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Stefano Vannucci

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Stefano Vannucci
Universita' di Siena,
Dipartimento di Economia Politica,
Piazza S. Francesco 7, 53110 Siena, Italia

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Abstract

The Galois lattice of an effectivity function is defined and its basic properties are studied. In particular, the use of its length and width as basic complexity measures of the underlying game correspondence is presented by means of several examples.

1 Introduction

In the last few decades, the develoment of game-theoretic methods has produced an enormous progress in our understanding of coalitional power in interactive decision processes. Indeed, game theory has contributed two basic -and very successful- models for the analysis of coalitional power, namely simple games and effectivity functions. A *simple game* amounts to a list of all those coalitions that can force any outcome by a suitable coordination of their members' behaviour (see e.g. Shapley(1962), and Ramamurthy(1990)). An *effectivity function* describes - for each coalition S - the set of subsets within which S can force the final outcome by means of some coordinate action of its members. Simple games are eminently suitable for the analysis of weighted majority voting games and related decision procedures. However, they are definitely inadequate to analyze interactive procedures whereby individual veto rights are allocated in some non-trivial way. By contrast, effectivity functions are most appropriate for the latter task (which is indeed the main reason why they were first introduced by Moulin,Peleg(1982)). On the other hand, they achieve their versatility at the cost of being somehow exceedingly dependent on the outcome space, and its description. Arguably, simple games and effectivity functions should be aptly supplemented with other models of coalitional power that rely on an intermediate amount of information on the underlying decision procedures.

This paper (which builds upon Vannucci(1998a,b,c)) is devoted to the description and discussion of one such model, namely the Galois lattice of a game correspondence. The Galois lattice of a game correspondence - a complete lattice with a single atom - is made up of pairs consisting of "closed" families of coalitions and outcome-subsets, ordered by set-inclusion (the interpretation being that the coalitions of a "closed" family attached to a certain "closed" family of outcome-subsets are precisely those coalitions that are able to force the final outcome within any set of the latter family). Thus, *lengths* and *widths* of Galois lattices of game correspondences provide two natural structural complexity measures of the latter, and allow some new interesting classifications of well known game forms, games, and solution concepts with respect to the allocation of decision power they embody. A few significant examples including bargaining game forms, 2×2 strategic game forms, voting game forms, constitutional effectivity functions, and solution concepts for private-good-economies will be presented and discussed below. The structure of the paper is as follows: section 2 provides some relevant background and motivation. Section 3 is devoted to the introduction of some basic results. Section 4 describes the Galois lattices of several prominent game correspondences, and discusses the ensuing classifications of the latter in terms of Galois-latticial parameters. Section 5 offers some short concluding remarks. The proofs are confined to an appendix.

2 Background and Motivation

Let us first consider two voting procedures, defined as follows. Under the first procedure (V), the voters are asked to make proposals at a preliminary stage, in some ordered way; then the proposals themselves are suitably ordered and the players veto any proposal they consider not acceptable. The first proposal that is not vetoed by anybody is chosen; if no such proposal exists then no action is taken, and the status quo obtains. Under the second procedure (V*), the voters are suitably ordered and asked to make one proposal at a preliminary stage; then they proceed to veto exactly one alternative from the set consisting of those proposals- including the status quo- that have not been previously vetoed. The unique alternative that fails to be vetoed is the final outcome.

In many relevant aspects (V) and (V*) are *very different.* Under (V) veto power is a free good. From a practical point of view, (V) is a recipe for inertia except perhaps for small homogeneous assemblies with no relevant conflicts of interest. By contrast, under (V*) veto power is a scarce good. The status quo is not a specially privileged alternative, and reasonable outcomes are to be expected - under several solution concepts- even when "positive action" is required (see e.g. Moulin(1983)). However, (V) and (V*) share the same set W of *winning* - i.e. all-powerful - coalitions, that reduce to the grand coalition of all voters. Hence, (V) and (V*) are plainly indistinguishable if one looks at their simple games, under the standard interpretation of the latter. By contrast, looking at their effectivity functions enables an adequate representation and analysis of the basic differences between (V) and (V*). Indeed, under (V) any single player can force the final outcome to be the status quo, and intermediate coalitions do not enjoy any supplementary decision power. Under (V*), no coalition except the grand one can enforce the status quo and intermediate-size coalitions do enjoy an intermediate amount of decision power.

Next, consider a *neutral* effectivity function E as defined on a finite outcome space X: namely, E is such that for any coalition S and any pair A, B of subsets of X having the same cardinality, S can force the outcome either in both A and B, or in none of them. Now, suppose a further outcome x° is added (due possibly to a certain change of description of the outcome space). Then, a new effectivity function E° can be defined by the following rule: for any coalition S and issue $B \subseteq X \cup \{x^{\circ}\}$, S can force B under E° if and only if S can force $B \backslash \{x^{\circ}\}$under E. Clearly enough, E and E° are two distinct non-isomorphic effectivity functions, but the structure of coalitional power they represent is arguably very much the same. Similar examples obtain from more general basic transformations of effectivity functions (see section 3 below). This is so because - while simple games focus on a single set of winning coalitions - effectivity functions amount to a family of sets of "locally winning" coalitions as parameterized by the entire set of non-empty subsets of the outcome space. In a sense, effectivity functions carry too many details for analyzing the structure of coalitional power. We claim that Galois lattices of effectivity functions (to be defined below) provide us with the "right" intermediate amount of information that is required in order to accomplish the latter task in a proper way.

3 Model and Basic Results

Let (N, X) be a pair of non-empty sets (the sets of players and outcomes, respectively; we also assume $\#N \geq 2$ and $\#X \geq 2$ in order to avoid trivialities). A *(monotonic) simple game* on N is a set W, $\emptyset \neq W \subseteq P(N)$, such that $S \in W$ and $S \subseteq T$ entail $T \in W$. The coalitions belonging to W are meant to represent the *winning* or all-powerful ones. An *effectivity function* (EF) on (N, X) is a function $E : P(N) \to P(P(X))$ such that :

$EF1)$ $E(N) \supseteq P(X) \backslash \{\emptyset\}$; $EF2)$ $E(\emptyset) = \emptyset$; $EF3)$ $X \in E(S)$ for any S, $\emptyset \neq S \subseteq N$.

Moreover, E is a *well-behaved* EF if

$EF4)$ $\emptyset \notin E(S)$ for any S , $\emptyset \subset S \subseteq N$ is also satisfied.

An EF E on (N, X) is *monotonic* if for any $S, T \subseteq N$ and any $A, B \subseteq X$

$[A \in E(S)$ and $S \subseteq T$ entail $A \in E(T)]$ and

$[A \in E(S)$ and $A \subseteq B$ entail $B \in E(S)]$.

In what follows we shall confine ourselves to *monotonic* EFs.

A monotonic EF E on (N, X) is *regular* if $\emptyset \neq A \in E(S)$ entails $X \backslash A \notin E(N \backslash S)$ for any $S \subseteq N$ and $B \subseteq X$, and *maximal* if $A \notin E(S)$ entails $(X \backslash A) \in E(N \backslash S)$ for any $\emptyset \neq S \subseteq N$ and $\emptyset \neq A \subseteq X$. Moreover, an EF E on (N, X) is *superadditive* if for any $S, T \subseteq N$ and $A, B \subseteq X$, $A \in E(S)$, $B \in E(T)$ and

$S \cap T = \emptyset$ entail $A \cap B \in E(S \cup T)$. Finally, an EF E on (N, X) is *simple* if a non-empty set $W \subseteq P(N)$ exists such that for any $S \subseteq N$, $A \subseteq X$, $A \in E(S)$ if and only if either $A = X$ and $S \neq \emptyset$ or $A \neq \emptyset$ and $S \in W$. Indeed, simple EFs amount to simple games as endowed with a fixed outcome set.

