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Abstract

The purpose of this paper is to study the influence of workers' savings on the dynamics of Goodwin's growth cycles. This is done by noticing, along Kaldorian-Pasinettian lines, that if workers save, then they hold part of the capital and earn some profits, which vary over the cycle. Thus, a correct specification of such a case requires the consideration of an extra variable – the share of capital held by workers. It is shown that, without having to impose any special conditions on the values of the parameters, a Hopf-Bifurcation analysis establishes the possibility of persistent and bounded cyclical paths for the resulting 3-dimensional dynamical system. The paper concludes with an investigation of the possibility of further bifurcations as a route to more complex behaviour.

1 Introduction[1]

Quietly — outside mainstream macroeconomics — Goodwin's growth cycle model [28] has emerged over the years as a powerful and fruitful "*system for doing macrodynamics*". This is hardly deniable given that still today, more than thirty years from its publication, one can often find new contributions that adopt its structure in the attempt to generalize it. Yet, despite the many — more than one hundred — existing contributions that have tried to generalize it in all possible directions, there is an aspect that still seems to deserve further investigation, namely, the problem of the proper way of relaxing Goodwin's extreme assumptions about savings behavior and wage rate dynamics.

As is well known, Goodwin crucially assumed (i) a classical saving function, according to which all wages are consumed and all profits saved and invested, and (ii) a Phillips curve in real terms ($\dot{w}/w = f(v) = -\gamma + \rho v$, $\gamma, \rho > 0$) according to which real wages (w) vary linearly with the employment rate (v).

Another way of describing Goodwin's assumption about savings behavior is to say that he assumed that the two propensities to save out of profits and out of wages (s_p and s_w, respectively) are different and such that $s_p > s_w$. Then, as a limiting and simpler case, he chose to work with $s_p = 1$ and $s_w = 0$.[2] An immediate possible generalization, thus, is to consider the case in which $0 < s_p \leq 1$ and $0 \leq s_w < 1$ and this is indeed what a number of authors have tried to do over the years.[3] To the best of my knowledge, however, only van der Ploeg (see, for example, [42]) has done it in a manner that tries to take account of Pasinetti's criticism of Kaldor's approach to differential savings.[4]

On the other hand, with regard to the assumption about the real wage dynamics, it is safe to say that Goodwin meant it only as a first approximation, required — we could add — to obtain in his model a dynamical system of the Lotka-Volterra type. It is therefore easy to think of possible

[1]The author is indebted to Fabio Petri who some time ago had the basic idea and sketched the mathematics required for the local stability analysis of the dynamical system of a model similar to the one investigated in this paper. Thanks are also due to Massimo Di Matteo for his helpful comments. I am however the only one responsable for the views expressed herein and, needless to say, for any errors.

[2]This interpretation seems correct given that Goodwin himself writes that the assumption could be relaxed to constant proportional savings, "*this changing the numbers but not the logic of the system*" [28, p. 54; my emphasis].

[3]The first author to mention the possibility of incorporating into the model a positive propensity to save out of wages is Atkinson [4], already in 1969. Other contributions that have tried to generalize the model in this direction are Velupillai [51], Glombowski-Krüger [24, 25, 26, 27], van der Ploeg [40, 41, 42, 43], Flaschel-Krüger [18], Fitoussi-Velupillai [17], Ferri-Greenberg [16], Sportelli-Cagnetta [48] and Sportelli [47], to mention only a few.

[4]See Kaldor [31] and, for example, Pasinetti [38].

generalizations of this assumption also and indeed, in this case too, this is what has happened over the years. Different authors have considered a number of different more general cases, for example, with a nonlinear $f(v)$ (see Velupillai [50]) or with the function f depending not only on v, but also on its variation rate (see Cugno-Montrucchio [8], Sportelli [47]).[5]

The purpose of this paper is to take the problem of differential savings and the problem of a more general formulation of the Phillips curve up again and to investigate the full consequences of them for the dynamics of Goodwin's model when they are *jointly* introduced into the original formulation. Our purpose in particular is to show that — if one takes account of Pasinetti's criticism of Kaldor's approach to differential savings — their joint consideration has important consequences for the dynamics of the model. Indeed, a persistent cyclical movement can emerge, that can be interpreted as a first step in a route to more complex (irregular) behavior. This appears to be a result that paves the way for further interesting research.

The paper is organized as follows. Section 2 contains a brief description of the original model and of the standard way in which the two more general assumptions just described have usually been introduced into it. The alternative formulation – along Kaldorian-Pasinettian lines – is then introduced in Section 3, where the 3-dimensional (3D) dynamical system of the modified model is derived and analyzed. Section 4 attempts to show – by use of the Hopf Bifurcation Theorem with s_w as the bifurcation parameter – that the model can produce persistent cyclical behavior. In doing this, we will also compare our approach and results with those obtained by other authors who have recently used the Hopf Bifurcation Theorem in the same or in a similar context. The section ends with an attempt to investigate the conditions for further bifurcations. A few concluding and summarizing results are finally given in Section 5.

2 The original model with $s_w \neq 0$ and a more general formulation of the Phillips curve

The following symbols and basic assumptions are used throughout the paper:[6]

q, output

[5] Above all, however, starting with Izzo [30] and then Desai [10], the Phillips curve has been modified by writing it in *money* rather than *real* terms and then introducing into the model an equation for price dynamics. Although we will not consider this kind of generalization of the original model, we will have something to say about it at the end of the paper.

[6] The dot over a variable ($\dot{x}$, for example) indicates the derivative with respect to time (dx/dt).

l, employment
$q/l = a = a_0 e^{\alpha t}$, $\alpha > 0$
$n = n_0 e^{\beta t}$, labor force, $\beta > 0$
$g_n = \alpha + \beta$
w, real wage
$u = wl/q = w/a$, share of wages
$v = l/n$, employment rate
k, capital stock
$k_c = \varepsilon k$, capital held by capitalists
$k_w = (1 - \varepsilon)\, k$, capital held by workers
$\sigma = k/q$, constant
P_c, capitalists' profits
P_w, workers' profits
P, total profits
$r = P/k$, rate of profit
s_p, propensity to save out of profits
s_w, S_w, workers' propensity to save and workers' savings respectively
s_c, S_c, capitalists' propensity to save and capitalists' savings respectively
S, total savings
$s_p - s_w > 0$
$\Delta s = s_c - s_w > 0$
$g = \dot{q}/q = \dot{k}/k = I/k = S/k = (1-u)/\sigma$, (warranted) rate of growth

In Goodwin's original model [28] — where it is assumed that $s_p = 1$ and $s_w = 0$ — a specification of the dynamic behavior of wages is all that is needed in order to obtain from these definitions and basic assumptions the dynamical system of the model.

Choosing, for example, as Goodwin does, a Phillips curve for the rate of growth of real wages:

$$\frac{\dot{w}}{w} = f(v), f'(v) > 0 \tag{2.1}$$

the dynamical system of the model becomes:

$$\frac{\dot{v}}{v} = \frac{\dot{l}}{l} - \frac{\dot{n}}{n} = g - g_n = \frac{1}{\sigma} - g_n - \frac{1}{\sigma}u \tag{2.2}$$

$$\frac{\dot{u}}{u} = \frac{\dot{w}}{w} - \alpha = f(v) - \alpha \tag{2.3}$$

which, with the linear $f(v)$ of the form $-\gamma + \rho v$ ($\gamma, \rho > 0$) used by Goodwin — in the case in which the parameters are such that $(1/\sigma) - g_n > 0$ — is equivalent to the dynamical system of the Lotka-Volterra prey-predator model:[7]

$$\begin{aligned}\dot{v} &= (a_1 - a_2 u)\, v \\ \dot{u} &= (-b_1 + b_2 v)\, u\end{aligned}$$

where $a_1 = (1/\sigma) - g_n > 0$, $a_2 = (1/\sigma)$, $b_1 = \alpha + \gamma > 0$ and $b_2 = \rho > 0$.

The unique positive equilibrium (v^e, u^e) of system (2.2)-(2.3) is such that $v^e = f^{-1}(\alpha)$ and $u^e = 1 - \sigma g_n$ and, moreover, as in Kaldor's theory of growth and income distribution, it implies a steady-state growth of the model economy at a (warranted) rate equal to the natural rate:[8]

$$g^e = g_n \tag{2.4}$$

As is well known, the roots of the characteristic equation at this equilibrium point are purely imaginary and such that:

$$\lambda_{1,2} = \pm i \sqrt{\frac{1}{\sigma} v^e u^e f'(v^e)} \tag{2.5}$$

where $i = \sqrt{-1}$.

As a consequence the local stability analysis cannot be used to decide on the type of dynamics of the original system. Thanks to the equivalence with the Lotka-Volterra model, however, we know that the equilibrium point is a *center*. This means that the model describes persistent fluctuations of v and u around the equilibrium, the amplitude of which fully depends on initial conditions.

This important qualitative feature of the model does not change if, *leaving unchanged the rest of it*, we take $0 \leq s_w < 1$ and $0 < s_p \leq 1$, with $s_p > s_w$.

In this case, we have for the warranted rate of growth:

$$g = \frac{s_p(1-u)q + s_w u q}{k} = \frac{s_p - (s_p - s_w)\, u}{\sigma} \tag{2.6}$$

so that the dynamical system of the model becomes:

$$\frac{\dot{v}}{v} = \frac{s_p}{\sigma} - g_n - \frac{(s_p - s_w)}{\sigma} u \tag{2.7}$$

$$\frac{\dot{u}}{u} = f(v) - \alpha \tag{2.8}$$

Clearly, equations (2.7)-(2.8), if we choose the same $f(v)$ as before, still form a dynamical system of the Lotka-Volterra type. The only difference

[7] See Goodwin [28, p. 54-55], Gandolfo [23, pp. 449-464] and Medio [34, pp. 34-38].

