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Knightian Uncertainty Causes Price Intervals in Financial Markets

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Abstract - In financial models it is assumed that an asset is equivalent to a replicating portfolio of marketed assets, which span the state space. To prevent any arbitrage possibility, the value of the asset must equal the market value of this portfolio and the unique market price of the asset may be defined common expectation. Introducing Knightian uncertainty it is possible to distinguish two quite different classes of agents, who have different subjective probability distributions. As a result, linear pricing is replaced by Choquet non-linear pricing and for any asset it is possible to determine a closed price interval in which there is partial inertia. This interval of rational prices only depends on the agents' Knightian uncertainty attitude.

Key words: Knightian uncertainty, capacity, Choquet asset pricing rule, no-trade prices

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1. INTRODUCTION

Given a set *S* of finite states of the world and a set of commodity *G*, such that each good $g \in G$ is characterized by physical characteristics, location, date of availability and state of the world, Arrow and Debreu define a two period (the present and the future) economy. In that economy with a large number of contingent markets for contingent goods, a Pareto-efficient equilibrium emerges.

With respect to a contingent commodity market equilibrium, which assumes that exists a market for contracts to deliver a particular commodity in each states of the world and uncertainty is generated exogenously by occurring of states of the world, security markets are an effective alternative. By assuming the existence of a security set that span the finite set of states of the world (there is one security for each state $s \in S$ of the world and it pays a return in that state only and nothing elsewhere), Arrow (1953) substitutes all contingent commodity markets with security markets. In this way he economizes on the number of markets required by equilibrium from $G(S+1)$ up to $G+S+1$. Securities also permit to convert an economy where trading of Arrow-Debreu goods occurs at once into a 'sequence economy' where there is trading at every date. Nevertheless, in the complete Arrow security economy the agents, that don't need to form price expectations of goods at future dates, act as if all trading took place at once and they face a single budget constraint, a sort of Arrow-Debreu budget constraint. The complete Arrow security economy is defined to be an inessential sequence economy, which is "isomorphic to the original Arrow-Debreu economy" (Hahn, 1992).

Radner (1968, 1972) defines a sequence economy for goods and securities in which there is trading at every date and realized state and points out the existence of equilibrium. Since agents have more than one budget constraint and trade more than once, Radner's sequence economy is defined essential by Hahn (1973). There are some aspects of Radner's arguments that I would like to point out because they provide a premise on my approach.

Radner (1968) assumes agents with asymmetric information about the states of the world, represented by their different partitions of S, and notes that this "source of moral hazard can only be avoided by trades conditioned on the common coarsest partition" (Hahn 1992).

Unlike Arrow (1953), who assumes the existence of perfect price foresight[1], Radner proposes the concept of common expectation. Common expectation requires that all agents "associate the same future price to the same future exogenous events, but does not require them to agree on the subjective probabilities associated with those events" (Radner 1972, p. 289). Assuming that the traders associate the same future prices to the same events, "does not necessarily imply that they agree on the joint probability distribution of future prices, since different traders might assign different subjective probabilities to the same events." (Ibidem, p. 289) Summing up, instead of assuming rational expectations Radner argues that prices both encompass and reveal all the information needed for trading and each agent improves her/his knowledge in a Bayesian manner. Radner demonstrates the existence of equilibrium in that exchange sequence economy.

In a competitive market with no transaction costs, if the number of linearly independent securities equals all the possible states of the world (there is a sufficient rich array of securities), markets securities are complete and portfolios of securities can replicate any pattern of returns across states.

Let a security $a:S\to R^S$ be defined by its vector of returns in different states of the world, such that $a_j=1$ if $s=j$ and $a_j=0$ otherwise, and let $q\in R_+^S$ be the price vector of securities. Any marketable portfolio $\Psi:S\to R^S$ can be constructed and it equals a finite list of marketed securities[2] and, with no transaction costs, the cost of the portfolio C(Ψ) is $\sum_{j=1}^{S} a_j q_j$. Such a portfolio can be considered equivalent to an asset β that exactly yields an equal amount. No arbitrage condition implies that two portfolios Ψ and Φ, which yield the same payoff, have the same cost, that is $C(\Psi)=C(\Phi)$.