In many relevant contexts, one may be focussed on EFs that treat "symmetrically" players and/or outcomes. Such requirements are embodied in the following properties. An EF E on (N, X) is *anonymous* if for any $A \subseteq X$, and $S, T \subseteq N$ such that $\#S = \#T$: $A \in E(S)$ if and only if $A \in E(T)$. An EF E on (N, X) is *neutral* if for any $S \subseteq N$ and any $A, B \subseteq X$ such that $\#A = \#B$: $A \in E(S)$ if and only if $B \in E(S)$.

We are mainly interested in those EFs that can represent the decision power of coalitions under a certain decision mechanism, or *game correspondence.* A *game correspondence* on (N, X) is a correspondence $G : D \rightarrow\rightarrow X$ where $D \subseteq \prod_{i \in N} S_i$, and S_i is the set of "interactive behaviours" available to player $i \in N$. A *game form* is a single-valued game correspondence.

Now, the notion of decision power admits at least two distinct interpretations, namely "guaranteeing power" and "counteracting power" that in turn correspond to the ability to force maximin and minimax outcomes, respectively. Thus, the allocation of "guaranteeing power" under game correspondence G with domain D is represented by the $\alpha - EF$ of G - denoted by $E_\alpha(G)$- as defined by the following rule:

for any non-empty $S \subseteq N$,

$$(E_\alpha(G))(S) = \left\{ \begin{array}{c} A \subseteq X\text{: a } t^S \in \prod_{i \in S} S_i \text{ exists such that } (t^S, s^{N \setminus S}) \in D \\ \text{and} \\ G(t^S, s^{N \setminus S}) \subseteq A \\ \text{for any } s^{N \setminus S} \in \prod_{i \in N \setminus S} S_i, \end{array} \right\}.$$

Conversely, the allocation of "counteracting power" under game correspondence G with domain D is represented by the $\beta - EF$ of G, denoted by $E_\beta(G)$ and defined as follows :

for any non-empty $S \subseteq N$

$$(E_\beta(G))(S) = \left\{ \begin{array}{c} A \subseteq X : \text{ for any } s^{N \setminus S} \in \prod_{i \in N \setminus S} S_i \text{ some } t^S \in \prod_{i \in S} S_i \\ \text{exists such that } (t^S, s^{N \setminus S}) \in D \\ \text{and } G(t^S, s^{N \setminus S}) \subseteq A \end{array} \right\}.$$

It is easily checked that $E_\alpha(G)$ is regular, $E_\beta(G)$ is maximal, and both of them are *monotonic* and - provided that G is non-empty valued- *well-behaved* . Also, it is well-known that superadditivity and monotonicity of an EF E imply that a game correspondence G exists such that $E = E_\alpha(G)$: see Moulin(1983), and Otten,Borm,Storcken,Tijs(1995)) . Indeed, monotonicity of $\alpha - EFs$ and $\beta - EFs$ of game correspondences is our main reason for confining the ensuing analysis to monotonic EFs (as mentioned previously). Furthermore, the foregoing distinction between $\alpha - EFs$ and $\beta - EFs$ brings us to the general notion of a *polarity operator* for EFs, implicitly defined as follows (see e.g. Abdou,Keiding(1991)): the *polar* $\hat{E}$ of a monotonic EF E on (N, X) is an EF on (N, X) such that :

i) $\hat{E}(\emptyset) = \emptyset$ and ii) for any non-empty $S \subseteq N$ and $A \subseteq X$, $A \in \hat{E}(S)$ if and only if $X \backslash A \notin E(N \backslash S)$.

(It should be noticed here that $E = \hat{E}$ if and only if E is both regular and maximal).

In Vannucci(1998a) it is observed that : i) the set of all EFs on (N, X) is bijective to a set of binary relations on $(P(N), P(X))$: hence any EF on (N, X) can be equivalently regarded as a binary relation ; ii) therefore, the classic Birkhoff theorem on so called Galois connections applies. It follows that the functions $f_E : P(P(N)) \to\to P(P(X))$, $g_E : P(P(X)) \to\to P(P(N))$ as defined by the rules

$f_E(\mathbf{S}) = \{A \subseteq X : A \in E(S)$ for any $S \in \mathbf{S}\}$ for any $\mathbf{S} \subseteq P(N)$, and

$g_E(\mathbf{A}) = \{S \subseteq N : A \in E(S)$ for any $A \in \mathbf{A}\}$

enjoy the following list of properties:

a) the functions $K_E = g_E \circ f_E$ and $K_E^* = f_E \circ g_E$ are *closure operators* on $(P(N), \supseteq)$ and $(P(X), \supseteq)$, respectively (we recall here that a (Moore) closure operator on a preordered set $(Y, \geq)$ is a function $K : Y \to Y$ such that for any $y, z \in Y : K(y) \geq y$; $y \geq z$ entails $K(y) \geq K(z)$; $K(y) \geq K(K(y))$).

b) the corresponding *closure systems* - i.e. sets of closed sets - $\mathbf{C}(K_E) = \{\mathbf{S} \subseteq P(N) : \mathbf{S} = K_E(\mathbf{S})\}$, $\mathbf{C}(K_E^*) = \{\mathbf{A} \subseteq P(X) : \mathbf{A} = K_E^*(\mathbf{A})\}$ are (dually isomorphic) *complete lattices* under the join and meet operations defined as follows:

for any $\{\mathbf{S}_i\}_{i \in I} \subseteq \mathbf{C}(K_E)$, $\{\mathbf{A}_i\}_{i \in I} \subseteq \mathbf{C}(K_E^*)$,

$\vee_{i \in I} \mathbf{S}_i = K_E(\cup_{i \in I} \mathbf{S}_i)$, $\wedge_{i \in I} \mathbf{S}_i = \cap_{i \in I} \mathbf{S}_i$, $\vee_{i \in I}^* \mathbf{A}_i = K_E^*(\cup_{i \in I} \mathbf{A}_i)$, $\wedge_{i \in I}^* \mathbf{A}_i = \cap_{i \in I} \mathbf{A}_i$

(we recall that a lattice is a partially ordered set $(L, \geq)$ such that for any pair $\{x, y\} \subseteq L$, both a greatest lower bound (glb) -or meet- $\wedge \{x, y\}$and a lowest upper bound (lub) - or join - $\vee \{x, y\}$ exist; a lattice is *complete* if any subset of L has both a glb and a lub).

c) the lattices under b) are *dense,* i.e. have a unique atom and - if E is well-behaved- *co-dense*, i.e. have a unique co-atom (an *atom* of a lattice $(L, \geq)$ is a $\geq$-minimal non-bottom element of L, and a *co-atom* is-dually- a $\geq$-maximal non-top element of L).