[8] As we shall stress at greater length later, however, it is worth noticing that, along such an equilibrium growth path, in Goodwin's model there is a constant rate of unemployment rather than full employment as in Kaldor's model. Moreover, the system persistently fluctuates around it rather than approaching it.

in comparison with the original model is that now the u-coordinate of the center is:

$$u^{ee} = \frac{s_p - \sigma g_n}{s_p - s_w} = 1 - \frac{\sigma g_n - s_w}{s_p - s_w} \tag{2.9}$$

Having noticed this, however, things change drastically if, in the attempt to go further in our generalization of the model, we add — either to the original model or to its extension with $s_w \neq 0$ — a more general formulation of the Phillips curve, according to which labor's bargaining power not only depends on the level of employment as in (2.1), but also on its rate of change.[9] Indeed, it is easy to show that, in this case, the positive equilibrium point becomes (locally) stable.

To prove this, let us assume that:[10]

$$\frac{\dot{w}}{w} = f\left(v, \frac{\dot{v}}{v}\right) = F(v, u) \tag{2.10}$$

where:

$$f_v > 0, f_{\hat{v}} > 0, \hat{v} = \dot{v}/v$$

so that:

$$\begin{aligned} F_v &= f_v + f_{\hat{v}} \frac{\partial \hat{v}}{\partial v} = f_v > 0 \\ F_u &= f_{\hat{v}} \frac{\partial \hat{v}}{\partial u} = -f_{\hat{v}} \frac{(s_p - s_w)}{\sigma} < 0 \end{aligned}$$

In this case the model becomes:

$$\frac{\dot{v}}{v} = \frac{s_p}{\sigma} - g_n - \frac{(s_p - s_w)}{\sigma} u \tag{2.11}$$

$$\frac{\dot{u}}{u} = F(v, u) - \alpha \tag{2.12}$$

with non-trivial equilibrium (v^{ee}, u^{ee}), where u^{ee} is still defined by (2.9), whereas v^{ee} is that value of the rate of employment for which:

$$f(v^{ee}, 0) = 0$$

[9] This is the case considered by Phillips himself, who, at the end of his 1958 contribution, writes that: "The statistical evidence in Sections II to IV above seems in general to support the hypothesis stated in Section I, that the rate of change of money wage rates can be explained by the level of unemployment and *the rate of change of unemployment.*" [44, p. 299; my emphasis]

[10] This formulation was introduced by Cugno and Montrucchio [8, pp. 97-98]. See also Sportelli [47, pp. 43-44].

The characteristic equation at (v^{ee}, u^{ee}) becomes:

$$\underbrace{\lambda^2}_{(+)} + \underbrace{\left[f_{\hat{v}}\left(v^{ee},0\right) \frac{(s_p - s_w)}{\sigma} u^{ee} \right]}_{(+)} \lambda + \underbrace{f_v\left(v^{ee},0\right) \frac{(s_p - s_w)}{\sigma} v^{ee} u^{ee}}_{(+)} = 0 \quad (2.13)$$

which, given the signs of the coefficients, has two negative roots. This proves that, *whatever the sign of its discriminant*, the movement is convergent.[11]

Thus, neither of the two extensions of the original model we have considered seems worth pursuing, the first simply "changing the numbers, but not the logic of the system", the second destroying its cyclical features.

Yet, the conclusion is drastically different if — taking account of Pasinetti's [38] criticism of Kaldor's [31] theory of growth and income distribution — we notice that a correct consideration of a positive propensity to save out of wages does not only simply imply a change in the equation for the warranted rate of growth — from $g = (1-u)/\sigma$ to (2.6) — but also *an increase in the dimensionality of the dynamical system.* The reason for this is that, given that workers save, they own a share of the capital stock and earn some profits from it. Clearly, this workers' share (or, alternately, the share owned by capitalists) is not constant, but rather varies with v and u over the cycle.

To investigate the implications of such a simple consideration is the main purpose of what follows.[12]

3 The model when workers save: A correct formulation

3.1 Derivation of the dynamical system

Using the notation introduced in the previous section, we can take account of Pasinetti's point about the implications of a positive propensity to save

[11] Indeed, this is the case considered by Lorenz [33, pp. 67-73], in his textbook on *Nonlinear Economic Dynamics and Chaotic Motion*, to show the structural instability of Goodwin's model; to prove, in other words, that the behavior of its dynamical system is very sensitive to variations in the functional structure such as a change from $f(v)$ to $F(v,u)$. We will have more to say on this at the end of the paper.

[12] In doing this, we are following very closely some of van der Ploeg's contributions, in particular van der Ploeg [42, 43]. In a sense our analysis, although based on a simpler extension of Goodwin's model, put together two van der Ploeg's insights, one taken from [42] — where the model, as in our case, becomes 3D because of the consideration of a varying workers' share of the capital stock, but the analysis is confined to local stability — and another one taken from [43] — where, with a constant workers' share of the capital stock, the increase in the dimensionality of the model, from 2 to 3, is due to a different reason (namely, to the fact that a more dynamic view of technical change is adopted, with the capital-output ratio varying with the cost of labor over the cycle), but, as in this paper, the Hopf bifurcation theorem is applied to the resulting dynamical system.

out of wages[13] by writing:

$$\begin{aligned} q &= wl + P_w + P_c & (3.1) \\ S_w &= s_w (wl + P_w) = s_w (wl + rk_w) & (3.2) \\ S_c &= s_c r k_c & (3.3) \\ r &= \frac{P}{k} = \frac{q - wl}{k} = \frac{(1-u)\, q}{k} = \frac{1-u}{\sigma} & (3.4) \end{aligned}$$

For this modified version of the model, we obtain:

$$g = \frac{s_w (wl + rk_w) + s_c r k_c}{k} = \frac{1}{\sigma} [s_w + \Delta s (1-u)\varepsilon] \quad (3.5)$$

from which:

$$\frac{\dot{v}}{v} = \left(\frac{s_w}{\sigma} - g_n\right) + \frac{\Delta s}{\sigma}\varepsilon - \frac{\Delta s}{\sigma} u\varepsilon \quad (3.6)$$

Thus, we now have:

$$\frac{\dot{u}}{u} = f(v, \frac{\dot{v}}{v}) - \alpha = F_1(v, u, \varepsilon) - \alpha \quad (3.7)$$

Then, with regard to the dynamics of the proportion of capital held by capitalists, we have:

$$\frac{\dot{\varepsilon}}{\varepsilon} = \frac{\dot{k}_c}{k_c} - \frac{\dot{k}}{k} = \frac{\dot{k}_c}{k_c} - g$$

where:

$$\frac{\dot{k}_c}{k_c} = \frac{s_c P_c}{k_c} = s_c r = s_c \frac{1-u}{\sigma}$$

so that, given (3.5):

$$\frac{\dot{\varepsilon}}{\varepsilon} = \frac{1}{\sigma} [(1-\varepsilon)\Delta s + u(-s_c + \Delta s \varepsilon)] \quad (3.8)$$

Thus, introducing the new notation:

$$\begin{aligned} \Delta s_\sigma &= \frac{\Delta s}{\sigma} \\ s_{w\sigma} &= \frac{s_w}{\sigma} \\ s_{c\sigma} &= \frac{s_c}{\sigma} \end{aligned}$$

the 3D dynamical system of this version of the model — in the variables v, u, and ε — can be written as:

$$\begin{aligned} \dot{v} &= (s_{w\sigma} - g_n + \Delta s_\sigma \varepsilon - \Delta s_\sigma u\varepsilon)\, v = \varphi_1 (v, u, \varepsilon) & (3.9) \\ \dot{u} &= [F_1(v, u, \varepsilon) - \alpha]\, u = \varphi_2 (v, u, \varepsilon) & (3.10) \\ \dot{\varepsilon} &= (\Delta s_\sigma - s_{c\sigma} u - \Delta s_\sigma \varepsilon + \Delta s_\sigma u\varepsilon)\varepsilon = \varphi_3 (v, u, \varepsilon) & (3.11) \end{aligned}$$

[13] See, for example, Pasinetti [38]. Notice, in particular, the change in notation — required in order to take account of Pasinetti's criticism — from s_p to s_c.

3.2 Singular points

Apart from the trivial singular point $(0,0,0)$, system (3.9)-(3.11) has a positive singular point (equilibrium) $(v^*, u^*, \varepsilon^*)$ such that:

$$\begin{aligned} s_{w\sigma} - g_n + \Delta s_\sigma \varepsilon^* - \Delta s_\sigma u^* \varepsilon^* &= 0 \qquad (3.12) \\ f(v^*, 0) - \alpha &= 0 \qquad (3.13) \\ \Delta s_\sigma - s_{c\sigma} u^* - \Delta s_\sigma \varepsilon^* + \Delta s_\sigma \varepsilon^* u^* &= 0 \qquad (3.14) \end{aligned}$$

From (3.12):

$$u^* = 1 - \frac{g_n - s_{w\sigma}}{\Delta s_\sigma \varepsilon^*} \qquad (3.15)$$

Inserting in (3.14) and solving for ε:

$$\varepsilon^* = \frac{s_c (g_n - s_{w\sigma})}{\Delta s g_n} \qquad (3.16)$$

Then, inserting (3.16) in (3.15):

$$u^* = 1 - \frac{\sigma g_n}{s_c} \qquad (3.17)$$

Finally, $v^* = v^{ee}$, as in the previous elaboration of the model, is that value of the employment rate that satisfies (3.13).