Under both no arbitrage and no transaction costs conditions, the market value of any asset is the expected value of its future dividends or flows of payments. Roughly speaking, there is none arbitrage opportunity if two portfolio of securities yielding the same revenue have the same price, that is the same formation cost. The main feature of this argument is that price at which an asset is traded is given by formation costs of portfolios replicating it. As a consequence, by no arbitrage and no transaction cost conditions, the pricing functional of the economy is unique, positive and linear.

In Arrow's model the security prices can be normalized so that they sum up to one and the summation of security prices may be interpreted as a probability distribution on the space of states. It is remarkable to note that the derived probability distribution is not a probability distribution (subjective or objective) of the agents on the set of states of the world, but it is simply "a weighting of the states made by prices which express an aggregation of agents behaviors towards uncertainty" (Chateauneuf 'et al.' 1992). From a theoretical point of view, all valuation models in finance, the most famous of which is the Black and Scholes (1973) one, can be considered as a generalization of the complete Arrow's model. In fact, given frictionless competitive markets[3], no arbitrage conditions[4] and asset price that follows a particular diffusion process (a geometric Wiener process[5]), in the Black and Scholes' model there is a unique probability distribution on the measurable space (S,Σ) such that market value of any asset is the expectation of its payments.[6] The unique additive probability distribution is "the analogue of Arrow's probabilities of the states defined by (Arrow) *security* price…that probability is revealed by market prices and has nothing to do with agents subjective beliefs" (Ami '.et al' 1991).

Some recent articles have shown that the valuation of an asset will be not a linear pricing rule (Lebesgue integral of the asset payments) but will be obtained by Choquet integral of the asset payments (non-linear pricing rule), if an agent has a non-additive measure or a capacity on (S,Σ). By a non-linear pricing rule, there might be portfolio inertia, that is an interval of prices within which each agent neither buys nor sells short the asset (Simonsen and Verlang 1991, Dow and Verlang 1992). The focus of this paper is the definition of a closed price interval, induced by expected market prices of an asset, in which there is a sort of *portfolio inertia* or a *thin market*. The definition of *two rational bounds* for price asset permits one to explain why trade are difficult in some range of prices, which class of agents are in the market if trading occurs, when all agents are willing to trade.

In this paper I assume that a finite set S of states of the world exists and agents exhibit Knightian uncertainty. All the agents face Knightian uncertainty about future events, but they have common expectation as in Radner (1972). Agents agree about the structure of the portfolio that generates a given asset, but they disagree with respect to probability of future states of the world. Roughly speaking, agents assume that an asset is completely defined by its flows of payments and take as given the structure of the replicating portfolio of securities. However, since the 'probability distribution' induced by the replicating portfolio does not represent the probability distribution on future events, they might have quite different probability distributions, that depend on their beliefs about these events.

While Chateauneuf 'et al.' (1992) assume a class of Knightian uncertainty seeking price-makers (brokers), who induce the bid and ask spread in an asset price, I assume that all agents are risk-neutral price-takers but they may be either Knightian uncertainty averse or seeking. It is assumed that all agents are split between the classes of uncertainty seeking (optimist) and uncertainty averse (pessimist).[7] The beliefs of the optimistic and pessimistic agents respectively determine the lower and upper bound of an asset price In this closed interval there is partial inertia and a thin market, outside the interval agents always trade and there is a thick market. This conclusion could have relevant application in financial markets, e.g. it permits to define the minimum price at which a hostile takeover can be bidden and the maximum price at which either the floating on the stock market can be made over or an initial public offer can be launched.

The paper is organized as follows. Section 2 introduces capacities in order to represent Knightian uncertainty. In section 3 the model is worked out and an interval of prices is defined. An example is shown in Section 4. The concluding remarks make up Section 5.

2. KNIGHTIAN UNCERTAINTY ATTITUDE AND CAPACITIES

Models of a sequence economy assume that states of the world have an additive probability of occurring, that is each agent's description of states of the world is exhaustive. Each agent has (explicitly or implicitly) a unique (common or not) and fully reliable probability distribution over events.

Consider a sequence economy in which states of the world included in the model do not exhaust the actual ones. A description of the world is considered as a misspecified model whenever that omitted states are not explicitly included in the model. When an agent does not know how many states are omitted (possibly missing markets), she/he can represent her/his beliefs by means of either a capacity (non-additive measure) or a convex set of probability distributions, none of which is considered fully reliable, on the set of events.[8] A situation that involves misspecified description of states of the world, missing states, ambiguous events represented by either a set of probability distributions or an interval of probabilities on each event, is referred as Knightian uncertainty or hard uncertainty.

