quaderni dell' istituto di economia

n. 26

Sandro Gronchi

On Karmel's criterion for optimal truncation

Facoltà di Scienze Economiche e Bancarie

Università degli Studi di Siena

Pubblicazione dell'Istituto di Economia
Facoltà di Scienze Economiche e Bancarie
Università degli Studi di Siena

Sandro Gronchi

On Karmel's criterion for optimal truncation

1984, Maggio
Stamperia della Facoltá

Il Professor Sandro Gronchi

insegna Istituzioni di Economia Politica

presso l'Istituto di Economia

della Facoltà di Scienze Economiche e Bancarie

dell'Università di Siena

*It is well known that P.H. Karmel | 7 | has integrated the technological-economic hypothesis of truncatability, with a behavioural hypothesis concerning the criterion by which an investor chooses the optimum duration of an investment project. Given that each possible duration of a project may yield zero, one or more internal rates of return, Karmel assumes that a rational investor, after having calculated them all, chooses that particular duration to which the maximum internal rate is attached[(1)].

In the Appendix to Karmel's article, the mathematician B.C. Rennie proves that no other internal rate is attached to that duration to which the maximum internal rate is attached. In other terms, the internal rate of return of a project whose lifetime has been chosen according to Karmel's criterion, is unique.

This allows Karmel to say that the objective of rendering unique the internal rate of any project, is spontaneously achieved by investors, where they are simply supposed to require maximum convenience[(2)].

(*) I wish to thank Prof A. Pasini of the Mathematics Dept. of Siena University, for useful comments on the first draft of the Mathematical Appendix.

(1) We do not agree with all those authors | 2, 9, 11 , et al. | who assimilate Karmel's (later) contribution with that of Soper. Soper's proposed criterion for optimal truncation is radically different from Karmel's. Moreover, Soper's analysis suffers from several serious errors, largely invalidating his conclusions. These errors are quite correctly and accurately pointed out by Karmel.

(2) It is well known that K.J. Arrow and D.V. Levhari | 2 | have proposed an approach which is conceptually different from that of Karmel. It is likewise known that the two approaches yield 'numerically' coincident results (C.J. Norstrom | 9 |).

Of course, it is true that an internal rate of return can be unambiguously used in decision-making procedures only if it is unique. For this reason Karmel's objective of rendering unique the internal rate of return for any feasible project, appears to be quite justifiable.

Nevertheless, we have recently argued(3) that uniqueness alone is not sufficient to guarantee a fully legitimate use of the internal rate of return. Our analysis can be summarized thus.

From the point of view of its economic significance, an internal rate of return may be of two types: a first-type internal rate of return means a pure lending rate earned (by the investor) on the funds invested in the project. A second-type internal rate of return has a mixed meaning, since it is both a lending rate earned on invested funds, and a borrowing rate also paid by the investor on funds financed by the project.

An internal rate of return can be significantly used for decision-making only if it is of the first type, i.e. if it is pure.

If $(a_0, \ldots, a_n)$ is a project and r_1 an internal rate of return attached to it, the mathematical condition under which r_1 is pure, is as follows:

$$\text{(I)} \quad \sum_{j=0}^{n} a_j (1 + r_1)^{i-j} \leq 0 \qquad (i = 0, \ldots, n-1) .$$

The same condition (I) has been proved by us to be sufficient but not necessary for the uniqueness of r_1 .

---

(3) S. Gronchi| 6| .

This implies that all internal rates which are pure, are also unique, while the converse is not true. It implies also that an internal rate of return can be fully available for legitimate (both unambiguous and significant) use in decision-making procedures if and only if condition (I) is satisfied.

Of course, these results apply also to the internal rate of a project whose lifetime has been chosen according to Karmel's criterion. Therefore, Rennie's theorem, stating that this internal rate is unique, does not guarantee that such a rate can be legitimately used for decision-making at all. To be sure of that, one needs to prove that condition (I) is satisfied. The main purpose of this note is just this.

Since we have said that condition (I) is sufficient for the uniqueness of the internal rate, while pursuing our purpose, we shall incidentally offer an alternative and much simpler proof of Rennie's theorem.

This note has one minor purpose. Though uniqueness-condition (I) generalizes slightly Soper's well known uniqueness-condition, the proof we have given elsewhere(4) is quite different from that of Soper(5). We want to offer here a new proof which follows closely Soper's procedure. The only unavoidable difference will be the use of an ad hoc generalization of Newton's theorem (concerning the existence of an upper bound to the roots of a polynomial), rather than to Newton's theorem as such. This generalization is proved in the Mathematical Appendix.

---

(4) S. Gronchi| 6 | .

(5) C.S. Soper| 11 | .