It is easy to check that these equilibrium values imply all steady-state results that follow from Kaldor-Pasinetti's theory of growth and income distribution; in particular:

- given (3.5), (3.16) and (3.17) guarantee a steady-state growth of the system at a warranted rate equal to the natural rate:

$$g^* = s_{w\sigma} + \Delta s_\sigma (1 - u^*) \varepsilon^* = s_{w\sigma} + \Delta s_\sigma \left(\frac{g_n}{s_{c\sigma}} \right) \frac{s_{c\sigma} (g_n - s_{w\sigma})}{\Delta s_\sigma g_n} = g_n$$

- unlike what was the case for the version of the model with differential savings we considered in the previous section, they imply the so-called *Cambridge equation*, according to which the steady-state rate of profit is determined by the natural rate of growth divided by the capitalists' propensity to save, independently of anything else:[14]

$$r^* = \frac{1 - u^*}{\sigma} = \frac{g_n}{s_c}$$

[14] See, for example, Pasinetti [39, pp. 121-122 and 127-128].

- in order to be economically meaningful (i.e., such that $0 < u^* < 1$ and $0 < \varepsilon^* < 1$), they require that:

$$0 \leq s_w < \sigma g_n < s_c \leq 1 \tag{3.18}$$

which is nothing else that the well-known condition that Kaldor's original model too must satisfy.[15]

Notice that, as already stressed, there is however a basic difference between the two approaches: whereas in the Kaldor-Pasinetti model the steady-state growth path continuously guarantees the full employment of labor, *in the Goodwin model it is characterized by a positive (constant) rate of unemployment equal to* $(1 - v^*)$. Moreover, and more importantly, it is possible to show that *Goodwin's model, in the more general case of differential savings and in the case in which condition (3.18) is satisfied, admits closed orbits solutions (limit cycles). In other words, it describes persistent fluctuations of the variables rather than a convergence to the steady-state solution.*

This can be rigorously established by applying to system (3.9)-(3.11) the Hopf Bifurcation Theorem (HBT, thereafter). As a preliminary step, we study the (local) stability of the model at the non-trivial singular point.

3.3 Local stability analysis

Linearizing system (3.9)-(3.11) at the non-trivial singular point $(v^*, u^*, \varepsilon^*)$, we obtain:

$$\begin{bmatrix} \dot{v} \\ \dot{u} \\ \dot{\varepsilon} \end{bmatrix} = \mathbf{J}^* \begin{bmatrix} v - v^* \\ u - u^* \\ \varepsilon - \varepsilon^* \end{bmatrix}$$

where the Jacobian matrix:

$$\mathbf{J}^* = \begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \tag{3.19}$$

given the basic assumptions introduced at the beginning of the previous section and condition (3.18), is such that *the signs of all its coefficients are unambiguously determined*:

$$\begin{aligned} a_{12} &= \left.\frac{\partial \varphi_1}{\partial u}\right|_{(v,u,\varepsilon)=(v^*,u^*,\varepsilon^*)} = -\Delta s_\sigma \varepsilon^* v^* < 0 \\ a_{13} &= \left.\frac{\partial \varphi_1}{\partial \varepsilon}\right|_{(v,u,\varepsilon)=(v^*,u^*,\varepsilon^*)} = \Delta s_\sigma \left(1 - u^*\right) v^* > 0 \end{aligned}$$

[15] See, for example, Pasinetti [38, p. 269].

$$a_{21} = \left.\frac{\partial \varphi_2}{\partial v}\right|_{(v,u,\varepsilon)=(v^*,u^*,\varepsilon^*)} = F^*_{1v}u^* = f^*_v u^* > 0$$

$$a_{22} = \left.\frac{\partial \varphi_2}{\partial u}\right|_{(v,u,\varepsilon)=(v^*,u^*,\varepsilon^*)} = F^*_{1u}u^* = -f^*_{\hat{v}}\Delta s_\sigma \varepsilon^* u^* < 0$$

$$a_{23} = \left.\frac{\partial \varphi_2}{\partial \varepsilon}\right|_{(v,u,\varepsilon)=(v^*,u^*,\varepsilon^*)} = F^*_{1\varepsilon}u^* = f^*_{\hat{v}}\Delta s_\sigma (1-u^*)u^* > 0$$

$$a_{32} = \left.\frac{\partial \varphi_3}{\partial u}\right|_{(v,u,\varepsilon)=(v^*,u^*,\varepsilon^*)} = (-s_{c\sigma} + \Delta s_\sigma \varepsilon^*)\varepsilon^* = -s_{c\sigma}\frac{s_{w\sigma}}{g_n}\varepsilon^* < 0$$

$$a_{33} = \left.\frac{\partial \varphi_3}{\partial \varepsilon}\right|_{(v,u,\varepsilon)=(v^*,u^*,\varepsilon^*)} = -\Delta s_\sigma (1-u^*)\varepsilon^* < 0$$

Thus, the characteristic equation of our system becomes:

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \tag{3.20}$$

where:

$$A = -\mathrm{tr}\,(\mathbf{J}^*) = -\underset{(-)}{a_{22}} - \underset{(-)}{a_{33}} > 0 \tag{3.21}$$

$$B = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= \underbrace{\underset{(-)\,(-)}{a_{22}a_{33}}}_{(+)} - \underbrace{\underset{(-)\,(+)}{a_{12}a_{21}}}_{(-)} - \underbrace{\underset{(+)\,(-)}{a_{23}a_{32}}}_{(-)} > 0 \tag{3.22}$$

$$C = -\det\,(\mathbf{J}^*) = \underbrace{\underset{(+)\,(-)\,(-)}{a_{21}a_{12}a_{33}}}_{(+)} - \underbrace{\underset{(+)\,(+)\,(-)}{a_{21}a_{13}a_{32}}}_{(-)} > 0 \tag{3.23}$$

Moreover:

$$AB - C \tag{3.24}$$

$$= \underbrace{\underbrace{(\underset{(-)}{a_{22}} + \underset{(-)}{a_{33}})}_{(-)}\underbrace{(\underbrace{\underset{(+)\,(-)}{a_{23}a_{32}}}_{(-)} - \underbrace{\underset{(-)\,(-)}{a_{22}a_{33}}}_{(+)})}_{(-)}}_{(+)} + \underbrace{\underset{(+)}{a_{21}}(\underbrace{\underset{(+)\,(-)}{a_{13}a_{32}}}_{(-)} + \underbrace{\underset{(-)\,(-)}{a_{22}a_{12}}}_{(+)})}_{(?)} \gtreqless 0$$

Given that — by (3.21), (3.22) and (3.23) — A, B, C are always positive, the implication of what we have obtained is that in the case in which condition (3.24) holds with the ">" sign, all Routh-Hurwitz (necessary and sufficient) conditions for the local stability of the equilibrium[16] are satisfied. This, as was the case for the extension of the model we have considered in section 2 (see equations (2.11)-(2.12)), would be the "end of the story" and we could conclude that the model economy (locally) converges toward $(v^*, u^*, \varepsilon^*)$. However, given that condition (3.24) can also hold with the equality sign, things are not so simple! Indeed, this implies that the dynamical behavior of the model can drastically change, from the qualitative point of view, as one or more of the parameters vary.

Our purpose is now to use the HBT to show that, choosing, for example, $s_{w\sigma}$ as the bifurcation parameter,[17] persistent cyclical behavior can emerge as $s_{w\sigma}$ varies.

3.4 Application of the Hopf Bifurcation Theorem

In the past few years, the HBT has been often utilized to prove — both in 2D and 3D continuous-time dynamical systems — the existence of closed orbits.

Restricting our attention to the applications of the HBT to 3D dynamical systems,[18] we can say that usually only the *existence part* of the theorem has been applied, the reason for this being that the *stability part* requires conditions (involving third order mixed partial derivatives) to which it is hard to give any economic interpretation. This, for example, is stressed by Asada [1, p. 48, f. 14], who writes that "...the stability of the closed orbit depends on the third order partial derivatives of the relevant functions, but the *economic implications of such stability conditions are quite ambiguous*" and by Franke [20, p. 251], who, with regard to the problem of the stability or instability of the closed orbit, stresses that "...which case holds (if any)

[16] See, for example, Gandolfo [23, pp. 221-223].

[17] Given that σ is assumed to be constant, that is tantamount to choosing the workers' propensity to save as the bifurcation parameter.

[18] This is the case — we believe — in which the HBT really adds value to the standard theory (e.g., Poincaré-Bendixson theorem), that can on the contrary be applied to show the existence of closed orbits only to 2D systems. Although we will not consider them, the interested reader, as examples of applications of the HBT to 2D dynamical systems, is referred to Torre [49], Benhabib-Nishimura [6], Medio [35], Semmler [46], Feichtinger [15], Flaschel-Groh [19], Sportelli [47] and Kind [32]. Moreover, Benhabib-Nishimura [7] and Franke-Asada [21] contain applications of the HBT to higher than 3D systems (nD and 4D respectively). There are then a number of contributions – notably by Farkas and co-workers (see, for example, Farkas-Kotsis [14]) and more recently by Fanti and Manfredi (see, for example, Fanti-Manfredi [11, 12, 13]) – in which the HBT is applied to higher dimension extensions of the Goodwin model. We notice, however, that in the latter extensions (see, for example, Fanti-Manfredi [13, pp. 383 and 385]), the higher dimensionality is the result of the consideration of lagged relations – together with the application of the so-called "linear-chain trick" – rather than of a truly more general structure of the model. Finally, with regard to applications to models formulated in discrete-time, we simply refer to Gandolfo [23, pp. 491-499] and Medio [36, pp. 67-69] and to the literature mentioned therein.

depends on the higher-order nonlinear terms in the Taylor expansion of the right-hand side (of the equations) of the dynamical system... This procedure is so complicated that *the resulting conditions for orbital stability would no longer be accessible to economic interpretation*".[19] However, what has hardly been noticed is that the simple application of the existence part of the theorem is not uninfluential for the specification of the model either. On the contrary, as shown by the contributions I have just mentioned, it often requires assumptions that are introduced for no other reason than that of satisfying the conditions of the theorem.[20]

In our application, on the contrary, this is not the case because, as shown in **Proposition 1** below, for the version of Goodwin's model we are considering, the application of the existence part of the HBT requires only that condition (3.18) — resulting from Kaldor-Pasinetti's theory of economic growth and distribution — is satisfied.