Let $S=\{s_1,\ldots,s_n\}$ be a non empty set of states of the world and let $\Sigma=2^S$ be the set of all events. A function $\upsilon:\Sigma \rightarrow R_+$ is a capacity or a non-additive signed measure if it has the following characteristics: $\upsilon(S)=1$, $\upsilon(\varnothing)=0$ (i.e. the capacity is normalized) and $\forall A,B\in\Sigma$ such that $A\subset B$, $\upsilon(A)\leq\upsilon(B)$ (i.e. the capacity is monotone). A capacity[9] is convex (concave) if for each $A,B\in\Sigma$ such that $A\cup B,\ A\cap B\in\Sigma$ $\upsilon(A\cup B)+\upsilon(A\cap B)\geq(\leq)\upsilon(A)+\upsilon(B)$. It is superadditive[10] (subadditive) if $\upsilon(A\cup B)\geq(\leq)\upsilon(A)+\upsilon(B)$ for all $A,B\in\Sigma$ such that $A\cup B\in\Sigma$, $A\cap B=\varnothing$.

Given a real-valued function $f:S\rightarrow R$, f is a measurable function if for every $t\in R$, $\{s|\, f(s)\geq t\}$ and $\{s|\, f(s)> t\}$ are elements of Σ. Since υ is non-additive, the integration of a real-valued function f with respect to υ is impossible in the Lebesgue sense. The proper integral for the non-additive measures is the Choquet integral, originally defined by Choquet (1954) and discussed in Schmeidler (1989). The Choquet integral of f with respect to a capacity υ is defined as

$$\int f d\upsilon = \int_0^{\infty} \upsilon(\{s | f(s) \geq t\}) dt + \int_{-\infty}^{0} [\upsilon(\{s | f(s) \geq t\}) - \upsilon(S)] dt$$

The Choquet integral coincides with the Lebesgue integral if υ is additive.

An agent expresses hard uncertainty aversion (preference) if she/he assigns larger probabilities to states when they are unfavorable (favorable), than when they are favorable (unfavorable), that is if her/his non-additive measure is convex (concave). As a consequence, a hard uncertainty averse decision-maker over-weights the worst consequence, on the contrary a hard uncertainty seeking decision-maker over-weights the best consequence. Since convexity (concavity) of the capacity, that implies superadditivity (subadditivity) of the Choquet integral, captures the agents attitude toward hard uncertainty, the optimists have a concave capacity and the pessimists have a convex one.

3. ASSET VALUATION AND PRICE INTERVALS

Let $S=\{s_1,...,s_n\}$ be a non empty set of states of the world and let $\Sigma=2^S$ be the set of all events. On the measurable space (S,Σ) let p be a probability or measure that is a function p: $\Sigma\rightarrow[0,1]$, then the triple (S,Σ, p) is a probability space.

It is assumed that an asset is fully defined by its future payments, which depend on which state of the world occurs. By no arbitrage condition, there exists a unique additive measure p, such that the value of any asset in L, the set of all marketed and marketable assets[11], is the expectation of its payments.[12] As a consequence an asset may be considered a random variable β:$S\rightarrow R$ of its payments $\beta(s_i)$, where $s_i\in S$, and assets are ranked with respect to their market values, that is $\forall\beta,\gamma\in L$ $\beta\geq\gamma$ if and only if [13] $\beta(s_i) \geq \gamma(s_i)$ and $\int_S \beta\partial p \geq \int_S \gamma\partial p$

The replicating portfolio defines the value of a given asset and it is assumed as the Radner's common expectation. Nevertheless agents face Knightian uncertainty and they are either hard uncertainty averse or hard uncertainty seeking.

Let υ: $\Sigma\rightarrow[0,1]$ be a normalized and monotone capacity on the measurable space (S, Σ)

(A 1) A capacity υ is said to be compatible with p if $\forall A,B\in\Sigma\ p(A)\leq p(B)$ implies $\upsilon(A)\leq\upsilon(B)$

(A 2) A capacity υ is said to be monotonically sequentially continuous if $\forall A\in\Sigma\ A_n\uparrow A$ implies $\upsilon(A_n)\uparrow\upsilon(A)$ and $A_n\downarrow A$ implies $\upsilon(A_n)\downarrow\upsilon(A)$