The *Galois lattice* of an EF E is $\mathbf{L}(E) = (Iso\,[\mathbf{C}(K_E) \times \mathbf{C}(K_E^*)]\,, \supseteq)$, where $Iso\,[\mathbf{C}(K_E) \times \mathbf{C}(K_E^*)]$ denotes the set of canonically isomorphic pairs of the closure systems of E,

and for any $\{(\mathbf{S}_i, \mathbf{A}_i)_{i \in I}\} \subseteq Iso\,[\mathbf{C}(K_E) \times \mathbf{C}(K_E^*)]$

$\bigvee_{i \in I} (\mathbf{S}_i, \mathbf{A}_i) = (K_E(\cup_{i \in I} \mathbf{S}_i), \cap_{i \in I} \mathbf{A}_i)$, $\bigwedge_{i \in I} (\mathbf{S}_i, \mathbf{A}_i) = (\cap_{i \in I} \mathbf{S}_i, K_E^*(\cup_{i \in I} \mathbf{A}_i))$.

Clearly enough, the Galois lattice $\mathbf{L}(E)$ (that is also sometimes called a *concept lattice)* is lattice-isomorphic to the closure systems of E. Hence, $\mathbf{L}(E)$ is complete, has a unique atom and, if E is well-behaved, a unique co-atom. Those basic facts concerning $\mathbf{L}(E)$ can be summarized by the following proposition (see Vannucci(1998a) for more details):

Proposition 1 *Let E be an EF on (N, X). Then, a complete lattice $\mathbf{L}(E)$ - the Galois lattice of E, uniquely defined up to isomorphisms- can be canonically attached to E. Moreover, i) $\mathbf{L}(E)$ is dense ; ii) if E is well-behaved, $\mathbf{L}(E)$ is co-dense; iii) $\mathbf{L}(E)$ is finite whenever either N or X is finite.*

Remark 2 *In view of Proposition 1 it must be the case that* $\mathbf{L}(E) = \mathbf{1} \oplus B(\mathbf{L}(E)) \oplus \mathbf{1}$ *for some lattice* $B(\mathbf{L}(E))$ *if* E *is well-behaved , and* $\mathbf{L}(E) = \mathbf{1} \oplus B(\mathbf{L}(E))$ *otherwise (where* $\mathbf{1}$ *denotes the degenerate 1-element lattice, and* $\oplus$ *denotes the linear or ordinal sum operation: see e.g. Birkhoff(1967), or Davey,Priestley(1990)). In any case, we shall refer to the lattice* $B(\mathbf{L}(E))$ *as the bulk of* $\mathbf{L}(E)$ *.*

Now, it is to be checked whether the Galois lattice of an EF does indeed satisfy - as claimed above- the outcome-invariance property described above in section 2. In order to accomplish this task we provide a formal definition of a suitable family of *uniform expansions* of an EF.

Definition 3 *Let* E *be an EF on* (N, X)*, and* Y *a set such that* $X \cap Y = \emptyset$*. The Y-uniform expansion of E- written* E_{+Y} *- is the EF on* $(N, X \cup Y)$ *defined by the following prescription : for any* $S \subseteq N$*,* $A \subseteq X \cup Y$ *,* $A \in E_{+Y}(S)$ *if and only if* $A \backslash Y \in E(S)$ *.*

The following proposition establishes the required property of Galois lattices of EFs.

Proposition 4 *Let* E *be an EF on* (N, X)*, and* E_{+Y} *the Y-uniform expansion of* E*. Then* $\mathbf{L}(E_{+Y}) = \mathbf{L}(E)$*.*

Thus, as mentioned above, Galois lattices of EFs - and their bulks- provide us with an algebraic invariant that allows some significant new classifications of game correspondences. (Of course, a game correspondence G is entitled to - at least- two Galois lattices, the α-Galois lattice and the β-Galois lattice, that correspond to $E_\alpha(G)$ and $E_\beta(G)$, respectively. We shall refer to *the* Galois lattice of game correspondence G when $E_\alpha(G) = E_\beta(G)$). Indeed, the Galois lattice $\mathbf{L}(E)$ also provides -at least- two natural complexity measures for the underlying EF E, namely its *length* and *width.* We recall here the relevant definitions (see again Birkhoff(1967), or Davey,Priestley(1990)).

Definition 5 *The length* $l(L)$ *of a lattice* L *is the least upper bound of the set of lengths of chains included in* L *(a chain is a totally ordered set; the length of a chain of* $k+1$ *elements is* k *).*

Definition 6 **.** *The width* $w(L)$ *of a lattice* L *is the size or cardinality of its largest antichain (an antichain is a set of pairwise incomparable elements).*

A most useful notion of *rank* for coalitions (and issues) can be introduced relying on the *length* $l(\mathbf{L}(E))$ of the Galois lattice of an EF E as defined above.

Definition 7 *Let E be an EF on (N, X). The height $h_E(x)$ of $x = (\mathbf{C}, \mathbf{C}') \in \mathbf{L}(E)$ is the least upper bound of the set of lengths of chains in $\mathbf{L}(E)$ having x as their maximum. The rank $r_E(S)$ of a coalition $S \subseteq N$ is the height $h_E(x)$ of the highest $x = (\mathbf{C}, \mathbf{C}') \in \mathbf{L}(E)$ such that $S \in \mathbf{C}$ (a dual definition obtains for an issue $A \subseteq X$).*

A few fundamental properties of an EF E on (N, X) - hence of a game correspondence G when $E_\alpha(G) = E_\beta(G)$ - can be readily expressed using the notion of rank (see again Vannucci(1998a)):

i) E is *consensual (CO)* if $r_E(N) > r_E(S)$ for any coalition $S \neq N$ (in words, E is *CO* if the grand coalition is uniquely endowed with maximum rank);

ii) E is *fully distributed (FD)* if $r_E(\{i\}) > 1$ for any $i \in N$ (in words, E is *FD* if each single player is endowed with some non-communal decision power);

iii) E is *unspecialized (USP)* if $r_E(S) = r_E(T)$ entails $E(S) = E(T)$ for any $S, T \subseteq N$, and *specialized (SP)* otherwise (in words, E is *US* if having the same rank entails having exactly the same decision power).

iv) E is *strictly hierarchical (SH)* if $r_E(S) = r_E(T)$ entails $S = T$ for any $S, T \subseteq N$ (in words, the coalitions are linearly ordered w.r.t. their rank in E).

Moreover, *simplicity* of an EF as defined above can also be characterized in terms of the rank function. Namely, an EF E is *simple (SI)* if $r_E(.)$ is two-valued.

Clearly enough, the foregoing properties are -generally speaking- not independent. In particular, *SH* entails *USP* and *not SI ; SI* entails *USP ; CO* and *FD* jointly entail *not SI.*

The following basic result can be easily established (see Vannucci(1998a)):

Proposition 8 *Let E be an EF on (N, X). Then, i) E is simple and consensual if and only if $B(\mathbf{L}(\hat{E})) = \mathbf{1}$; ii)E is simple and not consensual if and only if $B(\mathbf{L}(\hat{E})) = \mathbf{2}$ (the two-element Boolean lattice); iii) if E is simple, regular and not consensual, then $B(\mathbf{L}(E)) = B(\mathbf{L}(\hat{E})) = \mathbf{2}$.*

Thus, the *unanimity rule* (whose α-EF is the unique simple and consensual EF on (N, X)) turns out to be -somehow- uniquely connected to the degenerate lattice $\mathbf{1}$, while simple and not consensual EFs are -somehow- uniquely connected to the *simple* Boolean lattice $\mathbf{2}$ (we recall here that a lattice is *simple* if and only if its congruences reduce to the trivial ones i.e. the identity congruence and the universal congruence).

