_i$ are order filters i.e. $T \in \mathcal{C}_i$ if $T \supseteq S \in \mathcal{C}_i$, and $C \in \mathcal{D}_i$ if $C \supseteq B \in \mathcal{D}_i$, i=1,..,h . Now, let $S \subseteq T \subseteq N$ : then $T \in \mathcal{C}_i$ if $S \in \mathcal{C}_i$, for any i=1,..,h , whence $\mu(S) \leq \mu(T)$ by definition of $\mu$. Similarly, $A \subseteq B \subseteq X$ entails $X \backslash A \in \mathcal{D}_i$ whenever $X \backslash B \in \mathcal{D}_i$; thus $\min\{i : X \backslash B \in \mathcal{D}_i \backslash \mathcal{D}_{i+1}\} \leq \min\{ i: X \backslash A \in \mathcal{D}_i \backslash \mathcal{D}_{i+1}\}$ whence - by definition of $\nu$ - $\nu(A) \leq \nu(B)$.

Finally, let $\emptyset \neq S \subseteq N$, $\emptyset \neq B \subseteq X$, $i(S)=\min\{i : S \in \mathcal{C}_i \backslash \mathcal{C}_{i-1}\}$, $i(B)=\min\{i : B \in \mathcal{D}_i \backslash \mathcal{D}_{i+1}\}$ . If $(S,B) \in E$ then $i(S) \leq i(B)$, since $i(B) < i(S)$ entails $B \in \mathcal{D}_{i(B)} \backslash \bigcup_{j>i(B)} \mathcal{D}_j$, $S \notin \bigcup_{j \leq i(B)} \mathcal{C}_j$, whence $(S,B) \notin E$, a contradiction. But then, $\mu(S) > \nu(X \backslash B)$, by definition of $\mu$ and $\nu$ . On the other hand, $\mu(S) > \nu(X \backslash B)$ entails - by definition of $\mu$ and $\nu$ and monotonicity of E - $(S,B) \in \mathcal{C}_{i(S)} \times \mathcal{D}_{i(S)}$, whence $(S,B) \in E$ .

Conversely, let E be a capacity-representable EF on (N,X), and $(\mu,\nu)$ the relevant pair of capacities. Let us consider the set of pairs $(\mathcal{C}_i,\mathcal{D}_i)$ of closed sets which constitute the Galois lattice L(E) . For any $\mathcal{C}_i,\mathcal{C}_j$, $\mathcal{C}_i \neq \mathcal{C}_j$, and any $S,T \subseteq N$, if $S \in \mathcal{C}_i \backslash \mathcal{C}_j$ and $T \in \mathcal{C}_j \backslash \mathcal{C}_i$ then $\mu(S) > \nu(X \backslash B)$ for any $B \in \mathcal{D}_i$, $\mu(S) \leq \nu(X \backslash A)$ for some $A \in \mathcal{D}_j$, $\mu(T) > \nu(X \backslash B')$ for any $B' \in \mathcal{D}_j$, and $\mu(T) \leq \nu(X \backslash A')$ for some $A' \in \mathcal{D}_i$ . But then, $\mu(S) > \nu(X \backslash A') \geq \mu(T) > \nu(X \backslash A) \geq \mu(S)$, contradiction. Thus, either $\mathcal{C}_i \subseteq \mathcal{C}_j$ or $\mathcal{C}_j \subseteq \mathcal{C}_i$ ( and, dually, either $\mathcal{D}_i \supseteq \mathcal{D}_j$ or $\mathcal{D}_j \supseteq \mathcal{D}_i$ ); hence L(E) is a chain. □

**Proof of Proposition 5.** i) The thesis can be reduced to the following fact and claims .

***Fact .*** If L is a distributive lattice then $x \in L$ is prime if and only if it is meet-irreducible .

Proof. This is a well-known fact about distributive lattices ( see e.g. Birkhoff(1967), III.3, p.58 ).

***Claim 1.*** If K(E) - or $K^*(E)$ - is a topological closure operator then both L(E) and $L^*(E)$ are distributive lattices.