(A 3) Two assets $\beta,\gamma\in L$ are comonotonic if and only if for any $s_i,s_j\in S$ $[\beta(s_i)-\beta(s_j)]\,[\gamma(s_i)-\gamma(s_j)]\geq 0$

PROPOSITION 1. Consider the class of hard uncertainty averse agents who will take a long position in the asset β. Assume (A1), (A2), (A3), no transaction costs and common expectation, there exists a unique convex capacity μ on (S,Σ) such that the Choquet expectation of β with respect to μ is $\int_S \beta\partial\mu=\int_S \beta\partial p$

Proof (Chateauneuf 1991, *Theorem 3*). For every convex capacity μ on (S,Σ) and every function β: $S\rightarrow R$ there exists a set P of additive probabilities on (S,Σ), such that for all events $p(.)\geq\mu(.)$, and $\int_S \beta\partial\mu=\min\left\{\int_S \beta\partial p \,\middle|\, p\in P\right\}$

Pessimists agree that the value of a given asset is revealed by the replicating portfolio, but they are hard uncertainty averse and consider the value of the asset β as the minimum expected value consistent with their beliefs (worst expectation). This threshold value can be considered as the highest price at which the pessimistic agents will wish to buy a given asset.

Consider the case in which the pessimistic agents sell the asset β. The price from which the pessimists will wish to sell the asset β may be defined by the Choquet integral of β with respect to μ^*, indeed the conjugate or dual capacity for μ. The dual capacity μ^* may be considered to what extent an agent believes the negation of A, indeed $\mu^*(A)=\mu(S)-\mu(A^C)$. By the asymmetry of the Choquet integral (e.g. Denneberg 1994) $\int_S -\beta\partial\mu=-\int_S \beta\partial\mu^*$

PROPOSITION 2 Consider the class of hard uncertainty averse agents who will take a short position in the asset β. Assume (A1), (A2), (A3), no transaction costs and common expectation, there exists a unique conjugate capacity μ^* on (S,Σ) such that the Choquet expectation of β with respect to μ is

$$\int_S \beta \partial \mu^* = \max\left\{ \int_S \beta \partial p^* \middle| p^* \in P \right\}$$

Proof (Gilboa 1989, *Lemma 1*). The upper Choquet integral with respect to a convex measure on (S, Σ) may be computed as the Choquet integral with respect to its dual

The Choquet pricing of β with respect to μ^* reveals the best expectation of the pessimists, that is the upper price (*upper bound*) from which the pessimists will sell the asset. At prices between the lower and upper prices, the pessimists neither buy nor sell the asset, that is hard uncertainty aversion leads to *inertia*.

Consider the class of hard uncertainty seeking agents, who have a concave capacity v on (S,Σ). Common expectation also requests the optimists to agree with the value of a given asset revealed by the replicating portfolio. However, the optimists are hard uncertainty seeking and consider the value of the asset β as the maximum expected value consistent with their beliefs.

PROPOSITION 3 If common expectation, no transaction costs, (A1), (A2) and (A3) hold, the optimists have a unique concave capacity v on (S,Σ), such that

$$\int_S \beta \partial v = \max\left\{ \int_S \beta \partial p \middle| p \in P^\circ \right\}$$

Proof (Chateauneuf 1991, *Theorem 3'*) For every concave capacity v on (S,Σ) and every function $\beta: S \rightarrow R$ there exists a set P° of additive probabilities on (S, Σ), such that for all events $p(.) \leq v(.)$, and $\int_S \beta \partial v = \max\left\{ \int_S \beta \partial p \middle| p \in P^\circ \right\}$

The expected value of β with respect to v is the best expectation of the optimistic agents. This threshold value can be considered as the lowest price from which optimistic agents will wish to sell a given asset.

Consider the case in which the optimistic agents will buy the asset β. The optimists would buy at the lowest price and they define the maximum price up to which they will wish to buy a given asset. That minimum price is defined by the Choquet integral of β with respect to $v°$, indeed the conjugate or dual capacity of v, which is the minimum expected value of the asset β consistent with their beliefs.