For our model — with dynamical system (3.9)-(3.11) and characteristic equation (3.20) — to prove the HBT is tantamount to proving the following proposition.[21]

Proposition 1 *When condition (3.18) is assumed, there exists a critical value of the workers' propensity to save —* $s_{w\mathrm{H}} = \sigma s_{w\sigma\mathrm{H}}$ *— that satisfies it and is such that (3.20) evaluated at* (v^*, u^*, ε) *has the following properties:*

(HB$_1$*) it possesses a pair of pure imaginary roots (*$\lambda_{2,3} = \theta \pm i\omega$*); in other words, at* $s_{w\sigma} = s_{w\sigma\mathrm{H}}$*:*

$$A(s_{w\sigma\mathrm{H}})B(s_{w\sigma\mathrm{H}}) - C(s_{w\sigma\mathrm{H}}) = 0$$

(HB$_2$*) the real parts of the complex roots cross the real axis at non-zero speed; in other words, at* $s_{w\sigma} = s_{w\sigma\mathrm{H}}$*:*

$$\left.\frac{d\theta}{ds_{w\sigma}}\right|_{s_{w\sigma}=s_{w\sigma\mathrm{H}}} \neq 0$$

Proof.

(HB$_1$) In terms of the parameters of the model, given (3.24), we can write (see Appendix A.1 for calculations):

$$AB - C = \Delta s_\sigma \varepsilon^* f_{\hat{v}}^* \left[f_{\hat{v}}^* u^* + (1-u^*)\right] \left[\Delta s_\sigma u^* s_{w\sigma} \varepsilon^* \right. \tag{3.25}$$
$$\left. + \Delta s_\sigma \varepsilon^* u^* \Delta s_\sigma (1-u^*)\varepsilon^*\right] + \Delta s_\sigma f_v^* u^* \varepsilon^* \left(-v^* s_{w\sigma} + f_{\hat{v}}^* u^* \Delta s_\sigma \varepsilon^* v^*\right) = 0$$

[19] See also Asada [2], Asada-Semmler [3], Benhabib-Miyao [5], van der Ploeg [43], Sasakura [45], Gandolfo [23, pp. 476-477].

[20] See, for example, assumption (A.8) in Sasakura [45, p. 437] or assumptions 1 and 2 in Franke [20, pp. 246 and 250].

[21] See Gandolfo [23, pp. 257-258 and 475-488].

which is satisfied by the following value of $s_{w\sigma}$ (see Appendix A.2 for calculations):

$$\begin{aligned} s_{w\sigma\mathrm{H}} &= \frac{f_{\hat{v}}^* s_{c\sigma}\left[1+(f_{\hat{v}}^*-1)u^*\right]+f_v^* f_{\hat{v}}^* u^* v^*/(1-u^*)}{f_{\hat{v}}^* s_{c\sigma}\left[1+(f_{\hat{v}}^*-1)u^*\right]+f_v^* v^*+f_v^* f_{\hat{v}}^* u^* v^*/(1-u^*)} g_n \\ &= D g_n \end{aligned} \tag{3.26}$$

The denominator of the fraction in (3.26) is certainly always greater than the numerator, so that:

$$D < 1 \rightarrow s_{w\sigma\mathrm{H}} < g_n$$

Moreover:

$$\begin{aligned} f_{\hat{v}}^* s_{c\sigma}\left[1+(f_{\hat{v}}^*-1)u^*\right] &= f_{\hat{v}}^* s_{c\sigma}\left[1+\frac{(f_{\hat{v}}^*-1)(s_{c\sigma}-g_n)}{s_{c\sigma}}\right] \\ = f_{\hat{v}}^*\left[s_{c\sigma}+(f_{\hat{v}}^*-1)(s_{c\sigma}-g_n)\right] &= f_{\hat{v}}^*\left[s_{c\sigma}+f_{\hat{v}}^* s_{c\sigma}-s_{c\sigma}-f_{\hat{v}}^* g_n+g_n\right] \\ &= f_{\hat{v}}^*\left[f_{\hat{v}}^*(s_{c\sigma}-g_n)+g_n\right] > 0 \end{aligned}$$

so that:

$$D > 0 \rightarrow s_{w\sigma\mathrm{H}} > 0$$

This completes the proof of the first part of the Proposition.

(HB_2) By using the so-called *sensitivity analysis*, it is then easy to show that the second requirement of the Proposition is also met.

First of all, we notice that the coefficients of the characteristic equation are such that (see Appendix A.3 for calculations):

$$A = a(g_n - s_{w\sigma}), a > 0 \tag{3.27}$$
$$B = b(g_n - s_{w\sigma}), b > 0 \tag{3.28}$$
$$C = c(g_n - s_{w\sigma}), c > 0 \tag{3.29}$$

so that we also have:

$$\begin{aligned} \frac{\partial A}{\partial s_{w\sigma}} &= -a = \frac{A}{s_{w\sigma}-g_n} < 0 \\ \frac{\partial B}{\partial s_{w\sigma}} &= -b = \frac{B}{s_{w\sigma}-g_n} < 0 \\ \frac{\partial C}{\partial s_{w\sigma}} &= -c = \frac{C}{s_{w\sigma}-g_n} < 0 \end{aligned}$$

Second, we know that, apart from $A > 0$,$B > 0$ and $C > 0$ that is always true, for $s_{w\sigma} = s_{w\sigma\mathrm{H}}$, one also has $AB - C = 0$. Thus, when $s_{w\sigma} = s_{w\sigma\mathrm{H}}$, one root of the characteristic equation (λ_1) is real negative, whereas the other two are a pair of pure imaginary roots ($\lambda_{2,3} = \theta \pm i\omega$, with $\theta = 0$). This means that we have:

$$\begin{aligned} A &= -(\lambda_1 + \lambda_2 + \lambda_3) = -(\lambda_1 + 2\theta) \\ B &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 2\lambda_1\theta + \theta^2 + \omega^2 \\ C &= -\lambda_1\lambda_2\lambda_3 = -\lambda_1\left(\theta^2 + \omega^2\right) \end{aligned}$$

so that, differentiating with respect to $s_{w\sigma}$, in the case in which $\lambda_{2,3}$ are purely imaginary ($\theta = 0$), we obtain:

$$\begin{aligned} \frac{\partial A}{\partial s_{w\sigma}} &= -\frac{\partial \lambda_1}{\partial s_{w\sigma}} - 2\frac{\partial \theta}{\partial s_{w\sigma}} \\ \frac{\partial B}{\partial s_{w\sigma}} &= 2\lambda_1\frac{\partial \theta}{\partial s_{w\sigma}} + 2\omega\frac{\partial \omega}{\partial s_{w\sigma}} \\ \frac{\partial C}{\partial s_{w\sigma}} &= -\omega^2\frac{\partial \lambda_1}{\partial s_{w\sigma}} - 2\lambda_1\omega\frac{\partial \omega}{\partial s_{w\sigma}} \end{aligned}$$

or:

$$\begin{bmatrix} -1 & -2 & 0 \\ 0 & 2\lambda_1 & 2\omega \\ -\omega^2 & 0 & -2\lambda_1\omega \end{bmatrix} \begin{bmatrix} \partial\lambda_1/\partial s_{w\sigma} \\ \partial\theta/\partial s_{w\sigma} \\ \partial\omega/\partial s_{w\sigma} \end{bmatrix}$$
$$= \mathbf{A} \begin{bmatrix} \partial\lambda_1/\partial s_{w\sigma} \\ \partial\theta/\partial s_{w\sigma} \\ \partial\omega/\partial s_{w\sigma} \end{bmatrix} = \begin{bmatrix} -a \\ -b \\ -c \end{bmatrix}$$

such that at $s_{w\sigma} = s_{w\sigma\mathrm{H}}$:

$$|\mathbf{A}| = 4\omega(\lambda_1^2 + \omega^2) > 0 \tag{3.30}$$

and (see Appendix A.4 for calculations):

$$|\mathbf{A}_{s_{w\sigma}}| = 2a\omega^3 > 0 \tag{3.31}$$

Given that, by Cramer's rule:

$$\frac{|\mathbf{A}_{s_{w\sigma}}|}{|\mathbf{A}|} = \left.\frac{\partial\theta}{\partial s_{w\sigma}}\right|_{s_{w\sigma}=s_{w\sigma\mathrm{H}}}$$

conditions (3.30) and (3.31) prove the second part of the Proposition. ■

Thus, system (3.9)-(3.11) admits closed orbits solutions (persistent and bounded fluctuations of the variables) for values of $s_{w\sigma}$ in the neighborhood of $s_{w\sigma\mathrm{H}}$.

