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Market Equilibrium and Stability

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1 Introduction

This treatment of general equilibrium maintains the view, also held by Schumpeter, that the topic has essentially to do with aggregates and any modelling of underlying elements, in particular anything to do with utility, is secondary if not spurious. Familiarity here extends to the basic model, with excess supply depending on prices to the extent of their ratios, which determine *exchange rates*, joined with *continuity*, and subject to *Walras Law* expressing the aggregate budget constraint, that what is bought is paid for out of what is sold. An additional assumption is *Finite Supply Capacity*, that the excess supply for any good is bounded from above. This could come from physical limitation of production possibilities.

Then there is the *Intercept Condition*, that demand for a good must exceed supply if the price is small, or that demand overtakes supply of a good as its price falls, to make excess supply negative. This is consistent with the principle that the price of a good is an incentive to sell and a disincentive to buy, so that as it falls supply decreases and demand increases. The assumptions now are enough for the *existence of equilibrium prices*. The proof uses the KKM Lemma dealt with in an appendix.

One further assumption provides for *global stability* under a differential price adjustment process, and hence *uniqueness* of the equilibrium. This is the *Slope Condition*, which itself *implies the Intercept Condition* which suffices for the existence of an equilibrium. It requires that excess supply of a good be decreasing as a function of the other prices. The sense is that there is incentive to buy more and sell less of a good if the price of another good increases. Behind this is the idea that the less bought of the other good, as by the already mentioned incentive principle, is offset by more bought of the first; and similarly with the supply side.

The method of proof involves algebra of distribution matrices, given in the appendix. These occur as transition matrices in Markov process theory, where stability depends on convergence of infinite powers, whereas here the issue is for infinite products.

2 Markets

The market is a distinct general phenomenon, present in all kinds of circumstances. To understand it as such and as being unconditioned by very special circumstances, it should in some way be cut off for a separate treatment. Though one may then be in an imaginary world, demonstrating the market phenomenon in a well-defined, abstract world—one we know well because we invented it—should go some way towards providing an explanation.

The occurence of markets is completely commonplace, it is outstanding no planning committee or other agency is required to take part, they can just happen, almost anywhere. The conditions to be considered therefore should be rather primitive. For having a theory, some such conditions have to be explored.

The possibility of a market cannot be isolated from the process of original realization. An existing possible arrangement is tantalizing in the absence of a way of getting to it, from some, and perhaps any position. The *tâtonnement* or tentative groping process of Walras, or something like it, is therefore an essential part of the matter. The pure existence possibility has to be joined with some operational finding process. That requires a kind of roughness, overriding disturbances, so as to embody the stability of the self-regulated system without which it could not in practice exist.

The considerations are necessarily abstract, intended only to reflect something of the form of common experience. A simplest representation therefore could be of adequate value. This is not micro-economic theory that pretends to offers a picture of the system built from ultimate parts, but more on the surface.

That a good has a price P is demonstrated when a transaction takes place, in which it has that price. There is no real alternative sense in which a good can have a price. It means nothing that cabbages are $1 per lb if no one, or rather no two, are buying and selling at that price. Goods left on the shelf with aprice tag on them, like many a house for sale, do not have a price until they are sold. Those tags are at most offers, or just factors in the *tâtonnement*. There are countless other goods on the shelf, some of them not yet invented.

In the transaction there is a buyer and a seller, and, even though two are present, what is bought and what is sold are the same thing. There seems no escape from that. In other words, *supply* S equals *demand* D : $S = D$. This is not an equation with content. It just connects two names that have been given to the same thing. But why give the same object two different names? It would be more logical to give it one, say the *transaction quantity* Q.

We have a transaction price P and quantity Q for a good. Both are observable, and they are observed together. But it is very popular to give Q two different names, S and D, and then talk about S and D separately as if they were distinct entities, which could, moreover, exist apart. There is a puzzle here, or some confusion—possibly on the part of this writer.

It could be wondered (from limitless curiosity) why any price should be just P and not higher or lower. Since the market is central to economics, it is in some way understandable that there should be an attempt to give an explanation of price, even a complete one. But what kind of explanation can there be? To explain a change in a price is a more modest endeavour than explaining a price.The same is true with regard to the neck of the giraffe, the elongation is well understood in terms of evolutionary mechanisms (despite some debate about that), but no biologist attempts a complete explanation of the giraffe. Economists, of the fundamentalist kind, have greater courage. They offer an explanation of price, of all prices; or, rather, a form for an explanation has been proposed (perhaps more than one). But carrying out a realization of the form is a further matter having much less attention. One may ask what content there could be just in the proposal about form, and whether there is anything in it that can be known to be true or false, or neither.