PROPOSITION 4. If common expectation, no transaction costs, (A1), (A2) and (A3) hold, the optimists have a unique dual capacity $v°$ on (S,Σ), such that

$$\int_S \beta \partial v^\circ = \min\left\{ \int_S \beta \partial p^\circ \middle| p^\circ \in P^\circ \right\}$$

Proof (Denneberg 1994, *Proposition 5.1*) The lower Choquet integral of the asset β with respect to a concave v on (S, Σ) may be computed as the Choquet integral with respect to its dual

The Choquet pricing of β with respect to $v°$ reveals the worst expectation of the optimists, that is the highest price (*lower bound*) up to which the optimists will wish to buy the asset β At prices between the lower and upper price, the optimists neither buy nor sell the asset and there is inertia.

THEOREM. *By propositions 1, 2, 3 and 4 for each asset in the set of all marketed and marketable ones, there exists a price interval within which the market is thin because of partial inertia. The bounds of that price interval are given by the worst expectation of the optimists (lower bound) and the best expectation of the pessimists (upper bound)*

Proof a) By common expectation, the value of the asset β is revealed by the replicating portfolio and it equals $\int_S \beta \partial p$. Because of Knightian uncertainty aversion, the pessimistic agents have a unique superadditive measure μ such that $\int_S \beta \partial \mu = \int_S \beta \partial p$. By a well known theorem (Chateauneuf, 1991), the Choquet integral of β with respect to μ is equal to the minimum of a family of Lebesgue integrals with respect to a set P of probability distributions consistent with their beliefs and $\int_S \beta \partial \mu = \int_S \beta \partial p$ with $p \in P = core(\mu)$. From cooperative game theory, the capacity μ may be considered an unanimity game and the *core* of μ may be defined as $\{p | p$ additive on Σ, $\mu \leq p$, $\mu(S) = p(S)\}$. As a result, the pessimistic agents consider the Choquet expected value of the asset β with respect to μ equals to their worst expectation and it is the highest price at which they will buy the asset β. Assume that the pessimistic agents would like to sell the asset β. Short position may be represented by the lowest price from which the pessimists will wish to sell β. By the asymmetry of the Choquet integral $\int_S -\beta \partial \mu = -\int_S \beta \partial \mu^*$, where μ^* is the conjugate capacity[14] of μ. By a well known lemma (Gilboa 1989), there exists a unique capacity μ^* on (S,Σ), such that $\int_S \beta \partial \mu^* = \int_S \beta \partial p^*$ with $p^* \in P = core(\mu)$. The Choquet integral of β with respect to μ^* is equal to the maximum among the family of Lebesgue integrals with respect to P and it reveals the best expectation of the pessimists, that is the maximum expected value of the asset β with respect to all measures consistent with their beliefs. This threshold price is the upper price (*upper bound*) from which the pessimistic agents will sell β. At prices between the lower and the upper price do not hold the asset β

b) Consider the class of Knightian uncertainty seeking agents, by common expectation there exists (Chateauneuf 1991) a unique subadditive ν such that $\int_S \beta \partial \nu = \int_S \beta \partial p$ with $p \in P° = core(\nu)$. This threshold price can be considered as the lowest price from which the optimistic agents will wish to sell β. The Choquet integral of β with respect to ν equals the maximum of a family of Lebesgue integrals with respect to a family of probability distributions $P°$, such that the *core* of ν is defined as $\{p|\ p$ additive on $\Sigma, \nu \geq p,\ \mu(S)=p(S)\}$. The optimistic agents' long position may be represented by the maximum price up to which they will wish to buy the asset β. By the asymmetry of the Choquet integral $\int_S -\beta \partial \nu = -\int_S \beta \partial \nu°$ where $\nu°$ is the conjugate of ν defined in the usual way, there exist a unique capacity $\nu°$ on (S,Σ), such that $\int_S \beta \partial \nu° = \int_S \beta \partial p°$ with $p° \in P° = core(\nu)$. The Choquet expected value of the asset β with respect to $\nu°$ equals the minimum among the Lebesgue integrals with respect to the probability distributions in the *core* of ν. It reveals the worst expectation of the optimistic agents, that is the highest price (*lower bound)* up to which they will wish to buy the asset β. Between the upper and the lower prices there is inertia

Summing up, at prices between $[\int_S \beta \partial \nu°; \int_S \beta \partial \mu *]$ there is a *thin market* for the asset β because of *partial inertia*. In the subinterval $[\int_S \beta \partial \nu°; \int_S \beta \partial p]$ the optimists do not hold the asset and only the pessimists will wish to buy; vice versa when the price of β is in the sub-interval $[\int_S \beta \partial p; \int_S \beta \partial \mu *]$, the pessimists do not hold the asset and only the optimists will wish to sell. The price $\int_S \beta \partial p$, derived by replicating portfolio, is included in the finite interval $[\int_S \beta \partial \nu°; \int_S \beta \partial \mu *]$, which is revealed by agents' attitude with respect to Knightian uncertainty and it does not depend on attitudes to risk. If the price of β exceeds $\int_S \beta \partial \mu *$, all agents will sell the asset, if the price of β is lesser than $\int_S \beta \partial v^\circ$, all agents will buy the asset. Just as an implication, there is a *thick market* for the asset β, out of the price interval.