Proof. Firstly, recall that - as mentioned above - any K(E)-closed ( or $K^*(E)$-closed ) set is an order filter or upset of ( $\mathbb{PP}N$ , $\subseteq$ ) ( of( $\mathbb{PP}X$ , $\subseteq$ ) , respectively ) , by monotonicity of E . Thus, the closure system $\{ \mathcal{A} \subseteq \mathbb{P}N : (K(E))(\mathcal{A})=\mathcal{A} \}$ induced by K(E) is a subset of $\mathcal{F}(\mathbb{P}N)$, the set of all order filters of ($\mathbb{P}N$ , $\subseteq$ ) . Secondly, observe that - as it is checked - ($\mathcal{F}(\mathbb{P}N)$, $\subseteq$ ) is a distributive lattice with respect to the set-theoretic operations ( i.e. with inf=$\bigcap$ and sup=$\bigcup$ ) . Since any closure system is $\bigcap$-closed, $\bigcup$-additivity of K(E) makes the corresponding closure system a *sublattice* of $\mathcal{F}(\mathbb{P}N$, $\subseteq$ ) hence L(E) - and $L^*(E)$, which is isomorphic to L(E) as a lattice - is a distributive lattice. The same argument applies to $K^*(E)$.

***Claim 2.*** K(E) is a topological closure operator if and only if any singleton-generated $K^*(E)$-closed set is prime . Dually, $K^*(E)$ is a topological closure operator if and only if any singleton-generated $K(E)$-closed set is prime .

Proof. By definition $K(E)= g(E)\circ f(E)$ . First, notice that $K(E)$ is $\bigcup$-additive iff $g(f(A)\bigcap f(B)) \subseteq g(f(\mathcal{A}))\bigcup g(f(\mathcal{B}))$ . This fact is easily proved as follows : since $K(E)$, being a closure operator, is inflationary ( i.e. $\mathcal{A}\subseteq (K(E))(\mathcal{A})$ for any $\mathcal{A}\subseteq \mathbb{P}N$ ) ) it follows that $K=g\circ f$ is $\bigcup$-additive iff

$$(*) \quad (g\circ f)(\mathcal{A}\bigcup \mathcal{B}) \subseteq (g\circ f)(\mathcal{A}) \bigcup (g\circ f)(\mathcal{B}) .$$

and, dually, $K^*(E)$ is $\bigcup$-additive iff

$$(*') \quad (f\circ g)(\mathcal{A}^*\bigcup \mathcal{B}^*) \subseteq (f\circ g)(\mathcal{A}^*) \bigcup (f\circ g)(\mathcal{B}^*) .$$

Also, observe that, by antitonicity of f and g,

$f(\mathcal{A}\bigcap\mathcal{B}) \supseteq f(\mathcal{A})\bigcap f(\mathcal{B})$ and $g(\mathcal{A}^*\bigcap\mathcal{B}^*) \supseteq g(\mathcal{A}^*)\bigcap g(\mathcal{B}^*)$, for any $\mathcal{A},\mathcal{B}\subseteq \mathbb{P}N$, $\mathcal{A}^*,\mathcal{B}^*\subseteq \mathbb{P}X$.
Moreover, for any $\mathcal{A},\mathcal{B}\subseteq \mathbb{P}N$, $\mathcal{A}^*,\mathcal{B}^*\subseteq \mathbb{P}X$ ;

$$(**) \quad f(\mathcal{A}\bigcup\mathcal{B})=f(\mathcal{A})\bigcap f(\mathcal{B}) , \quad \text{and} \quad g(\mathcal{A}^*\bigcup\mathcal{B}^*)=g(\mathcal{A}^*)\bigcap g(\mathcal{B}^*)$$

( because $f(\mathcal{A}\bigcup\mathcal{B})\subseteq f(\mathcal{A})\bigcap f(\mathcal{B})$ , $g(\mathcal{A}^*\bigcup\mathcal{B}^*)\subseteq g(\mathcal{A}^*)\bigcap g(\mathcal{B}^*)$ by antitonicity of f and g, again, while for any $B\subseteq X$, $B\in f(\mathcal{A})\bigcap f(\mathcal{B})$ entails $(S,B)\in E$ for any $S\in\mathcal{A}$ and $(T,B)\in E$ for any $T\in\mathcal{B}$, by definition of $f=f(E)$, whence $(S',B)\in E$ for any $S'\in\mathcal{A}\bigcup\mathcal{B}$, and similarly for g ).