Before discussing at greater length the implications of this result, it is interesting to notice that, once the possibility of closed orbits has been proved, an important property of the cyclical behavior of the model follows, consisting in the fact that the u^*- and the ε^*-coordinate of the singular point are equal to the average values of the respective variables over a whole cycle.[22]

Proposition 2 *For the dynamical system (3.9)-(3.11) of Goodwin's model with differential savings, the average values of the income share of labor and of the share of capital owned by capitalists are equal to the respective coordinates of the singular point (i.e., to* u^* *and* ε^* *respectively).*

Proof. (see Gandolfo [23, pp. 463-464] for the analogous property of the original Lotka-Volterra model)

To show this, let us rewrite equations (3.9) and (3.11) as:

$$\frac{d}{dt}\log v = s_{w\sigma} - g_n + \Delta s_\sigma \varepsilon - \Delta s_\sigma u\varepsilon \tag{3.32}$$

$$\frac{d}{dt}\log \varepsilon = \Delta s_\sigma - s_{c\sigma} u - \Delta s_\sigma \varepsilon + \Delta s_\sigma u\varepsilon \tag{3.33}$$

from which, integrating over a period T equal to the period of the oscillations:

$$\left(s_{w\sigma} - g_n\right) T + \Delta s_\sigma \int_0^T \varepsilon dt - \Delta s_\sigma \int_0^T u\varepsilon dt = 0 \tag{3.34}$$

$$\Delta s_\sigma T - s_{c\sigma} \int_0^T u dt - \Delta s_\sigma \int_0^T \varepsilon + \Delta s_\sigma \int_0^T u\varepsilon dt = 0 \tag{3.35}$$

From (3.34)-(3.35):

$$\left(s_{w\sigma} - g_n\right) T + \Delta s_\sigma \int_0^T \varepsilon dt = -\Delta s_\sigma T + s_{c\sigma} \int_0^T u dt + \Delta s_\sigma \int_0^T \varepsilon$$

from which, in average over a cycle:

$$\frac{1}{T}\int_0^T u dt = \frac{\left[\Delta s_\sigma + \left(s_{w\sigma} - g_n\right)\right]}{s_{c\sigma}} = 1 - \frac{g_n}{s_{c\sigma}} = u^*$$

Then, from (3.34):

$$\left(s_{w\sigma} - g_n\right) T + \Delta s_\sigma \int_0^T \varepsilon \left(1 - u\right) dt = 0$$

Thus, in average over a cycle:

$$\frac{1}{T}\int_0^T \varepsilon dt = \frac{s_{c\sigma}\left(g_n - s_{w\sigma}\right)}{\Delta s_\sigma g_n} = \varepsilon^*$$

[22] As is well known, this is a property that is also satisfied by the original model. With regard to v^*, it is not possible to draw an analogous conclusion because we have not specified a functional form for the generalized Phillips curve.

This completes the proof of the Proposition. ■

Moving on to the economic interpretation of the result contained in **Proposition 1**, we can say that the bifurcational approach we have used in this paper seems to be very useful in that it has allowed us to go beyond the conventional steady state results one easily obtains from local stability analysis.[23] Although for a low workers' propensity to save — less than the critical value $s_{w\sigma\mathrm{H}}$ and in the limit equal to zero — the positive steady-state solution of the model is (locally) stable,[24] increasing such a propensity destabilizes the model. However, this does not lead to an unrealistic situation with fluctuations of ever increasing amplitude, neither does it require the use of the "saddle-point trick". Rather, through an Hopf bifurcation, the result seems to be that of persistent and bounded fluctuations of the employment rate, the income share of wages and the proportion of capital held by workers and capitalists. Having proved only the existence part of the HBT, however, we do not know whether the resulting closed orbits will be stable (*super-critical*) or unstable (*sub-critical*).

This does not appear to be too big a problem, though, for at least two reasons. First, as in a number of previous contributions,[25] following Benhabib and Miyao [5, p. 593], one can say that this does not matter too much because both cases are interesting and open to clear-cut economic interpretation. Second, and — we believe — more importantly, it is possible to say that which case prevails does not matter too much if the bifurcation from the stable steady-state growth to closed orbits around it is seen as only the first step in a route to more complex behavior rather than as the "goal" of the analysis. To investigate a bit longer the meaning of this second claim is the main purpose of the next section.

4 Numerical simulations

Traditionally, limit cycle behavior in extensions of Goodwin's growth cycle model has been the result of the quest for *structural stability*. Examples of this approach can be found in Medio [34, pp. 40-53], Funke [22] and van der Ploeg [43], to mention only a few. Rather than going into the details of the discussion of all the problems associated with this important property of dynamical systems, we prefer to take a different point of view and to stress that both the original Goodwin model and the extensions just mentioned (*including ours!*) are — in a sense — unsatisfactory. The reason

[23] On this point, see also Benhabib-Miyao [5, p. 591].

[24] From the calculations given in Appendix A.2, it follows that $AB - C \gtreqless 0$ as $s_{w\sigma} \lesseqgtr s_{w\sigma\mathrm{H}}$, so that for values of $s_{w\sigma}$ less than the critical value all Routh-Hurwitz conditions for the local stability of the singular point are satisfied.

[25] See, for example, Cugno-Montrucchio [9, p. 6], van der Ploeg [43, pp. 9-10] and Franke [20, p. 251].

for this is that they all imply fluctuations of the variables of the model that are of a *periodic nature*.[26] On the contrary, *it is a fact* that, although bounded, the economic fluctuations observed in the real world are highly irregular (aperiodic). These two observations, however, are reconciled when it happens that the Hopf bifurcation, the existence of which we have just proved, turns out to be *the first step in the route of the system from regularity to a chaotic regime*.

To end this paper, we try now to give an idea of how this may happen by resorting to numerical simulations. In doing this, our purpose is first to show that the results we have obtained in the qualitative analysis of the model are confirmed by the numerical simulations; and, second — in the attempt to understand which "route to chaos" prevails, if any — to see what happens when the parameter is further increased.[27]

First of all, we need to choose a functional form for $F_1(v, u, \varepsilon) = f(v, \hat{v})$. In the simplest case in which f is additive and linear in both variables, we can write:

$$f\left(v, \frac{\dot{v}}{v}\right) = -\gamma + \rho v + \delta \left[s_{w\sigma} - g_n + \Delta s_\sigma (1-u)\varepsilon\right] \tag{3.36}$$

which is such that:

$$v^* = \frac{\alpha + \gamma}{\rho} \tag{3.37}$$

Then, an aid for choosing plausible values of the parameter to be used in the simulations is given by a recent contribution by Harvie [29], where the author applies Goodwin's model to all major OECD countries. Choosing, for example, his results for the UK economy, the parameter estimates are the following [29, p. 24, Table 6]:

$$\begin{array}{ll} \alpha = 0.0221 & \beta = 0.0037 \\ \sigma = 2.57 & \gamma = 18.54 \\ \rho = 21.90 & \end{array}$$

From these values and condition (3.18) above, it also follows that the two propensities to save must satisfy:

$$s_w < 0.0663606 < s_c$$

[26] In the version of the model we have considered, for example, we know that the result of the Hopf bifurcation are closed orbits of period approximately equal to $2\pi/\sqrt{B}$.

[27] For a discussion of the concept of "bifurcation" and of typical "routes to chaos" — the so-called *scenarios* — see, for example, Medio [36, pp. 59-69 and 149-177, respectevely]. There (p. 68), the conditions for the *Hopf (or Neimark) bifurcation for maps* are also given. In the case in which the map is nothing else than the *Poincaré map* of a higher order continuous-time dynamical system, the Hopf bifurcation is the bifurcation (or loss of stability) of the limit cycle of the corresponding flow. For this reason, this bifurcation is often referred to as a *secondary Hopf bifurcation*.

s_w	v^*	u^*	ε^*
0.001	0.847584	0.889490	0.986563
0.002	0.847584	0.889490	0.973080
0.003	0.847584	0.889490	0.959553
0.004	0.847584	0.889490	0.945980
0.005	0.847584	0.889490	0.932362
0.006	0.847584	0.889490	0.918697
0.007	0.847584	0.889490	0.904987
0.008	0.847584	0.889490	0.891230
0.009	0.847584	0.889490	0.877427
0.0091	0.847584	0.889490	0.876044
0.00919	0.847584	0.889490	0.874799
0.009195	0.847584	0.889490	0.874730

Table 1: Convergence to the steady-state for values of the workers' propensity to save less than the critical value $s_{w\mathrm{H}}$

Finally, after having chosen a value for the parameter δ $(= f_{\hat{v}})$ — for example $\delta = 0.02$ — and for the capitalists' propensity to save — for example $s_c = 0.6$ — it is possible to use (3.26) to calculate the critical value of the workers' propensity to save, for which the Hopf bifurcation occurs. Doing this, we obtain:

$$s_{w\sigma\mathrm{H}} \approx 0.0035780$$

or:

$$s_{w\mathrm{H}} \approx 0.0091954 \tag{3.38}$$

Using these values for the parameters, the numerical simulations[28] confirm the qualitative results we have obtained in the previous section; in particular:

- for values of s_w less than the critical value (3.38), the system converges to the steady-state, with the values for v^*, u^* and ε^* shown in Table 1. As can easily be checked, these values are the same as those resulting from the qualitative analysis of the model and given by (3.16), (3.17) and (3.37). In particular, we notice that, as was to be expected, only the steady-state value of ε proves to depend on s_w;[29]

[28] For some information about the simulations, see Appendix B below.

[29] From the simulations, it also follows that the convergence to $(v^*, u^*, \varepsilon^*)$ is very slow (*the slower, the larger is* s_w) and *cyclical*.