4 Computing Galois Lattices of Game Correspondences: Some Examples

Computing the Galois lattice of a given game correspondence or effectivity function may well involve a heavy computational burden (see e.g. Grätzer (1998), especially Appendix H by Ganter and Wille). The present section is devoted to the computation of the (bulks of) Galois lattices of some prominent and well-known game correspondences that are simple enough to allow for "manual" calculations (including two solution concepts for private-good-economies, that can indeed be regarded as a special example of a "revelation" game correspondence). Comparing such Galois lattices, and their parameters, will enable us to classify such game correspondences according to the properties introduced in the previous section.

4.1 Bargaining Game Forms

A bargaining game form on (N, X) is a tuple $G^B = (N, (X_i)_{i \in N}\ , x^*)$ where $X_i = X$ for any $i \in N$, and $x^* \in X$ denotes the conflict outcome. The players in N can unanimously agree on any outcome in X. The conflict outcome x^* obtains if the players fail to agree on any other outcome.Therefore, $E_\alpha(G^B)$ is given by the following rule: for any S, B , $\emptyset \neq S \subseteq N$, $\emptyset \neq B \subseteq X$, $B \in (E_\alpha(G^\beta))(S)$ if and only if either $S = N$ or $x^* \in B$. The following result is easily established:

Proposition 9 *Let G^B be a bargaining game form as defined above. Then, i) $E_\alpha(G^B) = E_\beta(G^B)$, and $B(\mathbf{L}(G^B)) = \mathbf{2}$. Thus, G^B is simple (hence unspecialized) and consensual (but neither fully distributed nor strictly hierarchical).*

4.2 Voting Game Forms

As mentioned above, voting game forms have been largely studied by analysing their simple games, which amounts to regarding the EFs of the former as simple. It should be emphasized again that this is only appropriate for *certain* voting procedures. Indeed, majoritarian-like voting schemes rely on a sharp distinction between all-powerful - or *winning* - and powerless - or *losing*- coalitions. Hence, the EFs of such schemes *are indeed simple* (and their Galois lattices are as described above under Proposition 3). Unfortunately, it is well-known that majoritarian-like voting procedures are -generally speaking- *unstable* in that at many preference profiles their *core* is *empty*. By contrast, certain *voting-by-limited-veto* procedures (as briefly introduced in section 2 above, and thoroughly analyzed elsewhere, e.g. in Moulin,Peleg(1982), Moulin(1983), Peleg(1984), Danilov,Sotskov(1993)) enjoy several nice stability properties - including general non-emptiness of the core- and rely on a considerably more complex allocation of decision power. This is neatly reflected by the properties of their Galois lattices. As a prominent example of a voting-by-limited-veto procedure that shares anonymity and neutrality properties with majoritarian-like schemes we shall focus on a version of the *proportional veto procedure*, first introduced by Moulin (see e.g. Moulin(1983), Abdou,Keiding(1991)). Namely, we consider a *proportional veto procedure with endogenous agenda formation* that can be described as follows. A distinguished outcome x^*- the *"status quo"*- is identified. Then, each player *makes k proposals,* is informed on the resulting set of outcomes, and *issues k vetos* - according *to* a *prefixed order* - on non-vetoed alternatives. The unique non-vetoed outcome is selected. The corresponding EF E^{PV} (that is regular and maximal, hence unambiguously determined) is defined by the following rule:

for any $S \subseteq N$, $A \subseteq X$, $A \in E^{PV}(S)$ if and only if

$$\lceil (kn+1)\tfrac{s}{n} \rceil > kn + 1 - a$$

where $s = \#S$, $n = \#N$, and $a = \#A$.

Since each coalition-size corresponds to a distinctive "degree" of decision power, the Galois lattice $\mathbf{L}(E^{PV})$ is easily computed. Thus, it is straightforward to establish the following result:

Proposition 10 *Let E^{PV} be the proportional veto EF as defined above. Then, i) $B(\mathbf{L}(E^{PV})) = \mathbf{n}$ (where $\mathbf{n}$ denotes the chain of size n). Hence, in particular ii) E^{PV} is consensual and unspecialized (but not simple, fully distributed, or strictly hierarchical).*

It should be noticed that the Galois lattices of both bargaining and (neutral) voting game forms as considered above are invariably *chains*. The following example shows that there also exist elementary game forms whose Galois lattices are *not* chains.

4.3 2×2 Strategic Game Forms

Let us consider a 2×2 *strategic game form* $G^2 = (\{1,2\}, (S_1, S_2), h)$, where $S_1 = \{s_1, t_1\}$, $S_2 = \{s_2, t_2\}$, and $h(s_1, s_2) = a$, $h(s_1, t_2) = b$, $h(t_1, s_2) = c$, $h(t_1, t_2) = d$. Clearly enough, $E_\alpha(G^2) \neq E_\beta(G^2)$. Let us consider the "generic" case (i.e. $\#X = 4$). The following proposition summarizes the situation :

Proposition 11 *Let G^2be a 2×2 strategic game form as defined above. Then, i)$B(\mathbf{L}(E_\alpha(G^2))) = B(\mathbf{L}(E_\beta(G^2))) = \mathbf{2}^2$ (the 4-valued Boolean lattice) if G^2 is "generic" i.e.$\#X = 4$. Hence, in particular, both $E_\alpha(G^2)$ and $E_\beta(G^2)$ are consensual, fully distributed, specialized (and, of course, neither simple nor strictly hierarchical); ii) $B(\mathbf{L}(E_\alpha(G^2))) = B(\mathbf{L}(E_\beta(G^2))) = \mathbf{3}$ if $\#X = 3$ and the replicated outcome is* not *on a diagonal, whereas $B(\mathbf{L}(E_\alpha(G^2))) = B(\mathbf{L}(E_\beta(G^2))) = \mathbf{3}$ if $\#X = 3$ and the replicated outcome* is *on a diagonal; iii)$B(\mathbf{L}(E_\alpha(G^2))) = B(\mathbf{L}(E_\beta(G^2))) = \mathbf{2}$ if $\#X = 2$ and a diagonal exists which does not include replicated outcomes, whereas $B(\mathbf{L}(E_\alpha(G^2))) = \mathbf{2}$ and $B(\mathbf{L}(E_\beta(G^2))) = \mathbf{1}$ if $\#X = 2$ and each diagonal includes replicated outcomes.*

Therefore, a "generic" 2×2 strategic game form provides a first elementary example of an EF whose Galois lattice has a non-trivial width. More examples are to be described below.

4.4 Constitutional Effectivity Functions

Modern representative democracies rely on governance structures whose architectures may vary in many relevant respects. The most significant distinction is perhaps the one between *parliamentary* and *presidential* systems. Indeed, under *parliamentary* governance structures executive-termination can be prompted by a non-confidence vote on the part of the legislature. By contrast, under *presidential* systems the executive is *not* subject to non-confidence votes and -as a result- some degree of separation of powers between legislature and executive typically obtains. More often than not, under presidential systems the head of executive is directly appointed by means of general elections, while the opposite is the case with parliamentary systems. However, the nature of the executive-appointment procedures is, in our view, largely immaterial to the basic distinction between parliamentary and presidential systems. Therefore, in order to focus on the latter contrast, we shall single out for discussion a) the EF of a presidential system with perfect separation of powers and b) the EF of a parliamentary system with a directly elected premier and a fixed majority (see also Vannucci(1998b,1999)).