As a result, we may conclude from (*) and (**) that $K(E)$ is $\bigcup$-additive iff for any $\mathcal{A},\mathcal{B}\subseteq \mathbb{P}N$

$$(***) \quad g(f(\mathcal{A})\bigcap f(\mathcal{B}))\subseteq g(f(\mathcal{A}))\bigcup g(f(\mathcal{B}))$$

and similarly, in view of (*') and (**'), $K^*(E)$ is $\bigcup$-additive iff for any $\mathcal{A}^*,\mathcal{B}^*\subseteq \mathbb{P}X$

$$(***') \quad f(g(\mathcal{A}^*)\bigcap g(\mathcal{B}^*))\subseteq f(g(\mathcal{A}^*))\bigcup f(g(\mathcal{B}^*)) .$$

Now, let us suppose that $K(E)$ is $\bigcup$-additive. Let $\mathcal{C}^*\subseteq \mathbb{P}X$ be a singleton-generated $K^*(E)$-closed set, i.e. without loss of generality $\mathcal{C}^*=(K^*(E))(\mathcal{C}^*)$ and $\mathcal{C}^*=f(\{S\})$ for some $S\in \mathbb{P}N$ . We have to show that $\mathcal{C}^*$ is prime. In order to do that, consider a pair $(\mathcal{A}^*,\mathcal{B}^*)$ of $K^*(E)$-closed sets - i.e. w.l.o.g. $\mathcal{A}^*=(K^*(E))(\mathcal{A}^*)=f(g(\mathcal{A}^*))$, $\mathcal{B}^*=(K^*(E))(\mathcal{B}^*)=f(g(\mathcal{B}^*))$ - such that $\mathcal{C}^*\supseteq\mathcal{A}^*\bigcap\mathcal{B}^*$ . Then, $g(\mathcal{C}^*)\subseteq g(\mathcal{A}^*\bigcap\mathcal{B}^*)$ , by antitonicity of g , or equivalently $g(\mathcal{C}^*)\subseteq g( f(g(\mathcal{A}^*))\bigcap f(g(\mathcal{B}^*)) )$. It follows from (***) that $g(\mathcal{C}^*)\subseteq g(f(g(\mathcal{A}^*))) \bigcup g(f(g(\mathcal{B}^*)))$ . Recalling that - according to one basic property of Galois connections which has been mentioned above (see Section 2) - $g=g\circ f\circ g$ , the foregoing relationship can also be written $g(\mathcal{C}^*)\subseteq g(\mathcal{A}^*) \bigcup g(\mathcal{B}^*)$ . Since ( by hypothesis ) $\mathcal{C}^*=f(\{S\})$, $g(f(\{S\}))=(K(E))(\{S\})\supseteq\{S\}$ or $S\in g(\mathcal{C}^*)$ , whence either $S\in g(\mathcal{A}^*)$ or $S\in g(\mathcal{B}^*)$ . Let us assume w.l.o.g. $S\in g(\mathcal{A}^*)$ that is $\{S\}\subseteq g(\mathcal{A}^*)$ . Then, by antitonicity of f, $\mathcal{A}^*=f(g(\mathcal{A}^*))\subseteq f(\{S\})=\mathcal{C}^*$. Therefore $\mathcal{C}^*$ is indeed prime.

Conversely, let us suppose that any singleton-generated $K^*(E)$-closed set is prime . Consider now two $K^*(E)$-closed sets $\mathcal{A}^*,\mathcal{B}^*$ , and let $S\in g(\mathcal{A}^*\bigcap\mathcal{B}^*)$ i.e. $\{S\}\subseteq g(\mathcal{A}^*\bigcap\mathcal{B}^*)$ . Put $\mathcal{C}^*= f(\{S\})$ . Since $f\circ g\circ f=f$ ( as observed above ) it follows that $(K^*(E))(\mathcal{C}^*)=f(g(\mathcal{C}^*))=f(g(f(\{S\})))=\mathcal{C}^*$ hence

$C^*$ is a singleton-generated $K^*(E)$-closed set . Moreover, by antitonicity of f, $\mathcal{C}^* \supseteq f(g(\mathcal{A}^* \cap \mathcal{B}^*)) = \mathcal{A}^* \cap \mathcal{B}^*$ ( the last equality follows from the fact that the meet of two closed sets must be a closed set). Thus, either $\mathcal{C}^* \supseteq \mathcal{A}^*$ or $\mathcal{C}^* \supseteq \mathcal{B}^*$ since $\mathcal{C}^*$ must be prime, by our hypothesis. Let us assume w.l.o.g. $\mathcal{C}^* \supseteq \mathcal{A}^*$ . Then $g(\mathcal{C}^*) \subseteq g(\mathcal{A}^*)$ by antitonicity of g . Therefore $\{S\} \subseteq (K(E))(\{S\}) = g(f(\{S\})) = g(\mathcal{C}^*) \subseteq g(\mathcal{A}^*)$ . It follows that

(+) $g(\mathcal{A}^* \cap \mathcal{B}^*) \subseteq g(\mathcal{A}^*) \cup g(\mathcal{B}^*)$ for any pair of $K^*(E)$- closed sets .