The following definitions will be adopted throughout this note.

Definition 1. We define as a project, a vector $(a_0 , \dots , a_n)$ of expected net outputs such that $a_0 < 0$[6].

Definition 2. We define as internal rate of return attached to a project $(a_0 , \dots , a_n)$, an interest rate $r_1 > -1$ such that:

$$\sum_{j=0}^{n} a_j \, (1 + r_1)^{n-j} \;=\; 0 \;.$$

Section 1

In this section we shall prove that the internal rate of return attached to a project whose lifetime has been chosen according to Karmel's criterion, satisfies condition (I). More precisely, we shall prove the following proposition.

Prosition 1. Let:

_ $A := (a_0 , \dots , a_n)$ be a project;

_ $A_i := (a_0 , \dots , a_i)$ be the project which is obtained by truncating project A at time $i = 0 , \dots , n$[7];

_ R be the non-empty set[8] of all the internal rates of return which are attached to projects $A_0 , \dots , A_n$;

_ r_* be the maximum rate within set R ;

---

(6) We are forced to adopt a definition here which is more general than that which we have adopted in | 6 | . If we kept that definition, we would not be allowed to call all truncations 'projects' which can be obtained from a given project.

(7) We want to underline that $A_n = A$.

(8) It is easily proved that $R \neq \phi$ if (and only if) $a_i > 0$ for at least one $i = 1 , \dots , n$. Therefore we have $R \neq \phi$ for all economically relevant projects.

_ A_k be one[9] of projects $A_0 , \dots , A_n$ to which r_* is attached.

Then:

$$\sum_{j=0}^{i} a_j \, (1 + r_*)^{i-j} \;\leq\; 0 \qquad (i = 0 , \dots , k-1) \;.$$

Proof. Since $a_0 < 0$ (see Definition 1), obviously:

$$\lim_{r \to +\infty} \sum_{j=0}^{i} a_j \, (1 + r)^{i-j} \;<\; 0 \qquad (i = 0 , \dots , k-1) \;.$$

If for an $i < k$ we had:

$$\text{(II)} \qquad \sum_{j=0}^{i} a_j \, (1 + r_*)^{i-j} > 0 \;,$$

project A_i would yield an internal rate greater than r_* , since function:

$$\sum_{j=0}^{i} a_j \, (1 + r)^{i-j}$$

is continuous and would have different signs at the extremes of real interval $(r_* , +\infty)$. But this is impossible since r_* , by definition, is maximum within set R . Therefore, no $i < k$ can yield inequality (II). Q.E.D..

Section 2

We shall now offer the alternative proof of the following proposition.

Proposition 2. Given a project $A := (a_0 , \dots , a_n)$ and an

---

(9) There might well be more projects A_i to which r_* is attached. Nevertheless, the fact that A_0 yields obviously no internal rate of return, precludes that r_* can be attached to A_0 .

internal rate of return r_1 attached to it, project A has no internal rate of return other than r_1 if condition (I) is satisfied.

Proof. According to Definition 2, to prove the proposition we shall prove that polynomial

$$A(x) := \sum_{j=0}^{n} a_j x^{n-j}$$

has no positive root other than $x_1 := 1 + r_1$, if the following condition is satisfied:

$$\text{(III)} \quad \sum_{j=0}^{i} a_j x_1^{i-j} \leq 0 \qquad (i = 0, \dots, n-1) .$$

Applying Bezout's theorem[10], we can say that a unique (n-1).th degree polynomial exists, such that:

$$\text{(IV)} \quad A(x) = (x - x_1)\, Q(x) .$$

Equality (IV) implies that the roots of $A(x)$ other than x_1 are the same as the roots of polynomial $Q(x)$. According to Horner's theorem[11], we have:

$$\text{(V)} \quad Q(x) = \sum_{i=0}^{n-1} x^{n-i-1} \sum_{j=0}^{i} a_j x_1^{i-j} .$$

---

(10) See A. Kurosh| 8 |, p. 149.

(11) See A. Kurosh| 8 |, pp. 149-151.

Therefore, Fourier's sequence[12] relative to $Q(x)$ has the following elements, after the first[13]:

$$\text{(VI)} \quad Q^{(h)}(x) = \sum_{i=0}^{n-h-1} \left\{ \sum_{j=0}^{i} a_j x_1^{i-j} (n-i-1)! \, [(n-h-i-1)!]^{-1} \right\} x^{n-h-i-1}$$

$$(h = 0, \dots, n-1) .$$

Which implies that Fourier's complete sequence, at point zero, is as follows[14]:

$$a_0 x_1^{n-1} + a_1 x_1^{n-2} + \dots + a_{n-1}$$

$$1!\,(a_0 x_1^{n-2} + a_1 x_1^{n-3} + \dots + a_{n-2})$$

$$2!\,(a_0 x_1^{n-3} + a_1 x_1^{n-4} + \dots + a_{n-3})$$

$$(n-2)!\,(a_0 x_1 + a_1)$$

$$(n-1)!\, a_0 .$$

If condition (III) is satisfied, then all the elements of this sequen-

---

(12) Fourier's sequence relative to a v.th degree polynomial is formed by the polynomial itself and by its first v derivatives.