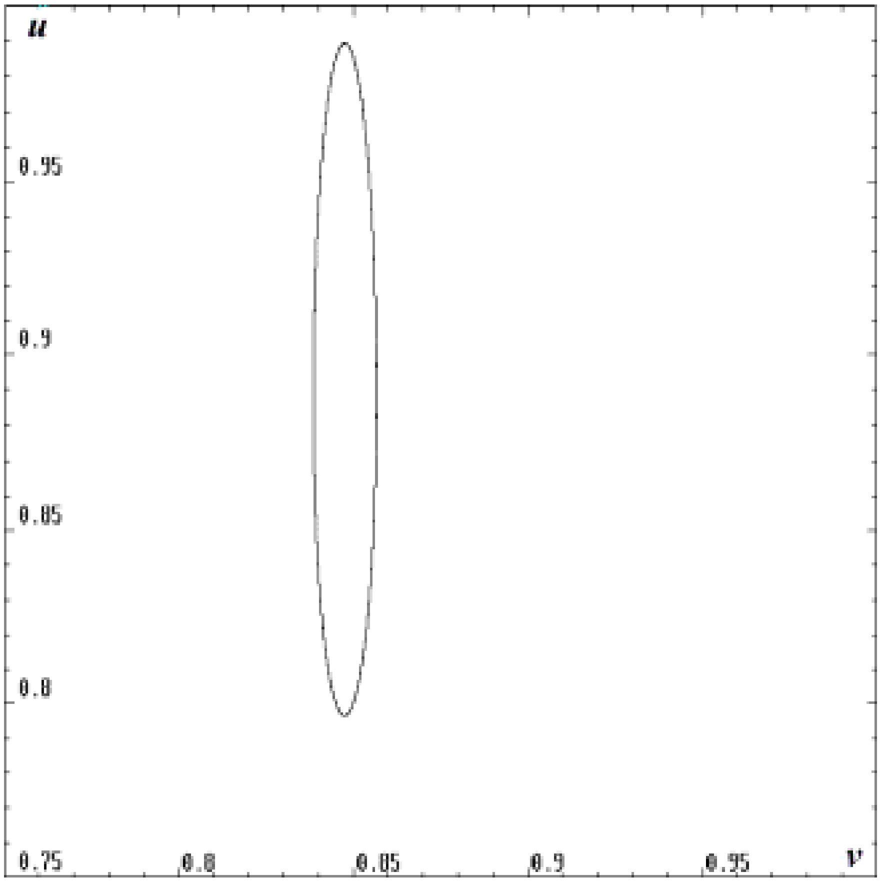

Figure 1: Limit cycle in the (v, u)-plane, with $s_w = 0.0092$

- for $s_w = 0.0092$, the numerical simulation given in Figure 1 confirms the implications of **Proposition 1**, namely the existence of persistent and bounded fluctuations of the variables.

In addition to this, the simulations also show that as s_w is further increased *new phenomena* appear. In particular, the bifurcation diagram given in Figure 2 suggests that, for higher values of the parameter, the original Hopf bifurcation is followed by other bifurcations in a route to *chaos*.

To confirm this result we have calculated the dominant Lyapunov exponent of our dynamical system for different values of s_w. The resulting Lyapunov exponent bifurcation diagram given in Figure 3 shows that, for values of the workers' propensity to save greater than a given critical value — approximately equal to 0.0095 — the Lyapunov exponent is positive, and therefore the system possesses sensitive dependence on initial conditions.

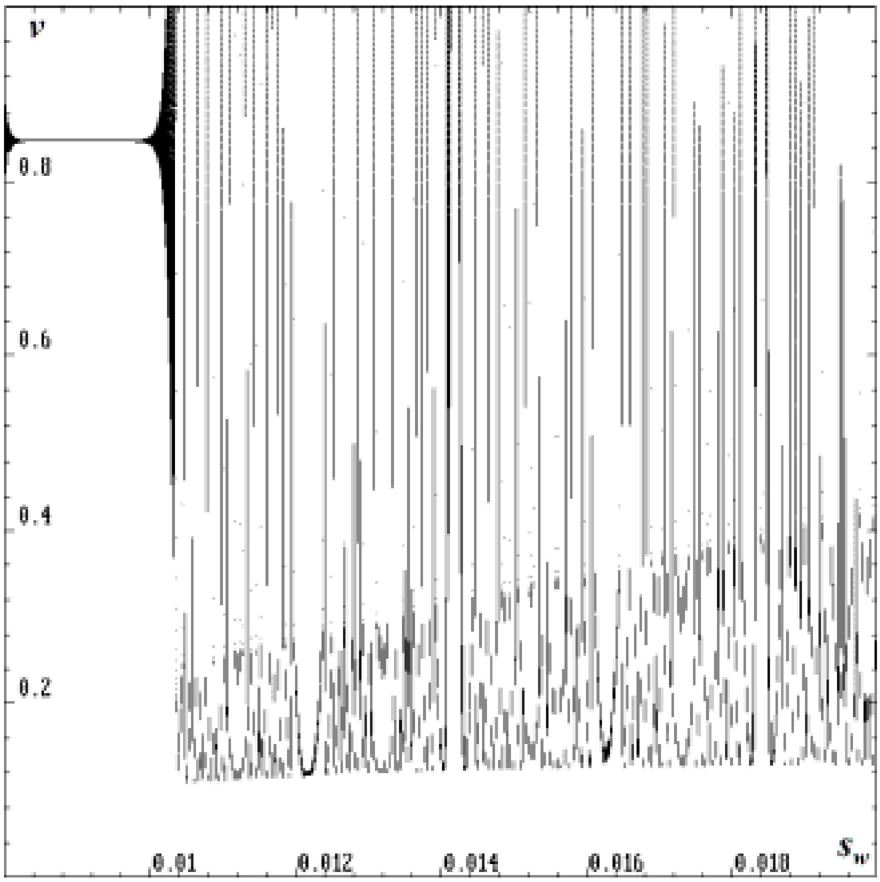

Figure 2: Bifurcation diagram for the variable v, with s_w in the range $(0.008, 0.02)$

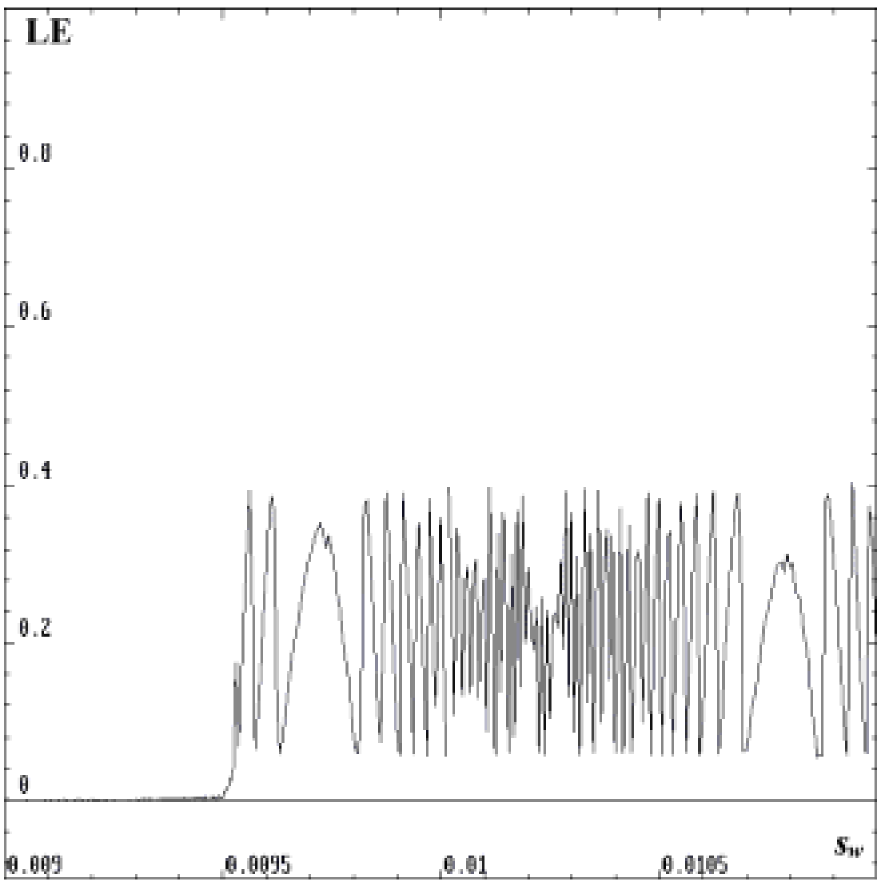

Figure 3: Lyapunov exponent bifurcation diagram with s_w in the range $(0.009, 0.011)$

5 Concluding remarks

In this paper, an extension of Goodwin's growth cycle model has been set out that considers the case of differential savings along Kaldorian-Pasinettian lines. Notwithstanding its simplicity, the exercise we have performed allows us to draw some interesting and encouraging conclusions, worthy, in our opinion, of further investigation.

In summary:

1. The extension of the model we have considered attempts to integrate Kaldorian and Goodwinian elements. From this point of view, the result we have obtained is that the steady-state features of the modified Goodwin model are the same as those of Pasinetti's version of Kaldor's model. However, interestingly enough, along the equilibrium growth path of our model there is not full employment but, rather, a constant rate of unemployment. Moreover, for values of the workers' propensity to save above a certain critical value, the system — either periodically or aperiodically — persistently fluctuates around it, rather than converging to it.