4. AN EXAMPLE

Given a set of states of the world $S=s_1,s_2,s_3$ and a set of prices $q_i=2,1,-1$ with $i= s_1,s_2,s_3$ for the asset β, let p be a probability distribution on $\Sigma=2^S$ such that:

$p(\varnothing)=0$; $p(s_1)=p(s_2)=p(s_3)=1/3$; $p(s_1\cup s_2)=p(s_1\cup s_3)=p(s_2\cup s_3)=2/3$; $p(S)=1$

The expected value of the asset β given by the replicating portfolio that is constructed in a market without uncertainty and risk is $E_p^\beta = (1/3)\times(2)+(1/3)\times(1)+(1/3)\times(-1)=2/3$

Let uncertainty aversion (pessimism) express with a superadditive capacity μ on Σ, such that $\mu(\varnothing)=0$; $\mu(s_1)=\mu(s_2)=\mu(s_3)=1/6$; $\mu(s_1\cup s_2)=\mu(s_1\cup s_3)=\mu(s_2\cup s_3)=3/4$; $\mu(S)=1$; the Choquet expected value of asset β is $E_\mu^\beta = (1/6)\times(2)+[(3/4)-(1/6)]\times(1)+[1-(3/4)]\times(-1)=2/3$

The distribution μ is obtained by holding conditions defined in Section 3, then fixing, by trial and error, $\mu(A)=1/6$ and solving the simple determined system in two unknowns:

$(\varsigma - (1/6))(1) = \xi + (1/3)$ and $1 - \varsigma = \xi$, where $\varsigma=\mu(A\cup B)$ and $\xi=1-\mu(A\cup B)$.

It is possible to evaluate the upper expected value of the asset β, that is the minimum price from which the pessimistic agents will sell the asset β, by considering that the conjugate capacity μ^* is $\mu^*(\varnothing)=0$; $\mu^*(s_1)=\mu^*(s_2)=\mu^*(s_3)=1/4$; $\mu^*(s_1\cup s_2)=\mu^*(s_1\cup s_3)=\mu^*(s_2\cup s_3)=5/6$; $\mu^*(S)=1$.

As a consequence the Choquet expected value of the asset β with respect to μ^* is $E_{\mu^*}^\beta = (1/4)\times(2)+[(5/6)-(1/4)]\times(1)+[1-(5/6)]\times(-1)=11/12$

According to this result, the pessimists will invest in the asset β if and only if its market price is lesser than 2/3 and will wish to sell when market price exceeds 11/12. In the price interval [2/3, 11/12] the pessimists neither buy nor sell short the asset .

Let uncertainty seeking (optimism) express with a subadditive capacity v on Σ, such that $v(\varnothing)=0$; $v(s_1)=v(s_2)=v(s_3)=5/12$; $v(s_1\cup s_2)= v(s_1\cup s_3)= v(s_2\cup s_3)=15/24$; $v(S)=1$

The Choquet expected price of the asset β with respect to ν is

$$E_{\nu}^{g} = (5/12)\times(2) + [(15/24)-(5/12)]\times(1) + [1-(15/24)]\times(-1) = 2/3$$

It is possible to evaluate the lower expected value of the asset β, that is the maximum price up to which the optimistic agents will buy the asset β, by considering that the conjugate capacity for a capacity ν is $\nu°(\varnothing)=0$; $\nu°(s_1)=\nu°(s_2)=\nu°(s_3)=9/24$; $\nu°(s_1\cup s_2)=\nu°(s_1\cup s_3)=\nu°(s_2\cup s_3)=7/12$; $\nu°(S)=1$.