A main doctrine about a price in settled times was that it should be settled. A price was part of the order of things like fowls of the air and beasts of the field; there was such a thing as a proper chicken and it had a proper price. The price could be known and counted on; it could enter into plans for a dinner or the allocation of a budget. This is a rational position, and practical, but it is not one that can always be enjoyed, because unsettled times produce changes also in prices. The interest then is still practical; not why prices should all be absolutely what they are, but how they might change. This is as with the giraffe. There can still be the question of why there should be any offer to explain all prices, and even whether there really is an explanation.

It has been said “Teach a parrott to say Supply and Demand

and there you have an economist" (Stephen Leacock, *Literary Lapses*). Whether or not this statement deserves approval, it suggests a pleonasm, since we decided that supply and demand are the same thing. They are not equal, but are indistinguishable. That is a matter of inescapable logic, since what means buying something to a buyer means selling something to a seller, and these two things happen to be identical. All the same, though the two things are one, it is outstanding that the same cannot be said of the buyer and the seller. These are two separate individuals. In fact, since the transaction is voluntary on both sides, it seems something of a coincidence that they got together. For theory here is not just the chance but the matter to be explained.

This apparent chance has an extension to the aggregate, where it looks as if an unlikely coincidence must occur in order to have a price at all, because those willing to buy must be exactly matched by those willing to sell. It seems implausible that everyday economic life, where the price phenomenon is a commonplace, should be based on such a precarious balance. Theory should deal with the possibility of the fine arrangement; and, since feasibility is of incomplete interest without an idea of how it is realized, it should deal also with the process for arriving at it. Just now, however, the concern is with elements rather than aggregates.

The encounter between buyer and seller might in reality have some significant effect on each. If this is of a special and individual kind it must be ignored, and for purposes of theory we should deal with simple automata. A simplicity in the encounter is assured if the only recognizable interface between buyer and seller is the price. For a start, it is supposed that the buyer and the seller in their separateness each have a definite potentiality for their part in the transaction which takes place, already existing and then realized from the encounter. A shape for such assumptions is provided by the following:

> (i) Anyone willing to buy a unit of a good at some price would be willing to buy it also at any lower price.
>
> (ii) Anyone willing to sell a unit of a good at some price would be willing to sell it also at any higher price.

These are norms consistent with price being an incentive to sell and a disincentive to buy, and not absolute laws. We know from Veblen about ostentatious expenditure, and ostentatious charity or other dispositions that might be imagined could be an addition to that. The situation is not different from Newton's when he made uniform motion a norm; non-uniform motion causes attention to forces, if any can be found, and here ostentation is to Veblen what gravitation was to Newton. These assumptions express *free disposal*: the buyer who pays less is free to dispose of the difference leaving a situation which is the same and therefore as acceptible as when P has been paid, and similarly on the side of a seller.

If P_b is the upper limit to prices at which the buyer would be willing to buy a unit and P_s is the lower limit at which the seller would sell then simultaneous willingness both sides requires

$$P_s < P < P_b.$$

Hence

$$P_s < P_b,$$

for simultaneous consent to a transaction quantity at some price P, which then can be anywhere between P_s and P_b. Otherwise at least one refuses and there is no transaction.

When successive further units are brought in we get declining and rising step functions, and if the units are small and numerous these become general monotonic curves, the supply and demand curves for buyer and seller. This line can be taken further by bringing in notions about a market and these individuals being in one.

Important data of the sensible world of economics concern form, not hard to get but evident from ordinary experience. No measurements are needed to know it, or read it. Also, things on paper are a part of economics because they affect behaviour and events; they are a real part of the real world. There is a voluminous recording and manipulation of data, but nothing of what in the main passes for economic theory really depends on it; in part of its nature it is near to ritual, in some form indispensible to social decision making as from time immemorial. The empiricism of natural scientists is different from so-called ‘empirical work’ in economics.