2. The existence part of the HBT we have employed for the qualitative analysis of the dynamics of the model is a powerful tool, by now standard in economic dynamics. Yet, there are *three* aspects of our application that are worth stressing. *First*, as we have seen, our application does not require the introduction of any *ad hoc* assumption about the values of the parameters, apart from the one about the relative values of the two propensities to save implied by Kaldor-Pasinetti's theory: in our model this condition is all that is needed to ensure both that the steady-state values of the variables are economically significative and that the conditions of the HBT are satisfied. *Second*, the results we have obtained strongly suggest the importance of "going from local to global analysis". From this point of view, the limit cycle — the existence of which we have proved by using the HBT — must be seen as the starting point rather than as the "goal" of the analysis. Indeed, more important then is, as we have tried to do with the numerical simulation, to investigate the existence of a "route to chaotic behavior" as the crucial parameter is further increased. *Third*, we have introduced the hypothesis of differential savings in the original formulation of the model. In doing this, we have neglected a number of interesting extensions of the model — available in the existing literature on the topic — which give rise to a higher order dynamical system, for example all those extensions in which the Phillips curve is written in monetary terms and then an equation for price dynamics is introduced into the model. However, as is well known — and as is testified to by some of the recent contributions quoted above — the existence part of the

HBT can be easily applied to higher than 3D systems as well, and the same, clearly, is true for the many numerical simulation techniques nowadays easily applicable even with an ordinary personal computer. For this reason, then, there are no limits to the analysis of further generalizations of the model along the lines suggested and sketched in this paper.

A Appendix: Basic calculations

A.1 Derivation of (3.25)

$$AB - C = (a_{22} + a_{33})(a_{23}a_{32} - a_{22}a_{33}) + a_{21}(a_{13}a_{32} + a_{22}a_{12})$$

$$= [-f_{\hat{v}}^{*}\Delta s_{\sigma}\varepsilon^{*}u^{*} - \Delta s_{\sigma}(1-u^{*})\varepsilon^{*}]\,[f_{\hat{v}}^{*}\Delta s_{\sigma}(1-u^{*})u^{*}(-\frac{s_{c\sigma}}{g_n}s_{w\sigma})\varepsilon^{*}$$
$$-f_{\hat{v}}^{*}\Delta s_{\sigma}\varepsilon^{*}u^{*}\Delta s_{\sigma}(1-u^{*})\varepsilon^{*}] + f_{v}^{*}u^{*}[\Delta s_{\sigma}\,(1-u^{*})\,v^{*}(-\frac{s_{c\sigma}}{g_n}s_{w\sigma})\varepsilon^{*}$$
$$+f_{\hat{v}}^{*}\Delta s_{\sigma}\varepsilon^{*}u^{*}\Delta s_{\sigma}\varepsilon^{*}v^{*}]$$

$$= \Delta s_{\sigma}\varepsilon^{*}f_{\hat{v}}^{*}\,[f_{\hat{v}}^{*}u^{*} + (1-u^{*})]\,[\Delta s_{\sigma}u^{*}s_{w\sigma}\varepsilon^{*} + \Delta s_{\sigma}\varepsilon^{*}u^{*}\Delta s_{\sigma}(1-u^{*})\varepsilon^{*}]$$
$$+f_{v}^{*}u^{*}\,(-\Delta s_{\sigma}v^{*}s_{w\sigma}\varepsilon^{*} + f_{\hat{v}}^{*}\Delta s_{\sigma}\varepsilon^{*}u^{*}\Delta s_{\sigma}\varepsilon^{*}v^{*})$$

$$= \Delta s_{\sigma}f_{\hat{v}}^{*}\varepsilon^{*}\,[f_{\hat{v}}^{*}u^{*} + (1-u^{*})]\,[\Delta s_{\sigma}s_{w\sigma}u^{*}\varepsilon^{*} + \Delta s_{\sigma}u^{*}\varepsilon^{*}\Delta s_{\sigma}(1-u^{*})\varepsilon^{*}]$$
$$+f_{v}^{*}\Delta s_{\sigma}u^{*}\varepsilon^{*}\,(-v^{*}s_{w\sigma} + f_{\hat{v}}^{*}u^{*}\Delta s_{\sigma}\varepsilon^{*}v^{*})$$

A.2 Derivation of (3.26)

$$AB - C \gtreqless 0 \leftrightarrow \Delta s_{\sigma}\varepsilon^{*}f_{\hat{v}}^{*}\,[f_{\hat{v}}^{*}u^{*} + (1-u^{*})]\,[\Delta s_{\sigma}u^{*}s_{w\sigma}\varepsilon^{*}$$
$$+\Delta s_{\sigma}\varepsilon^{*}u^{*}\Delta s_{\sigma}(1-u^{*})\varepsilon^{*}] + f_{v}^{*}u^{*}\varepsilon^{*}\Delta s_{\sigma}\,(-v^{*}s_{w\sigma} + f_{\hat{v}}^{*}\Delta s_{\sigma}\varepsilon^{*}u^{*}v^{*}) \gtreqless 0$$

from which:

$$f_{\hat{v}}^{*}\,[f_{\hat{v}}^{*}u^{*} + (1-u^{*})]\,[\Delta s_{\sigma}u^{*}s_{w\sigma}\varepsilon^{*} + \Delta s_{\sigma}\varepsilon^{*}u^{*}\Delta s_{\sigma}(1-u^{*})\varepsilon^{*}]$$
$$+f_{v}^{*}u^{*}[-v^{*}s_{w\sigma} + f_{\hat{v}}^{*}\Delta s_{\sigma}\varepsilon^{*}u^{*}v^{*}] \gtreqless 0$$

or:

$$f_{\hat{v}}^{*}\,[f_{\hat{v}}^{*}u^{*} + (1-u^{*})]\,[\Delta s_{\sigma}\left(\frac{s_{c\sigma} - g_n}{s_{c\sigma}}\right)s_{w\sigma}\frac{s_{c\sigma}(g_n - s_{w\sigma})}{\Delta s_{\sigma}g_n}$$
$$+\Delta s_{\sigma}\frac{s_{c\sigma}(g_n - s_{w\sigma})}{\Delta s_{\sigma}g_n}\left(\frac{s_{c\sigma} - g_n}{s_{c\sigma}}\right)\Delta s_{\sigma}(1-u^{*})\frac{s_{c\sigma}(g_n - s_{w\sigma})}{\Delta s_{\sigma}g_n}]$$
$$+f_{v}^{*}u^{*}[-v^{*}s_{w\sigma} + f_{\hat{v}}^{*}\Delta s_{\sigma}\frac{s_{c\sigma}(g_n - s_{w\sigma})}{\Delta s_{\sigma}g_n}\left(\frac{s_{c\sigma} - g_n}{s_{c\sigma}}\right)v^{*}] \gtreqless 0$$

or:

$$f_{\hat{v}}^{*}\left[f_{\hat{v}}^{*}u^{*}+(1-u^{*})\right]\frac{(g_n-s_{w\sigma})}{g_n}[(s_{c\sigma}-g_n)\,s_{w\sigma}+(s_{c\sigma}-g_n)\,g_n$$
$$-(s_{c\sigma}-g_n)\,s_{w\sigma}]+f_{v}^{*}u^{*}[-v^{*}s_{w\sigma}+f_{\hat{v}}^{*}\frac{(g_n-s_{w\sigma})}{g_n}(s_{c\sigma}-g_n)\,v^{*}]\gtreqqless 0$$

or:

$$f_{\hat{v}}^{*}\left[1+(f_{\hat{v}}^{*}-1)u^{*}\right](s_{c\sigma}-g_n)\,(g_n-s_{w\sigma})$$
$$+f_{v}^{*}u^{*}[-v^{*}s_{w\sigma}+f_{\hat{v}}^{*}\frac{(g_n-s_{w\sigma})}{g_n}(s_{c\sigma}-g_n)\,v^{*}]\gtreqqless 0$$

or:

$$f_{\hat{v}}^{*}\left[1+(f_{\hat{v}}^{*}-1)u^{*}\right]s_{c\sigma}u^{*}(g_n-s_{w\sigma})$$
$$+f_{v}^{*}u^{*}[-v^{*}s_{w\sigma}+f_{\hat{v}}^{*}\frac{(g_n-s_{w\sigma})}{g_n}(s_{c\sigma}-g_n)\,v^{*}]\gtreqqless 0$$

or:

$$f_{\hat{v}}^{*}s_{c\sigma}\left[1+(f_{\hat{v}}^{*}-1)u^{*}\right](g_n-s_{w\sigma})$$
$$+f_{v}^{*}[-v^{*}s_{w\sigma}+f_{\hat{v}}^{*}\left(1-\frac{s_{w\sigma}}{g_n}\right)s_{c\sigma}\left(1-\frac{g_n}{s_{c\sigma}}\right)v^{*}]\gtreqqless 0$$

or:

$$f_{\hat{v}}^{*}s_{c\sigma}\left[1+(f_{\hat{v}}^{*}-1)u^{*}\right](g_n-s_{w\sigma})$$
$$+f_{v}^{*}[-v^{*}s_{w\sigma}+f_{\hat{v}}^{*}\left(1-\frac{s_{w\sigma}}{g_n}\right)s_{c\sigma}u^{*}v^{*}]\gtreqqless 0$$

or:

$$f_{\hat{v}}^{*}s_{c\sigma}\left[1+(f_{\hat{v}}^{*}-1)u^{*}\right]g_n-f_{\hat{v}}^{*}s_{c\sigma}\left[1+(f_{\hat{v}}^{*}-1)u^{*}\right]s_{w\sigma}$$
$$-f_{v}^{*}v^{*}s_{w\sigma}+f_{v}^{*}f_{\hat{v}}^{*}s_{c\sigma}u^{*}v^{*}-f_{v}^{*}f_{\hat{v}}^{*}\frac{s_{w\sigma}}{g_n}s_{c\sigma}u^{*}v^{*}\gtreqqless 0$$

or:

$$\left\{f_{\hat{v}}^{*}s_{c\sigma}\left[1+(f_{\hat{v}}^{*}-1)u^{*}\right]+f_{v}^{*}v^{*}+f_{v}^{*}f_{\hat{v}}^{*}\frac{s_{c\sigma}}{g_n}u^{*}v^{*}\right\}s_{w\sigma}$$
$$\lesseqqgtr\left\{f_{\hat{v}}^{*}s_{c\sigma}\left[1+(f_{\hat{v}}^{*}-1)u^{*}\right]+f_{v}^{*}f_{\hat{v}}^{*}\frac{s_{c\sigma}}{g_n}u^{*}v^{*}\right\}g_n$$