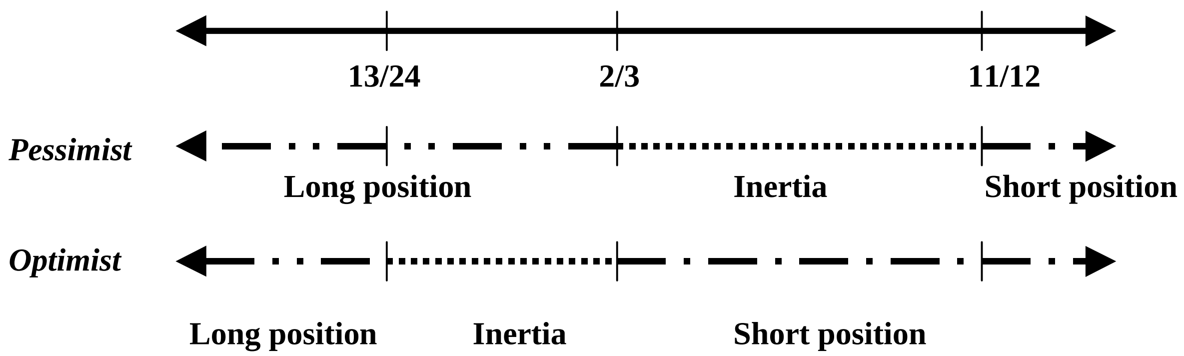
The lower bound equals the Choquet integral of the asset β with respect to $\nu°$ and

$$E_{\nu^*}^{\beta} = (9/24)\times(2) + [(7/12)-(9/24)]\times(1) + [1-(7/12)]\times(-1) = 13/24$$

According to this result, the optimistic agents will invest in the asset β if and only if its market price is lesser than 13/24 and they will wish to sell it when market price exceeds 2/3. In the interval [13/24; 2/3] the optimistic agents neither buy nor sell short the asset β.

Summing up, 2/3, which is the price of the asset β derived by replicating portfolio, is included in a finite interval of rational prices [13/24;11/12]. When the price of β is in the sub-interval [13/24; 2/3], the optimists do not hold the asset and only the pessimists will wish to buy. On the contrary, when the price of β is in the sub-interval [2/3; 11/12], the pessimists do not hold the asset and only the optimists will wish to sell. Out of the price interval all agents trade and the market of the asset β is thick

Figure 1. Price interval for the asset β



In this example, the sub-intervals of rational prices have different length and thinness. The subinterval within which the optimistic agents are not trading is shorter and thinner than the subinterval within which pessimistic agents are not trading. The optimists, that is professional traders, have a greater deal of experience in forecasting price movements than any other agent and have a more reliable beliefs.

5. CONCLUDING REMARKS

Differently from standard models in which different individuals have the same subjective probability distribution, at least two quite different classes of agents are distinguished, indeed the optimists and the pessimists, who have different non-additive subjective probability. This paper shows the existence of a price interval for any asset in which there is partial inertia, even if all individuals have common expectation. Introducing Knightian uncertainty attitude, it is possible to determine the lower and upper bounds of the price interval for either marketed or marketable assets. The value of an asset, derived by valuation models, is included in this price interval and it defines two subintervals each with only a class of agents will wish to trade. The lower and upper bounds of the asset price are defined by the Choquet integral with respect to the dual capacities, the first for optimists and the second for pessimists, consistent and coherent with common expectation. Since the price interval is derived using the techniques of contingent claims analysis, no question about updating, indeed each class of agents updates the market price as a Bayesian decision-maker and then defines the reservation price interval.

REFERENCES

Ami, D., Kast R., and Lapied A. (1991), Generalizing Arrow Pricing to Understand Financial Markets, *Document de Travail* G.R.E.Q.E. 91A04, Universites d'Aix-Marseille II et III.

Arrow, K.J. (1953), Le Role des Valeurs Boursieres pour la Repartition la Meillure des Risques, *Econometrie*, 40: 41-47; English translation (1964), *Review of Economic Studies*, 31: 91-96.

Black, F., and Scholes M. (1973),The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, 81: 637-654.

Chateauneuf, A. (1991), On the Use of Capacities in Modeling Uncertainty Aversion and Risk Aversion, *Journal of Mathematical Economics*, 20: 343-369.

Chateauneuf, A., Kast R., and Lapied A. (1992): Pricing in Slack Markets, *Document de Travail* G.R.E.Q.E. 92A05, Universites d'Aix-Marseille II et III.

Choquet, G. (1953): Theorie des Capacites, *Annales de l'Institut Fourier*, 5: 131-233.