These possible views are entertained in connection with price theory based on supply and demand, in order to evaluate rather than object to it. The theory relies on ideas of the kind already described that have obvious reference to experience and we know what they mean. But there is a transition at some point, and there can be questions about the result. For instance, if the final theory were true there would be no way of knowing it, and so there can be question about the sense of offering it as true, or even as possibly true. As for its being false, in particular economies we know that it is, at least to some extent, because for instance prices are subject to various regulations. Also, time is a complication that makes the theory even difficult to interpret. None the less, and perhaps properly, the theory of prices and their equilibrium (in an unknown and unknowable framework) is given an important place in economic theory. Whether or not there should be a complaint exactly here, there can be one of another order about the 'welfare' appendages to this theory.

The matters up to now have a local and individual reference, but dealing with an economy signifies a global framework of information and competition. Stephen Leacock brings that out in "Boarding House Geometry", another of his *Literary Lapses*. He sets out the argument in the manner of Euclid, a few Postulates and then Propositions. The former go somethinglike this:

> POSTULATES:
> A landlady is an angular figure equal to anything.
> Boarding house sheets produced however far each way will not meet.
> A pie is produced any number of times.
> . . .

and so forth. Then the first Proposition and its Proof.

> PROPOSITION:
> All boarding house rents are equal.

The proof is by the method of *reduction ad absurdum* by which an hypothesis is impossible if it has impossible consequences:

> PROOF:
> Suppose, if possible, that the rents are not equal. Then one is greater than the other. Then the other is less than it might have been, which is absurd, &c. QED

The absurdity is from the landlady's side and it could have gone just as well from the the tenant's side—*i.e.* the other is greater than it might have been . . . Leacock takes for granted our stated assumptions about buyers and sellers. While there are transactions in a good at price P no seller will sell at a lower price and no buyer will buy at a higher one. Since joint consent is needed, there will be no transactions at a price above or below P, and the good has a single price P. There is a global equalization, a global information situation having been presupposed.

Should buyers willing to buy at the price be exausted before sellers willing to sell, then if further transactions take place they will be at a price that brings in more buyers and thereby a lower price, though not one so low as to cut out all sellers. Those sellers who are within their threshold and prepared to sell at a lower price will take part, and those others who were at or beyond it will not. In going to a lower price demand rises and supply falls; similarly with the reverse situation, in which further transactions would take place, if at all, at a higher price. At any point the price is what it is because all who would be buyers at that price find sellers and all sellers, buyers. There might be none of either. But it makes no sense to say cabbages are £1 per kilo if there is a would-be buyer at the price who cannot find a seller, or a seller who cannot find a buyer. The price being P depends on the balance $S = D$. If both sides of this equation are zero, as when buyers and sellers are far apart and there are none for a range of prices, it would not be precarious; otherwise it is, and there are movements. This is a dynamic picture giving sense to movements. It does not depend on a knowledge a the total numbers that would buyor sell at whatever price; whatever these might be, should it make sense even to refer to them, they can be recognized to be continually changing. At any moment one can in principle know the prevailing price P and the transaction quantity Q which is simultaneously both supply and demand, and that is all.

The theory of price as determined by the equality of supply and demand postulates that supply and demand are functions $S = S(P)$, $D = D(P)$ of the possible price P, and the prevailing price P is determined by the condition $S(P) = D(P)$. If this theory is offered as having an empirical basis, there is a problem about knowledge of these functions, since only one point is observable on each, the point (P, Q). If the price P ever changes, it must be because the functions have changed and so are no longer observable; instead, a new single point can be observed on the new 'functions'. If the functions do not change they cannot be observed, and if they do change they cannot be observed either.

The market in which goods have a price at which they are bought and sold is a commonplace phenomenon. With it there can be the idea, even if not the exact experience, of a settled market where, day after day, the prices and transaction quantities are the same. If we can think of a market as we do because of experience, we can also think of a settled market. It is a logical possiblity arising from the terms of description of markets, and it makes an ideal reference.

With a settled market, the possibility can be entertained of its being made up of settled individuals with fixed supply and demand functions. That is an imaginable possibility, in fact, one that is already imagined in price theory. A market of some sort is known to be a real possibility in some real circumstances. Now it can be asked if a market is a possibility in some ideally possible circumstances with fair provisors that do not ask anything positively wrong from experience. This is an important first question for a theoretical understanding of the market system. A negative answer would be a great surprise and would make the market phenomenon thoroughly mysterious. Markets are found everywhere, in principle self created and self governed without any centralized intervention; and one would have to wonder what it is in the real world and not in the imagined world that makes them possible. With a positive answer there is, in addition to peace of mind, a central finding about the nature of markets, showing the known real possibility matched by an intrinsic theoretical possibility.