(13) All that one has to do, is to apply formula (IX) of the Mathematical Appendix, keeping in mind that in the present context we have: $P(x) = Q(x)$, and: $v = n-1$.

(14) For $x = 0$ sums (VI) and sum (V) reduce to their respective first terms.

ce are non-positive. Applying the generalization of Newton's theorem which is proved in the Mathematical Appendix, we can therefore state that $Q(x)$ has no positive root. Q.E.D..

University of Siena, June 1983 S.G.

MATHEMATICAL APPENDIX

The standard version of Newton's theorem is as follows[15].

Newton's theorem. Let:

$$\text{(VII)} \qquad P(x) := \sum_{i=0}^{v} p_i \; x^{v-i}$$

be a polynomial, and let b be an arbitrary real number. Then all the roots of $P(x)$ are less than b if Fourier's sequence[16] relative to $P(x)$ is strictly positive at point b.

In this Mathematical Appendix we shall prove the following slight generalization.

Newton's generalized theorem. Let us consider polynomial (VII) and let b be an arbitrary real number. Then all the roots of $P(x)$ are not greater than b if Fourier's sequence relative to $P(x)$ has no sign inversion at point b.

Proof. Applying Budan and Fourier's theorem[17], we can state that all the roots of $P(x)$ are not greater than b if Fourier's sequence at point b and Fourier's sequence at point $+\infty$, that is the two following sequences:

$$\text{(VIII)} \qquad P(b)\,,\; P'(b)\,,\; \dots\,,\; P^{(v)}(b)$$

$$\text{(IX)} \qquad P(+\infty)\,,\; P'(+\infty)\,,\; \dots\,,\; P^{(v)}(+\infty)\,,$$

---

(15) See, for example, W.H. Turnbull| 12 |, p. 97.

(16) See footnote 12.

(17) See A. Kurosh| 8 |, p. 263.

have the same number of sign inversions. It is easy to check that the last v elements of Fourier's sequence relative to $P(x)$ are the following:

$$\text{(X)} \qquad P^{(h)}(x) := \sum_{i=0}^{v-h} p_i \,(v-i)!\,[\,(v-h-i)!\,]^{-1}$$

$$(h=1, \ldots, v).$$

Therefore the leading coefficients of polynomials $P^{(h)}(x)$ do not differ in sign from p_0 , which is the leading coefficient of polynomial $P(x)$. This implies that sequence (IX) is either:

$$+\infty \;, \ldots , +\infty \;, n!\, p_0$$

or:

$$-\infty \;, \ldots , -\infty \;, n!\, p_0$$

according to whether $p_0 > 0$ or $p_0 < 0$. In any case, sequence (IX) has no sign inversion. Therefore sequences (VIII) and (IX) have the same number of sign inversions, if sequence (VIII) has no sign inversion. Q.E.D..

REFERENCES

|1| Andretta A., "Nota sul Teorema del Troncamento e sul Tasso Interno di Rendimento", Ricerche Economiche, luglio-dicembre 1977, vol. 31, pp. 375-388.

|2| Arrow K.J. e Levhari D.V., "Uniqueness of the Internal Rate of Return with Variable Life of Investment", The Economic Journal, September 1969, vol. 79, pp. 560-566.

|3| De Faro C., "A Sufficient Condition for a Unique Non-negative Internal Rate of Return: Further Comments", Journal of Financial and Quantitative Analysis, September 1978, vol. 13, pp. 577-584.

|4| Filippini C. e Filippini L., "Nota Critica al Teorema del Troncamento", Ricerche Economiche, gennaio-dicembre 1974, vol. 28, pp. 3-18.

|5| Flemming J.S. e WRIGHT J.F., "Uniqueness of the Internal Rate of Return: a Generalization", The Economic Journal, June 1971, vol. 81, pp. 256-263.

|6| Gronchi S., "On Investment Criteria Based on the Internal Rate of Return", Quaderni dell'Istituto di Economia dell'Università di Siena, aprile 1984, nº 25.

|7| Karmel P.H., "The Marginal Efficiency of Capital", Economic Record, December 1959, vol. 35, pp. 429-434.