Definition 12 *(The EF of a presidential system with perfect separation of powers) Let 0^* denote the elected president of the executive, and $N = \{1, .., n\}$ the set of parties- or voting blocs- of a legislature of size h. The parties have weights- or number of seats- w_i, $i = 1, .., n$. We also suppose that the weight profile $\mathbf{w} = (w_i)_{i \in N}$ is strong (i.e. for any $S \subseteq N$ either $\sum_{i \in S} w_i \geq \lfloor \frac{h}{2} \rfloor + 1$ or $\sum_{i \in N \setminus S} w_i \geq \lfloor \frac{h}{2} \rfloor + 1$). Moreover, we assume a sharp distinction between the respective "jurisdictions" of the executive and legislature. Therefore, the outcome space is $X = Y \times Z$, where Y denotes the "jurisdiction" of the executive, and Z the "jurisdiction" of the legislature. Then, the EF $E^{PS}(\mathbf{w})$ of a presidential system with perfect separation of powers and weight profile $\mathbf{w}$ is defined by the following rule: for any $S \subseteq N \cup \{0^*\}$, $A \subseteq X$, $A \in (E^{PS}(\mathbf{w}))(S)$ if and only if one of the conditions i)-iv) listed below is satisfied : i) $A \neq \emptyset$, $0^* \in S$ and $\sum_{i \in S} w_i \geq \lfloor \frac{h}{2} \rfloor + 1$; ii) $A \supseteq \{y\} \times Z$ for some $y \in Y$, and $0^* \in S$; iii) $A \supseteq Y \times \{z\}$ for some $z \in Z$, and $\sum_{i \in S} w_i \geq \lfloor \frac{h}{2} \rfloor + 1$; iv) $A = X$ and $S \neq \emptyset$.*

Definition 13 *(The EF of a parliamentary system with a directly elected premier and a fixed majority) Let 0^* denote the elected premier, $N = \{1, .., n\}$ the set of parties -or voting blocs- of a legislature of size h (whose allocation of seats is represented by a n-dimensional strong weight profile $\mathbf{w}$ as under the previous* *definition), $M = M(\mathbf{w}) \subseteq N$ a (possibly minimal) majority coalition, X the outcome set, and $x^* \in X$ a "deadlock" outcome that corresponds to legislature-termination, i.e. new elections. Then, the EF $E^{PA}(\mathbf{w}, M)$ of a parliamentary system with directly elected premier and fixed majority $M = M(\mathbf{w})$ at weight profile $\mathbf{w}$ is defined by the following rule: for any .$S \subseteq N \cup \{0^*\}$, $A \subseteq X$, $A \in (E^{PA}(\mathbf{w}, M))(S)$ if and only if one of the clauses i)-iii) described below is satisfied : i) $A \neq \emptyset$ and $S \supseteq M \cup \{0^*\}$; ii) $x^* \in X$ and $S \cap (M \cup \{0^*\}) \neq \emptyset$; iii) $A = X$ and $S \neq \emptyset$.*

We are now ready to state the next result on Galois lattices, which admits a quite straightforward proof.

Proposition 14 *Let $E^{PS}(\mathbf{w})$ and $E^{PA}(\mathbf{w}, M)$ be the presidential and parliamentary EFs as defined above.*
Then i) $B(\mathbf{L}(E^{PS}(\mathbf{w}))) = \mathbf{2}^2$;
ii) $B(\mathbf{L}(E^{PA}(\mathbf{w}, M))) = \mathbf{3}$ (where **3** *denotes the three-sized chain); hence, in particular, $E^{PS}(\mathbf{w})$ is specialized whereas $E^{PA}(\mathbf{w}, M)$ is unspecialized (but neither of them is consensual, fully distributed, strictly hierarchical, or simple).*

4.5 Solution Correspondences for Private-Good Economies

The computation of Galois lattices is easily extendable to solution concepts and correspondences. This is so because whenever the "objects" to be "solved" include a description of non-verifiable individual characteristics (e.g. preferences), the latter can be regarded as the output of strategic behaviour. As a result, the solution concept under consideration can be aptly interpreted as a revelation-game correspondence. The Galois lattice of such a solution correspondence provides, once again, a succinct description of the structure of coalitional power when the actual behaviour of players is well predicted by the given solution concept (hence, the coalitional power discussed here is of a *conditional* sort) . This subsection is devoted to an application of those ideas to some domains of pure exchange private-good economies. In particular, we shall focus on a) core correspondences- and related solution concepts- on an unconventionally large domain and b) a solution concept that combines Pareto-efficiency with *undominated diversity (* a mild fairness requirement that generalizes the no-envy criterion) on a standard domain.

To begin with, a few basic definitions are to be recalled. A *pure exchange private-good-economy* is a tuple $\mathbf{e} = (N, (X_i)_{i \in N}, (\succcurlyeq_i)_{i \in N}, (\omega_i)_{i \in N})$ where for each agent $i \in N$, $X_i = \mathbb{R}^k_+$ is her consumption set whose dimension k denotes the number of available private goods, $\succcurlyeq_i$is the total preference preorder of i on the allocation space $X = \prod_{i \in N} X_i$, and ω_i is her endowment. We denote by $\mathbb{E}(\omega)$ the set of all n-agent pure exchange private-good economies with endowment profile $\omega = (\omega_i)_{i \in N}$. (We should emphasize here that we are *not* imposing the usual selfishness, monotonicity, continuity, or convexity restrictions on preferences: therefore- as mentioned previously - we are considering a *much* *larger* domain of economies than the standard one). A *feasible allocation* of an economy $\mathbf{e} \in \mathbb{E}(\omega)$ is a profile of consumption programs $\mathbf{x} = (x_i)_{i \in N} \in X$ such that $\sum_{i \in N} x_i \leq \sum_{i \in N} \omega_i$. The set of feasible allocations of an economy $\mathbf{e} \in \mathbb{E}(\omega)$ will be denoted by $F(\omega)$. We are now ready to introduce the solution concepts mentioned above.

Definition 15 *(Consistent core correspondence on $\mathbb{E}(\omega)$) A consistent core allocation of an economy $\mathbf{e} = (N, (X_i)_{i \in N}, (\succcurlyeq_i)_{i \in N}, (\omega_i)_{i \in N}) \in \mathbb{E}(\omega)$ is a feasible allocation $\mathbf{x} \in F(\omega)$ such that i) for any S, $\emptyset \neq S \subseteq N$, and any $\mathbf{y} \in F(\omega)$ with $\sum_{i \in S} y_i \leq \sum_{i \in S} \omega_i$ an allocation $\mathbf{y}' \in F(\omega)$ exists such that $y'_i = y_i$ for any $i \in S$, and $\mathbf{x} \succcurlyeq_j \mathbf{y}'$ for some $j \in S$, and ii) for any S, $\emptyset \neq S \subseteq N$, and any $\mathbf{y} \in F(\omega)$ such that $y_i = x_i$ for every $i \in N \backslash S$, $\mathbf{x} \succcurlyeq_j \mathbf{y}$ for some $j \in S$. The set of consistent core allocations of $\mathbf{e}$ is denoted by $core(\mathbf{e})$. The consistent core correspondence $C : \mathbb{E}(\omega) \to\to F(\omega)$ is defined by the rule $C(\mathbf{e}) = core(\mathbf{e})$ for any $\mathbf{e} \in \mathbb{E}(\omega)$.*