Thus:

$$AB-C\gtreqqless 0$$

according as to whether:

$$s_{w\sigma\mathrm{H}}\lesseqqgtr\frac{f_{\hat{v}}^{*}s_{c\sigma}\left[1+(f_{\hat{v}}^{*}-1)u^{*}\right]+f_{v}^{*}f_{\hat{v}}^{*}u^{*}v^{*}/(1-u^{*})}{f_{\hat{v}}^{*}s_{c\sigma}\left[1+(f_{\hat{v}}^{*}-1)u^{*}\right]+f_{v}^{*}v^{*}+f_{v}^{*}f_{\hat{v}}^{*}u^{*}v^{*}/(1-u^{*})}g_n$$

A.3 Derivation of (3.27), (3.28) and (3.29)

$$A = -a_{22} - a_{33} = f_{\hat{v}}^* \Delta s_\sigma \varepsilon^* u^* + \Delta s_\sigma (1-u^*)\varepsilon^*$$

$$\begin{aligned} &= f_{\hat{v}}^* \Delta s_\sigma \frac{s_{c\sigma}(g_n - s_{w\sigma})}{\Delta s_\sigma g_n} u^* + \Delta s_\sigma (1-u^*) \frac{s_{c\sigma}(g_n - s_{w\sigma})}{\Delta s_\sigma g_n} \\ &= f_{\hat{v}}^* \frac{s_{c\sigma}(g_n - s_{w\sigma})}{g_n} u^* + (1-u^*) \frac{s_{c\sigma}(g_n - s_{w\sigma})}{g_n} \\ &= \left[f_{\hat{v}}^* \frac{s_{c\sigma}}{g_n} u^* + (1-u^*) \frac{s_{c\sigma}}{g_n} \right] (g_n - s_{w\sigma}) \end{aligned}$$

$$= \underbrace{\left(\frac{f_{\hat{v}}^* u^*}{1-u^*} + 1 \right)}_{a>0} (g_n - s_{w\sigma}) = a(g_n - s_{w\sigma})$$

$$B = a_{22}a_{33} - a_{12}a_{21} - a_{23}a_{32}$$

$$\begin{aligned} &= f_{\hat{v}}^* \Delta s_\sigma \varepsilon^* u^* \Delta s_\sigma (1-u^*) \varepsilon^* + \Delta s_\sigma \varepsilon^* v^* f_v^* u^* + f_{\hat{v}}^* \Delta s_\sigma (1-u^*) u^* \frac{s_{c\sigma}}{g_n} s_{w\sigma} \varepsilon^* \\ &= f_{\hat{v}}^* \Delta s_\sigma \frac{s_{c\sigma}(g_n - s_{w\sigma})}{\Delta s_\sigma g_n} u^* \Delta s_\sigma (1-u^*) \frac{s_{c\sigma}(g_n - s_{w\sigma})}{\Delta s_\sigma g_n} \\ &+ \Delta s_\sigma \frac{s_{c\sigma}(g_n - s_{w\sigma})}{\Delta s_\sigma g_n} v^* f_v^* u^* + f_{\hat{v}}^* \Delta s_\sigma (1-u^*) u^* \frac{s_{c\sigma}}{g_n} s_{w\sigma} \frac{s_{c\sigma}(g_n - s_{w\sigma})}{\Delta s_\sigma g_n} \\ &= f_{\hat{v}}^* (g_n - s_{w\sigma}) \frac{u^*}{1-u^*} (g_n - s_{w\sigma}) + (g_n - s_{w\sigma}) v^* f_v^* \frac{u^*}{1-u^*} \\ &+ f_{\hat{v}}^* \frac{u^*}{1-u^*} s_{w\sigma} (g_n - s_{w\sigma}) \\ &= \frac{u^*}{1-u^*} (g_n - s_{w\sigma}) \left[f_{\hat{v}}^* (g_n - s_{w\sigma}) + v^* f_v^* + f_{\hat{v}}^* s_{w\sigma} \right] \end{aligned}$$

$$= \underbrace{\frac{u^*}{1-u^*} \left(f_{\hat{v}}^* g_n + v^* f_v^* \right)}_{b>0} (g_n - s_{w\sigma}) = b(g_n - s_{w\sigma})$$

$$C = a_{21}a_{12}a_{33} - a_{21}a_{13}a_{32}$$

$$
\begin{aligned}
&= f_v^* u^* \Delta s_\sigma \varepsilon^* v^* \Delta s_\sigma (1-u^*)\varepsilon^* + f_v^* u^* \Delta s_\sigma \left(1-u^*\right) v^* \frac{s_{c\sigma}}{g_n} s_{w\sigma} \varepsilon^* \\
&= f_v^* u^* \Delta s_\sigma \frac{s_{c\sigma}(g_n - s_{w\sigma})}{\Delta s_\sigma g_n} v^* \Delta s_\sigma (1-u^*) \frac{s_{c\sigma}(g_n - s_{w\sigma})}{\Delta s_\sigma g_n} \\
&\quad + f_v^* u^* \Delta s_\sigma \left(1-u^*\right) v^* \frac{s_{c\sigma}}{g_n} s_{w\sigma} \frac{s_{c\sigma}(g_n - s_{w\sigma})}{\Delta s_\sigma g_n} \\
&= f_v^* u^* (g_n - s_{w\sigma}) v^* \frac{s_{c\sigma}(g_n - s_{w\sigma})}{g_n} + f_v^* u^* v^* s_{w\sigma} \frac{s_{c\sigma}(g_n - s_{w\sigma})}{g_n} \\
&= f_v^* u^* v^* g_n \frac{s_{c\sigma}(g_n - s_{w\sigma})}{g_n} - f_v^* u^* v^* s_{w\sigma} \frac{s_{c\sigma}(g_n - s_{w\sigma})}{g_n} \\
&\quad + f_v^* u^* v^* s_{w\sigma} \frac{s_{c\sigma}(g_n - s_{w\sigma})}{g_n} \\
&= \underbrace{f_v^* u^* v^* s_{c\sigma}}_{c>0} (g_n - s_{w\sigma}) = c(g_n - s_{w\sigma})
\end{aligned}
$$

A.4 Derivation of (3.31)

$$
\begin{aligned}
|\mathbf{A}_{s_{w\sigma}}| &= \begin{vmatrix} -1 & -a & 0 \\ 0 & -b & 2\omega \\ -\omega^2 & -c & -2\lambda_1\omega \end{vmatrix} \\
&= -\begin{vmatrix} -b & 2\omega \\ -c & -2\lambda_1\omega \end{vmatrix} + a \begin{vmatrix} 0 & 2\omega \\ -\omega^2 & -2\lambda_1\omega \end{vmatrix} \\
&= -\left(2b\omega\lambda_1 + 2\omega c\right) + 2a\omega^3
\end{aligned}
$$

At $s_{w\sigma} = s_{w\sigma\mathrm{H}}$:

$$
AB - C = 0, \theta = 0, A = -\lambda_1, B = \omega^2
$$

Thus:

$$
\begin{aligned}
|\mathbf{A}_{s_{w\sigma}}| &= -\left(2b\omega\lambda_1 + 2\omega c\right) + 2a\omega^3 \\
&= -2\omega(-bA + c) + 2a\omega^3 \\
&= -2\omega\left(\frac{B}{s_{w\sigma} - g_n} A - \frac{C}{s_{w\sigma} - g_n}\right) + 2a\omega^3 \\
&= \frac{2\omega}{g_n - s_{w\sigma}} (\underbrace{AB - C}_{=0!}) + 2a\omega^3 = 2a\omega^3 > 0
\end{aligned}
$$

B Appendix: Simulations

To perform the numerical simulation, we have used the computer program *DYNAMICS* 2, contained in the second edition of the book by Nusse and

York [37]. To do this, we had, first of all, to add our own process following the procedure described in Ch. 13 ("Adding your own process to *DYNAMICS"*) of the book [37, pp. 491-518]. The resulting file (with extension ".dd"), which stores our model, is the following:

OWN /* process defined below; 1 for map and 0 for Diff. Eqn.: */
0
"GOODWIN 1967 WITH KALDOR-PASINETTI DIFFERENTIAL SAVINGS
GKP model
X'=((c1/c2)-c3+((c4-c1)/c2)*Z*(1-Y))*X
Y'=(-(c5+c6)+c7*X+c8*((c1/c2)-c3+((c4-c1)/c2)*Z*(1-Y)))*Y
Z'=(((c4-c1)/c2)-(c4/c2)*Y-((c4-c1)/c2)*Z*(1-Y))*Z
"

"s' := 1 ! this is time
x' := ((c1/c2)-c3+((c4-c1)/c2)*z*(1-y))*x
y' := (-(c5+c6)+c7*x+c8*((c1/c2)-c3+((c4-c1)/c2)*z*(1-y)))*y
z' := (((c4-c1)/c2)-(c4/c2)*y-((c4-c1)/c2)*z*(1-y))*z"

"t := 0 x := .5 y := .5 z := .5
X_upper := 1; X_lower := 0; Y_upper := 1; Y_lower := 0; Z_upper := 1; Z_lower := 0
c1 ($\equiv s_w$)
c2 ($\equiv \sigma$)
c3 ($\equiv g_n$)
c4 ($\equiv s_c$)
c5 ($\equiv \gamma$)
c6 ($\equiv \alpha$)
c7 ($\equiv \rho$)
c8 ($\equiv \delta$)
step := .01"

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