|8| Kurosh A., Cours d'Algebre Superieure, Moscou: Editions Mir, 1971.

| 9 | Norstrom C.J., "Uniqueness of the Internal Rate of Return with Variable Life of Investment: a Comment", The Economic Journal, December 1970, vol. 80, pp. 983-984.

| 10| Nuti D.M., "On the Truncation of Production Flows", Kyklos, 1973, vol. 26, fasc. 3, pp. 485-496.

| 11| Soper C.S., "The Marginal Efficiency of Capital: a Further Note", The Economic Journal, March 1959, vol. 69, pp. 174-177.

| 12| Turnbull W.H., Theory of Equations, London: Oliver & Boyd, 1947.

RIASSUNTO

Il maggior risultato raggiunto in questa nota è che il tasso interno di rendimento associato a un progetto la cui durata sia stata scelta secondo il criterio di Karmel, non solo è unico, ma possiede anche il significato economico di puro tasso di crescita dei fondi investiti. In base a risultati discussi dall'autore in altra sede, ciò significa che tale tasso può essere usato a fini decisionali in modo pienamente legittimo.

ABSTRACT

The main purpose of this note is to point out that the internal rate of return attached to a project whose lifetime has been chosen according to Karmel's criterion, is a pure rate of growth of the funds invested in the project. According to the results which have been achieved by the author in a previous paper, this implies that such an internal rate is available for fully legitimate use in decision-making procedures.

ELENCO DEI QUADERNI PUBBLICATI

N. 1. MASSIMO DI MATTEO
Alcune considerazioni sui concetti di lavoro produttivo e improduttivo.

N. 2. MARIA L. RUIZ
Mercati oligopolistici e scambi internazionali di manufatti. Alcune ipotesi e un'applicazione all'Italia.

N. 3. DOMENICO MARIO NUTI
Le contraddizioni delle economie socialiste: una interpretazione marxista.

N. 4. ALESSANDRO VERCELLI
Equilibrio e dinamica del sistema economico-semantica dei linguaggi formalizzati e modello keynesiano.

N. 5. A. RONCAGLIA-M. TONVERONACHI
Monetaristi e neokeynesiani: due scuole o una?

N. 6. NERI SALVADORI
Mutamento dei metodi di produzione e produzione congiunta.

N. 7. GIUSEPPE DELLA TORRE
La struttura del sistema finanziario italiano: considerazioni in margine ad un'indagine sull'evoluzione quantitativa nel dopoguerra (1948-1978).

N. 8. AGOSTINO D'ERCOLE

Ruolo della moneta ed impostazione antiquantitativa in Marx: una nota.

N. 9. GIULIO CIFARELLI

The Natural Rate of Unemployment with Rational Expectations Hypothesis. Some Problems of Estimation.

N. 10. SILVANO VICARELLI

Note su ammortamenti, rimpiazzi e tasso di crescita.

N. 11. SANDRO GRONCHI

A Meaningful Sufficient Condition for the Uniqueness of the Internal Rate of Return.

N. 12. FABIO PETRI

Some Implications of Money Creation in a Growing Economy.

N. 13. RUGGERO PALADINI

Da Cournot all'oligopolio: aspetti dei processi concorrenziali.

N. 14. SANDRO GRONCHI

A Generalized Internal Rate of Return Depending on the Cost of Capital.

N. 15. FABIO PETRI

The Patinkin Controversy Revisited.

N. 16. MARINELLA TERRASI BALESTRIERI

La dinamica della localizzazione industriale: Aspetti teorici e analisi empirica.

N. 17. FABIO PETRI

The Connection between Say's Law and the Theory of the Rate of Interest in Ricardo.

N. 18. GIULIO CIFARELLI

Inflation and Output in Italy: a Rational Expectations Interpretation.

N. 19. MASSIMO DI MATTEO

Monetary Conditions in a Classical Growth Cycle

N. 20. MASSIMO DI MATTEO - MARIA L. RUIZ

Effetti dell'interdipendenza tra paesi produttori di petrolio e paesi industrializzati: un'analisi macrodinamica.

N. 21. ANTONIO CRISTOFARO

La base imponibile dell'IRPEF: un'analisi empirica.

N. 22. FLAVIO CASPRINI

L'efficienza del mercato dei cambi. Analisi teorica e verifica empirica

N. 23. PIETRO PUCCINELLI

Imprese e mercato nelle economie socialiste: due approcci alternativi.

N. 24. BRUNO MICONI

Potere prezzi e distribuzione in economie mercantili caratterizzate da diverse relazioni sociali

N. 25. SANDRO GRONCHI

On Investment Criteria Based on the Internal Rate of Return