Remark 16 *The consistent core correspondence as defined above is an extension to $\mathbb{E}(\omega)$ of the usual notion of core allocations, which is typically defined for economies with monotonic - and selfish- preferences. Thus, consistency of core allocations - i.e. core-stability of core-suballocations within their appropriate reduced economy, a property that is automatically satisfied by core allocations of economies with selfish monotonic preferences- has to be explicitly incorporated into the definition. It should also be mentioned that the foregoing definition implicitly assumes -as usual- that any coalition can be freely formed. Obvious adaptations can be introduced in order to cover the case of restricted sets of feasible coalitions. In particular, it can be shown that a general version of the results on Galois lattices of consistent core correspondences as presented below obtain, provided that the set of feasible coalitions is a complemented lattice (see Vannucci(1998c)).*

Another solution concept for private-good economies - perhaps the most widely used- is provided by the notion of a *Walrasian equilibrium.*

Definition 17 *(Consistent core Walrasian-extension on $\mathbb{E}^{\circ}(\omega)$) A Walrasian equilibrium of an economy $\mathbf{e} \in \mathbb{E}(\omega)$ is a pair $(\mathbf{p}^*, \mathbf{x}^*) \in \mathbb{R}^k_+ \times X$ such that $\mathbf{p}^* \neq \mathbf{0}$, $\mathbf{x}^* \in F(\omega)$ and $\mathbf{x}^* \succcurlyeq_i \mathbf{y}$ for any $i \in N$ and any $\mathbf{y} \in F(\omega)$ with $\mathbf{p}^* \cdot \mathbf{y} \leq \mathbf{p}^* \cdot \omega$. A feasible allocation $\mathbf{x} \in F(\omega)$ is a Walrasian allocation of the economy $\mathbf{e} \in \mathbb{E}(\omega)$ if $(\mathbf{p}, \mathbf{x})$ is a Walrasian equilibrium of $\mathbf{e}$ for some non-null non-negative price vector $\mathbf{p}$. The set of Walrasian allocations of an economy $\mathbf{e} \in \mathbb{E}(\omega)$ is denoted by $w(\mathbf{e})$. It is easily checked that Walrasian allocations of $\mathbf{e}$ are in the core of $\mathbf{e}$. Hence, a Walrasian subcorrespondence of the consistent core correspondence - or, equivalently, a consistent core Walrasian extension W^C - can be defined by the following rule : $W^C(\mathbf{e}) = w(\mathbf{e})$ if $w(\mathbf{e}) \neq \emptyset$, and $W^C(\mathbf{e}) = C(\mathbf{e})$ otherwise. Of course, both C and W^C are locally empty-valued. Therefore, their α-EFs and β-EFs are not well-behaved (see the definitions in section 3 above).*

The consistent core correspondence and its Walrasian subcorrespondence are meant to capture the operation of two distinct coordination devices that may be available to optimizing agents, but none of them is explicitly concerned with fairness criteria. By contrast, a combination of efficiency and fairness under *selfish* preferences is the main rationale for the *undominated efficient correspondence* to be defined below (see e.g. Roemer(1996)). We recall here that the set $\mathbb{E}^{\circ}(\omega) \subseteq \mathbb{E}(\omega)$ of economies with *selfish* preferences consists of all economies $\mathbf{e} = (N, (X_i)_{i \in N}, (\succcurlyeq_i)_{i \in N}, (\omega_i)_{i \in N})$ such that for any $i \in N$, and $\mathbf{x}, \mathbf{y} \in F(\omega)$, $\mathbf{x} \succcurlyeq_i \mathbf{y}$ if and only if $(x_i, (w_j)_{j \neq i}) \succcurlyeq_i (y_i, (z_j)_{j \neq i})$ for any $(w_j)_{j \neq i}$, $(z_j)_{j \neq i} \in \prod_{j \in N \setminus \{i\}} X_j$ (indeed, under selfish preferences $\mathbf{x} \succcurlyeq_i \mathbf{y}$ can also be written $x_i \succcurlyeq_i y_i$).

Definition 18 *(Undominated efficient correspondence on $\mathbb{E}^{\circ}(\omega)$) Let $\mathbf{e} = (N, (X_i)_{i \in N}, (\succcurlyeq_i)_{i \in N}, (\omega_i)_{i \in N}) \in \mathbb{E}^{\circ}(\omega)$ be an economy with selfish preferences. An allocation $\mathbf{x} \in F(\omega)$ is undominated in $\mathbf{e}$ if for no $j \in N$ both $[x_i \succcurlyeq_i x_j$ for all $i \in N]$ and $[x_j \not\succcurlyeq_i x_i$ for all $i \in N]$ hold true. An allocation $\mathbf{x} \in F(\omega)$ is Pareto-efficient in $\mathbf{e}$ if for no allocation $\mathbf{y} \in F(\omega)$ both $[y_i \succcurlyeq_i x_i$ for all $i \in N]$ and $[x_j \not\succcurlyeq_j y_j$ for some $j \in N]$ obtain. The set of undominated and Pareto-efficient allocations of economy $\mathbf{e}$ is denoted by $\mathbf{u}^{PE}(\mathbf{e})$. The undominated efficient correspondence U^{PE} : $\mathbb{E}^{\circ}(\omega) \to\to F(\omega)$ is defined by the rule $U^{PE}(\mathbf{e}) = \mathbf{u}^{PE}(\mathbf{e})$ for any $\mathbf{e} \in \mathbb{E}^{\circ}(\omega)$.*

We are now ready to state our results concerning the Galois lattices of the foregoing solution correspondences.

Proposition 19 *Let C, W^C, U^{PE} the solution correspondences for private-good-economies as described above. Then, i) $B(\mathbf{L}(E_\alpha(C))) = B(\mathbf{L}(E_\beta(C))) = B(\mathbf{L}(E_\alpha(W^C))) = B(\mathbf{L}(E_\beta(W^C))) = \mathbf{2}^n$; ii)$B(\mathbf{L}(E_\alpha(U^{PE}))) = B(\mathbf{L}(E_\beta(U^{PE})))$ $= \mathbf{2}$.*

Hence, Proposition 8i) provides a further example of a *specialized EF* (of width n) .Obviously, the underlying environment consisting of private-good economies is somehow conducive to EFs of this sort. Proposition 8ii) however confirms that a private-good-economy setting is also consistent with *simple* EFs.

5 Concluding Remarks

The present paper has—hopefully— made a case for the relevance of Galois lattices of effectivity functions as basic invariants for the classification and analysis of game correspondences. The emphasis of the paper has been on the "structural" complexity measures for game correspondences that are provided by such lattices. Further insights on the interplay between those "structural" properties and (core-)stability may arise from studying the Galois lattices of *convex* effectivity functions (as done in Vannucci(1998a)). It is also worth noting that - while EFs having non-distributive Galois lattices can certainly be devised (see again Vannucci(1998a))- distributivity seems to be the rule for typical game correspondences (just as with the lattices of truth values of most logics). Those arguments however are best left as a possible topic for further research.