Notice, however, that (+) entails (***) above since $f \circ g \circ f = f$ implies that f-images are in fact $K^*(E)$-closed sets. As a result, K(E) is $\bigcup$-additive .

ii) It follows from a step by step "dualization" of the argument sub i) . □

**Proof of Proposition 6.** i) Let us assume w.l.o.g. that s=n=#N . Then, observe that every chain of L(E) is made up of ( pairs of ) *nested* order filters ( of $(\mathbb{P}N, \subseteq)$ and $(\mathbb{P}X, \subseteq)$ , respectively ) , by monotonicity of E. But then we may take advantage of the well-known bijection between an order filter $\mathcal{C}$ of, say, $(\mathbb{P}N, \subseteq)$ and the $\subseteq$-antichain $a(\mathcal{C})$ of its minimal elements or generators, and recall that for any pair $\mathcal{C},\mathcal{D}$ of order filters of $(\mathbb{P}N, \subseteq)$ , $\mathcal{C} \subseteq \mathcal{D}$ iff $a(\mathcal{C}) \preceq^* a(\mathcal{D})$ where $\preceq^*$ is a partial order on antichains as defined by the prescription $a(\mathcal{C}) \preceq^* a(\mathcal{D})$ iff for any $C \in \mathcal{C}$ a $D \in \mathcal{D}$ exists such that $C \subseteq D$ ( see e.g. Anderson(1987) , chpt. 13 ). Therefore our thesis amounts to showing that a monotonic convex EF E can be devised such that the ( pairs of) order filters constituting the closure systems of L(E) correspond to a ( $1+\sum^{s-1}_{h=1}(s-h)$ )-sized chain of $\preceq^*$-nested antichains. This can be easily proved , using the following construct : first choose a permutation $\pi$ on N, and take the trivial antichain $\{\{\pi(1)\}\}$ : let us call it -for ease of reference- "layer 1"; then take the antichain of all two-element subsets of N comprising $\pi(1)$, followed by its subsets $\{\{\pi(1),\pi(2)\},...,\{\pi(1),\pi(k)\}\}$ with k=n,...,2 ( let us call this subchain of antichains made up of two-element subsets "layer 2") , and so on ( i.e. "layer k+1" consists of nested antichains of k+1-subsets comprising $\{\pi(1),\pi(2),..,\pi(k)\}$ defined as follows :

$\{\{\pi(1),\pi(2),..,\pi(k),\pi(k+1)\},...,\{\pi(1),\pi(2),..,\pi(k),\pi(k+h)\}\}$ , h= n − k, ..,h=1 ).

That the size of this chain is $S(s)=1+\sum^{s-1}_{h=1}(s-h)$ is easily shown by induction on s. Indeed, the thesis is clearly true for s=2 . Then, suppose that it is true for for s=m, and consider s=m+1. But adding one element to the ground set of the construct under consideration implies that i) one "layer" is added ( the last one, i.e. the m+1-th, which is made up of one one-element antichain, i.e. the ground m+1-element set ) , and for any pre-existing "layer k", 1<k<m one element is added to each antichain of layer k to the effect of adding one antichain to the "layer" ; "layer 1" is not changed at all. As a result the size of the resulting chain with s=m+1 is $S(m+1)=1+[\sum^{m-1}_{h=1}((m-h)+1)]+1= 1+\sum^{m}_{h=1}((m+1)-h)$ .

It remains to be shown that a *convex* EF E on (N,X) can be defined such that the chain of nested antichains or order filters of ($\mathbb{P}$N, $\subseteq$) just defined is the closure system attached to its closure operator K(E) . In order to see this, we may define E as follows . Let $(\mathcal{C}_1,...,\mathcal{C}_{S(s)})$ the nested chain of order filters of ($\mathbb{P}$N, $\subseteq$) implicitly defined by the previous construct. Clearly, $\mathcal{C}_i \supseteq \mathcal{C}_{i+1}$ and $\mathcal{C}_{S(s)}=\{N\}$, by construction. Take $X=\{x_0, x_1,....., x_{S(s)}\}$ and define an increasing sequence $(\mathcal{D}_1,...,\mathcal{D}_{S(s)})$ of order filters of ($\mathbb{P}$X, $\subseteq$) by the following rule :

$\mathcal{D}_1=\{B \subseteq X : B \supseteq X\backslash\{x_0\}\}$ , $\mathcal{D}_2=\{B \subseteq X : B \supseteq X\backslash\{x_0,x_1\}\}$, and, generally speaking, $\mathcal{D}_i=\{B \subseteq X: B \supseteq X\backslash\{x_0, x_1, x_{i-1}\}\}$ , i=1,.., S(s) − 1 , and $\mathcal{D}_{S(s)}=2^X$ .