DennebergG, D. (1994), *Non-additive measure and integral*. Dordrecht: Kluwer Academic Publishers.

Dow, J., and Werlang S. (1992),Uncertainty Aversion, Risk Aversion, and the Optimal Choice of Portfolio, *Econometrica*, 37: 197-204.

Duffie, D.(1988), *Security Markets: Stochastic Models*. San Diego: Academic Press.

Dunford, N and Schwartz J. T. (1957), *Linear Operators*. New York: Interscience.

Fox, C., Rogers, B. and Twersky A. (1996), Option Traders Exhibit Subadditive Decision Weights, *Journal of Risk and Uncertainty*, 3: 5-17.

Gilboa, I. (1989), Duality in Non-Additive Expected Utility Theory, *Annals of Operations Research*, 19: 405-414.

Hahn, F. H. (1973), On Transaction Costs, Inessential Sequence Economies and Money, *Review of Economic Studies*, 40: 449-462.

Hahn, F. H. (1992), Sequence Economies and Incomplete Markets, *mimeo*, University of Siena.

Hahn, F. H. (1995), On Economies with Arrow Securities, *mimeo*, University of Siena.

Markowitz, H. and Uschmen N. (1996), The Likelihood of Various Stock Market Return Distribution, Part 1: Principles of Inference, *Journal of Risk and Uncertainty*, 11: 207-219.

Radner, R. (1968), Competitive Equilibrium Under Uncertainty, *Econometrica*, 39: 31-58.

Radner, R. (1972), Existence of Equilibrium of Plans, Prices, and Price Expectations in a Sequence of Markets, *Econometrica*, 40: 289-303.

Savage, L. J. (1972): *The Foundation of Statistics*. New York: Willey; 2nd edn New York: Dover.

Schmeidler, D. (1989), Subjective Probability and Expected Utility Without Additivity, *Econometrica*, 57: 571-587.

Simonsen, M. and Werlang S. (1991), Subadditive Probabilities and Portfolio Inertia, *Revista de Econometrica*, 11: 1-19.

[1] There is perfect price foresight if "the price vector p(s,t) for state s and date t is known with certainty at the previous date at which security portfolios are chosen" (Hahn, 1995).

[2] An asset is called marketable when it is not traded in markets but is tradable by trading the marketed securities.

[3] Frictionless markets implies no taxation and no transaction cost.

[4] In a stochastic economy no arbitrage conditions imply both tight markets and asset market values positive and linear on the set of securities.

[5] If price of asset can never fall below zero, it may be assumed that changes in asset prices are lognormally distributed. If the price distribution is assumed lognormal, the diffusion process of an asset price follows a geometric Wiener process.

[6] Proof of these theorem is in Chateauneuf 'et al.' (1992).

[7] Without loss of generality it can be assumed that optimistic agents are professional stocks or option traders and pessimist are all other ones, see Simonsen and Verlang (1991), Fox 'et al.' (1996), Markowitz and Uschmen (1996).

[8] Talking about the way in which some probability relations are felt relatively sure as compared with others, Savage observed that "one tempting representation of the unsure is to replace the person's single probability measure P by a set of such measures, especially a convex set" (Savage, 1972, p. 58).

[9] It is worth noting that a capacity is additive or a probability measure if for all $A,B\in\Sigma$ such that $A\cup B\in\Sigma$ and $A\cap B=\emptyset$, $\upsilon(A\cup B)=\upsilon(A)+\upsilon(B)$

[10] A capacity υ is supermodular (submodular) if $A,B\in\Sigma$ such that $A\cup B$, $A\cap B\in\Sigma$, $\upsilon(A\cup B)+\upsilon(A\cap B)\geq(\leq)\ \upsilon(A)+\upsilon(B)$.

[11] The set of all asset L can be the normed space $L^2(S, \Sigma, p)$ endowed with the norm topology. For instance see Chateauneuf 'et al.' (1992), Duffie (1988).

[12] See for instance Dunford and Schwartz (1957), Chateauneuf 'et al.' (1992).

[13] The $\geq$ is in the usual way and the induced ranking of assets is monotonic and respects monotonic uniform convergence (e.g. Chateauneuf 'et al.', 1992).

[14] It is worth noting that μ^* is supermodular iff μ is submodular and $\mu^*=\mu$ if μ is additive, that is if an agent is Knightian uncertainty neutral, see Denneberg (1994).