6 Appendix

Proof of Proposition 3. Obvious, since the relevant closure systems — e.g. $\mathbf{C}(E)$ and $\mathbf{C}(E_{+Y})$— are isomorphic. Indeed, let $\mathbf{S} \in \mathbf{C}(E)$, i.e. $S \in \mathbf{S}$ if $A \in E(S)$ for any $A \subseteq X$ such that $A \in E(T)$ for each $T \in \mathbf{S}$. Now, let $S' \subseteq N$, $B \in E_{+Y}(S')$ for any $B \subseteq X \cup Y$ such that $B \in E_{+Y}(T)$ for each $T \in \mathbf{S}$, and $S' \notin \mathbf{S}$. Then, $A \notin E(S')$ for some $A \subseteq X$ such that $A \in E(T)$ for each $T \in \mathbf{S}$. Since —by hypothesis— $A \cap Y = \emptyset$, it follows that $A \in E_{+Y}(T)$ for each $T \in \mathbf{S}$, and $A \notin E_{+Y}(S')$ (by definition of E_{+Y}), a contradiction. Hence, $\mathbf{S} \in \mathbf{C}(E_{+Y})$. Conversely, let $\mathbf{S} \in \mathbf{C}(E_{+Y})\backslash\mathbf{C}(E)$. Hence, a $S' \subseteq N$ exists such that $A \in E(S')$ for any $A \subseteq X$ such that $A \in E(T)$ for each $T \in \mathbf{S}$,and $S' \notin \mathbf{S}$, while for any $S \subseteq N$ if $B \in E_{+Y}(T)$ for any $B \subseteq X \cup Y$ such that $B \in E_{+Y}(T)$ for any $T \in \mathbf{S}$ then $S \in \mathbf{S}$. Now, take $B \subseteq X \cup Y$ such that $B \in E_{+Y}(T)$ for any $T \in \mathbf{S}$.Then $B\backslash Y \in E(T)$ for any $T \in \mathbf{S}$, whence $B\backslash Y \in E(S')$, i.e.$B \in E_{+Y}(S')$. It follows that $S' \in \mathbf{S}$, a contradiction. As a result, $\mathbf{C}(E_{+Y}) = \mathbf{C}(E)$,and the identity function Id is the required latticial isomorphism. □

Proof of Proposition 8. i) It is well-known that for any strategic game form $E_\alpha(G) = E_\beta(G)$ iff $E_\alpha(G)$ is maximal (see Peleg(1984), lemma 5.1.17). Now, take $\emptyset \neq A \subseteq X, \emptyset \neq S \subseteq N$ such that $A \notin E_\alpha(G^B)(S)$. Then —by definition of G^B— $S \neq N$ and $x^* \notin A$: hence $N\backslash S \neq \emptyset$ and $x^* \in X\backslash A$ i.e. $X\backslash A \in E_\alpha(G^B)(N\backslash S)$. Moreover, the definition of G^B clearly entails that $S = N$ is the sole coalition such that $A \in E_\alpha(G^B)(S)$ for each $A \in P(X)\backslash\{\emptyset\}$ (this fact also implies that G^B is consensual). Indeed, $S = N$ is the sole coalition such that $A \in E_\alpha(G^\beta)(S)$ for *some* $A \in \{B \subseteq X : x^* \notin B\}$ while —by definition of G^B—$A \in E_\alpha(G^B)(S)$ for any coalition $S \neq \emptyset$ and any $A \subseteq X$ with $x^* \in A$ (hence in particular $r_{E_\alpha(G^B)}(\{i\}) = 1$ for any $i \in N$). It follows that $B(\mathbf{L}(G^B)) = ((\{N\}, P(X)\backslash\{\emptyset\}), (P(N)\backslash\{\emptyset\}, \{B \subseteq X : x^* \in B\})) = \mathbf{2}$ (modulo isomorphisms) . □

Proof of Proposition 9. i) Notice that for any $S \subseteq N$ such that $\#S = n - h$, $\lceil (kn+1)\frac{s}{n} \rceil > kn+1-a$ if and only if $a > kh$, i.e. $A \in E^{PV}(S)$ iff $\#A > kh$. It follows that $B(\mathbf{L}(E^{PV})) = ((\{S \subseteq N : \#S \geq n-h\}, \{A \subseteq X : \#A > kh\}), h = 0, .., n-1) = \mathbf{n}$ (modulo isomorphisms) .

In particular, E^{PV} turns out to be consensual (N is the only maximum rank coalition) and unspecialized(since its Galois lattice is a chain). □

Proof of Proposition 10. Let G^2 be a 2×2 strategic game form with outcome function h as described in the text. Clearly enough, by choosing rows and columns, respectively, 1 can ($\alpha-$ and $\beta-$)enforce $\{a, b\}$,$\{c, d\}$ (and supersets), whereas 2 can ($\alpha-$and $\beta-$)enforce $\{a, c\}$,$\{b, d\}$ (and supersets).

i) It is easily checked by direct inspection that the only difference between $E_\alpha(G^2)$ and $E_\beta(G^2)$ — in the "generic" case i.e. with $\#X = 4$— reduces to the fact that

$$\{\{a,d\},\{b,c\}\} \in [E_\beta(\{1\}) \cap E_\beta(\{2\})] \setminus [E_\alpha(\{1\}) \cup E_\alpha(\{2\})].$$

It follows that

$$B(\mathbf{L}(E_\alpha(G^2))) = \left\{ \begin{array}{c} (\{N\},\{A \subseteq X : A \neq \emptyset\}), \\ (\{N,\{1\}\},\{A \subseteq X : A = \{a,b\}, A = \{c,d\} \text{ or } \#A \geq 3\}), \\ (\{N,\{2\}\},\{A \subseteq X : A = \{a,c\}, A = \{b,d\} \text{ or } \#A \geq 3\}), \\ (\{N,\{1\},\{2\}\},\{A \subseteq X : \#A \geq 3\}) \end{array} \right\}$$

and

$$B(\mathbf{L}(E_\beta(G^2))) = \left\{ \begin{array}{c} (\{N\},\{A \subseteq X : A \neq \emptyset\}), \\ (\{N,\{1\}\},\{A \subseteq X : \#A \geq 2, A \neq \{a,c\}, A \neq \{b,d\}\}), \\ (\{N,\{2\}\},\{A \subseteq X : \#A \geq 2, A \neq \{a,b\}, A \neq \{c,d\}\}), \\ (\{N,\{1\},\{2\}\},\{A \subseteq X : A = \{a,d\}, A = \{b,c\}, \text{ or } \#A \geq 3\}) \end{array} \right\}$$

But then, modulo (latticial) isomorphisms, $B(\mathbf{L}(E_\alpha(G^2))) = B(\mathbf{L}(E_\beta(G^2) = \mathbf{2}^2$.

ii) When $\#X = 3$,three subcases may be distinguished: a) $a = b$, or $c = d$ (i.e. the identical outcomes are in the same row), b) $a = c$, or $b = d$ (i.e. the identical outcomes are in the same column), c) $a = d$, or $b = c$ (i.e. the identical outcomes are on a diagonal) . It should also be noticed that under both a) and b) $E_\alpha(G^2) = E_\beta(G^2)$: this is so because whenever a (2-dimensional) row or column has (two) identical entries —say x— the following facts are easily checked by direct inspection of the game(-form) matrix: 1) x is an element of both diagonals, 2) each diagonal replicates a distinct row (column) if the double x is on a column (a row). But then, $\{\{a,d\},\{c,d\}\} \subseteq E_\alpha(G^2)(\{1\}) \cap E_\alpha(G^2)(\{2\})$. Thus, $E_\alpha(G^2) = E_\beta(G^2) = E(G^2)$ follows from our previous observation on the relationship between $E_\alpha(G^2)$ and $E_\beta(G^2)$ in the "generic" case: we denote by $\mathbf{L}(E(G^2))$ their Galois lattice, which is determined as follows. Under case a) it is checked by direct inspection that if we denote by x the replicated outcome, and by y, z the remaining outcomes then

$$E(G^2)(\{1\}) = \{A \subseteq X : A \supseteq \{x\}\} \cup \{\{y,z\}\}, \text{ and}$$
$$E(G^2)(\{2\}) = \{A \subseteq X : A \supseteq \{x,y\} \text{ or } A \supseteq \{x,z\}\}.$$