Finally, posit $(S,B) \in E$ iff either $\max\{i : S \in \mathcal{C}_i\} \geq \min\{i : B \in \mathcal{D}_i\}$, or $S \neq \emptyset$ and B=X ..

To check convexity of E, consider S,T,A,B such that $(S,A) \in E$ , $(T,B) \in E$ . If $\{S,B\} \cap (\mathbb{P}N\backslash \bigcup_i \mathcal{C}_i) \neq \emptyset$ then either A=X or B=X , whence $A \cap B \in \{A,B\}$ and $A \cap B \in E(S \cup T)$ immediately follows. So let us assume $\{S,B\} \subseteq \bigcup_i \mathcal{C}_i$ , $j=\max\{i : S \in \mathcal{C}_i\}$ , $k=\max\{i : T \in \mathcal{C}_i\}$ . If $j=k=i^*$, then $A \cap B \supseteq X\backslash\{x_0,..,x_{i^*-1}\}$ while $S \cup T \in \mathcal{C}_{i^*+1}$ if $S \neq T$ (or $S \cup T \in \mathcal{C}_{i^*}$ if S=T)) whence $(S \cup T, A \cap B) \in E$ . If ( w.l.o.g.) $j > k$ , $A \cap B \supseteq X\backslash\{x_0,x_1,..,x_{j-1}\}$ , and $S \cup T \in \mathcal{C}_j$, whence again $(S \cup T, A \cap B) \in E$ .

ii) Let $s=2 \cdot h+1$ , for some positive integer h, $\#N > 1$, $\#X > 1$ , and (w.l.o.g.) s=n=#N . Now, a well-known extension of the classic Sperner's theorem on antichains establishes that Y is an n-element set with n odd, the set of all $\frac{1}{2}(n+1)$-element subsets of Y is an antichain of maximum size of ($\mathbb{P}$Y, $\subseteq$) ( see e.g. Anderson(1987), chpt.1, Theorem 1.2.2 ). Thus, take the family $\mathbf{S}=\{S_i : S_i \subseteq N, \#S_i=\frac{1}{2}(n+1)\}_{i \in I}$ of all $\frac{1}{2}(n+1)$-element subsets of N, and posit $X=\{x_i : i \in I\}$ , with $x_j \neq x_h$ for any $j,h \in I$ , $j \neq h$. Then define an EF E on (N,X) as follows : for any $S \subseteq N$, $A \subseteq X$ , $(S,B) \in E$ iff one of the following clauses applies :

a) $S \supseteq S_i$ and $A \supseteq X\backslash\{x_i\}$ ; b) S=N and $A \neq \emptyset$ ; c) $S \neq \emptyset$ and A=X .

It is immediately checked that $B(L(E))=\mathbf{M}_m$ where m=#I , i.e. B(L(E)) is the (non-distributive) lattice having exactly m atoms, and such that each one of them is also a co-atom . Therefore w(L(E))=m=max{ w(L(E)) : E is an EF on (N,X) with s(N,X)=k } .

To check convexity of E, assume $(S,A) \in E$ , and $(T,B) \in E$ . The following cases can be distinguished : $\alpha$) either ( $S \cap (N\backslash S_i) \neq \emptyset$ for any $S_i \in \mathbf{S}$) or ( $T \cap (N\backslash S_i) \neq \emptyset$ for any $S_i \in \mathbf{S}$) . In this case, $X \in \{A,B\}$ hence $A \cap B \in \{A,B\}$ and then $(S \cup T, A \cap B) \in E$ .

$\beta$) An $S_i \in \mathbf{S}$ exists such that both $S \supseteq S_i$ and $T \supseteq S_i$ . If $A \cap B \in \{A,B\}$, then again $(S \cup T, A \cap B) \in E$ . Otherwise, $A \cup B=X$ ( by definition of E ) . Since $S \cap T \supseteq S_i \neq \emptyset$, $(S \cap T, A \cup B) \in E$ .

$\gamma$) $S_i,S_j \in \mathbf{S}$ exist such that $S \supseteq S_i$ , $T \supseteq S_j$ , but $(S \cap T) \cap (X\backslash S_h) \neq \emptyset$ for any $S_h \in \mathbf{S}$ . In this case $A \cup B=X$ . Therefore, $(S \cap T, A \cup B) \in E$, since clearly $S \cap T \neq \emptyset$ , by definition of $\mathbf{S}$ . □