Hence,

$$B(\mathbf{L}(G^2)) = \left\{ \begin{array}{c} (\{N\},\{A \subseteq X : A \neq \emptyset\}), \\ (\{N,\{1\}\},\{A \subseteq X : A = \{y,z\} \text{ , or } A \supseteq \{x\}\}), \\ (\{N,\{1\},\{2\}\},\{A \subseteq X : A \supseteq \{x,y\} \text{ or } A \supseteq \{x,z\}\}) \end{array} \right\}.$$

Similarly, under case b)

$$B(\mathbf{L}(G^2)) = \left\{ \begin{array}{c} (\{N\},\{A \subseteq X : A \neq \emptyset\}) \\ (\{N,\{2\}\},\{A \subseteq X : A = \{y,z\}, \text{ or } A \supseteq \{x\}\}), \\ (\{N,\{1\},\{2\}\},\{A \subseteq X : A \supseteq \{x,y\} \text{ or } A \supseteq \{x,z\}\}) \end{array} \right\}$$

where, again, x denotes the replicated outcome. Thus, under cases a) and b) $B(\mathbf{L}(E(G^2)) = \mathbf{3}$.

Under case c), it is immediately seen that for each row there is (exactly) one column that includes the same elements, and vice versa. Moreover, no row or column amounts to a pair of duplicated outcomes, and none of the diagonals can possibly be replicated by a row or a column (because that would entail $\#X < 3$). Hence, if we denote by x the replicated outcomes, and by y, z the other outcomes

$$E_\alpha(G^2)(\{1\}) = E_\alpha(G^2)(\{2\}) = \{A \subseteq X : \#A \geq 2, A \neq \{y,z\}\}\text{ , and}$$
$$E_\beta(G^2)(\{1\}) = E_\beta(G^2)(\{2\}) = \{A \subseteq X : A \neq \emptyset, A \neq \{y\}, A \neq \{z\}\}.$$

Thus, $B(\mathbf{L}(E_\alpha(G^2))) = B(\mathbf{L}(E_\beta(G^2))) = \mathbf{2}$.

iii) If $\#X = 2$, four cases may be distinguished : a) an outcome $x \in X$ and a strategy profile $(u,v) \in S_1 \times S_2$ exist such that $x = h(r,w)$ iff $(r,w) \neq (u,v)$; b) each row is made up of replicated outcomes ; c) each column is made up of replicated outcomes; d) each diagonal is made up of replicated outcomes.

Under case a) it is immediately checked that

$$E_\alpha(G^2)(\{1\}) = E_\alpha(G^2)(\{2\}) = E_\beta(G^2)(\{1\}) = E_\beta(G^2)(\{2\}) =$$
$$=\{A \subseteq X : A \neq \emptyset, A \neq \{x\}\}.$$

It follows that $B(\mathbf{L}(E_\alpha(G^2))) = B(\mathbf{L}(E_\beta(G^2))) = \mathbf{2}$.

Under case b) ,

$$E_\alpha(G^2)(\{1\}) = E_\beta(G^2)(\{1\}) = \{A \subseteq X : A \neq \emptyset\}\text{, and}$$
$$E_\alpha(G^2)(\{2\}) = E_\beta(G^2)(\{2\}) = \{X\}.$$

Hence, $B(\mathbf{L}(E_\alpha(G^2))) = B(\mathbf{L}(E_\beta(G^2))) = \mathbf{2}$.

Similarly, under case c),

$$E_\alpha(G^2)(\{2\}) = E_\beta(G^2)(\{2\}) = \{A \subseteq X : A \neq \emptyset\}\text{, and}$$
$$E_\alpha(G^2)(\{1\}) = E_\beta(G^2)(\{1\}) = \{X\}.$$

Thus, again, $B(\mathbf{L}(E_\alpha(G^2))) = B(\mathbf{L}(E_\beta(G^2))) = \mathbf{2}$.

Under case d),

$$E_\alpha(G^2)(\{1\}) = E_\alpha(G^2)(\{2\}) = \{X\}\text{ , and}$$
$$E_\beta(G^2)(\{1\}) = E_\beta(G^2)(\{2\}) = \{A \subseteq X : A \neq \emptyset\}.$$

It follows that $B(\mathbf{L}(E_\alpha(G^2))) = \mathbf{2}$, while $B(\mathbf{L}(E_\beta(G^2))) = \mathbf{1}$. □

Proof of Proposition 13. i) It is immediately checked that — by definition of $E^{PS}(\mathbf{w})$— $B(\mathbf{L}(E^{PS}(\mathbf{w}))) =$

$$=\left\{\begin{array}{c} (\{S \subseteq N \cup \{0^*\} : 0^* \in S,\text{ and } \sum_{i \in S} w_i \geq \lfloor \frac{h}{2} \rfloor + 1\}, \{A \subseteq X : A \neq \emptyset\}) \\ (\{S \subseteq N \cup \{0^*\} : 0^* \in S\}, \{A \subseteq X : A \supseteq \{y\} \times Z\text{ for some } y \in Y\}), \\ (\{S \subseteq N \cup \{0^*\} : \sum_{i \in S} w_i \geq \lfloor \frac{h}{2} \rfloor + 1\}, \{A \subseteq X : A \supseteq Y \times \{z\}\text{ for some } z \in Z\}) \\ (\{S \subseteq N \cup \{0^*\} : S \neq \emptyset\}, \{X\}) \end{array}\right\}$$

Hence $B(\mathbf{L}(E^{PS}(\mathbf{w}))) = \mathbf{2}^2$ (modulo isomorphisms).

ii) It follows from the definition of $E^{PA}(\mathbf{w}, M)$ that $B(\mathbf{L}(E^{PA}(\mathbf{w}, M))) =$

$$=\left\{\begin{array}{c} (\{S \subseteq N \cup \{0^*\}\}, \{A \subseteq X : A \neq \emptyset\}), \\ (\{S \subseteq N \cup \{0^*\} : S \cap (M \cup \{0^*\}) \neq \emptyset\}, \{A \subseteq X : x^* \in X\}), \\ (\{S \subseteq N \cup \{0^*\} : S \neq \emptyset\}, \{X\}) \end{array}\right\}$$

Hence $B(\mathbf{L}(E^{PA}(\mathbf{w}, M)) = \mathbf{3}$ (modulo isomorphisms). □

Proof of Proposition 19. i) see Vannucci (1998c) ;

ii) It is easily checked that for any $i \in N$, and any $\mathbf{x} \in F(\omega)$, $F(\omega)\setminus\{\mathbf{x}\} \notin (E_\alpha(U^{PE}))(N\setminus\{i\}) \cup (E_\beta(U^{PE}))(N\setminus\{i\})$. Indeed, let $(\succcurlyeq_j)_{j\in N\setminus\{i\}}$any $N\setminus\{i\}$-profile of (selfish) total preorders on $X = \prod_{i\in N} X_i$, and $\succcurlyeq_i$a selfish total preorder on X such that $x_h \succcurlyeq_i y$ for any $h \in N$, and $y \in \mathbb{R}^k_+$. Then, $\mathbf{x}$ is both Pareto-efficient and undominated, and. the thesis follows immediately from monotonicity of both $E_\alpha(U^{PE})$ and $E_\beta(U^{PE})$ $\square$

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