To develop the question, consider functions which give the vector S of differences between aggregate supply and aggregate demand for all goods as determined by the vector p of all the prices. These are *market functions*, given in the form of excess supply functions, so $D(p) = -S(p)$ would be excess demand functions.

In principle, the economy is composed of individuals each with such a function $s(p)$, and the market function S is a sum of all the individual functions s. At any prices, each individual buys some goods and sells others, paying for purchases with receipts from sales, so that demands match supplies in exchange value and there is the individual *budget constraint* $ps(p) = 0$. Then, for their sum, $pS(p) = 0$, which is called *Walras' Law*.

The prices $p > o$ are significant only as determining exchange rates between goods from their ratios. Since the functions s depend only on the ratios, we have $s(tp) = s(p)$ for all $t > 0$. Summing, the function $S(p)$ is defined for all $p > o$ and such that

$$pS(p) = 0,\ S(tp) = S(p)\ (t > 0).$$

The market feasibility question, or the existence of some feasible prices, is now the question of whether or not there exists some prices $p > o$ for which aggregate supply equals aggregate demand for every good, or the excess supplies are simultaneously all zero, that is

$$S(p) = O.$$

It is not enough to know that such prices should exist; a further issue is how they would be found. After all, no one is doing the computing but the economy itself. The *Law of Supply and Demand*, that the price of a good falls if it is in excess supply and rises if it is in excess demand, is an available principle which, put in a suitable form, should amount to a computational algorithm.

Excess supply or demand is known not by a fixed identifiable individual or agency, but in a scattered decentralized way, as soon as a buyer of a good cannot find a seller or a seller a buyer at the price of the last transaction.That price then is no longer the price. The possible price rises or falls, in the minds and readiness of unfulfilled buyers or sellers in the one case or the other, until a transaction takes place, when it becomes the actual price; and so forth. A consistency with this picture is required; it is not exactly the *tâtonnement* of Walras, but more or less the same—and there is no need for the usual auctioneer.

3 Market functions

Market *excess supply* functions $S(p) \in \mathbb{R}^n$ are defined for all $p > 0$, $p \in \mathbb{R}_n$. The market as a sum of independent individuals requires the function S for the market to be a sum of functions s for the individuals,

$$S(p) = \sum s(p),$$

where, since prices are significant to the extent of determining exchange rates,

$$s(tp) = s(p) \ (t > 0), \tag{i}$$

and since $s(p)$ represents an exchange of goods at these rates,

$$ps(p) = 0. \tag{ii}$$

Hence, by summation,

$$S(tp) = S(p) \ (t > 0), \tag{i$'$}$$

$$pS(p) = 0 \text{ (Walras' law).} \tag{ii$'$}$$

For the vector $e = e(p)$ of *market value functions*

$$e_j = p_j S_j(p),$$

which give excess supplies in money or exchange value terms, it follows that

$$e(tp) = te(p) \ (t > 0), \tag{i$''$}$$

$$\textstyle\sum_j e_j(p) = 0. \tag{ii$''$}$$

From (i)′ the market functions, defined for $p > 0$, are fully specified from their values in the interior of the *normalized price simplex*

$$\Delta = \{p : p \geq o, \sum_i p_i = 1\},$$

where prices sum to 1. The *supply capacity limits* are

$$u_j = \sup S_j(p),$$

and the *finite supply capacity* condition requires

$$u_j < \infty.$$

Joining (i)′ and (ii)′ with this assumption, together with *continuity* of the S_j, the value functions e are *continuous and bounded above* in Δ. But from (ii)″,

$$e_j = -\sum_{i \neq j} e_i,$$

so *also they are bounded below*; this still allows the S_j to be unbounded below, from demand being insatiable. Hence, being bounded and continuous, the functions e have *unique continuous extensions to the closure of* Δ.

The *Intercept Condition* requires that, for any p_i $(i \neq j)$,

$$S_j < 0 \text{ for small } p_i,$$

equivalently, on Δ,

$$p_j = 0 \Rightarrow e_j < 0.$$

The sense of this condition is that demand overtakes supply of a good as its price goes to zero. Then from (ii)″,

$$p_j = 1 \Rightarrow p_i = 0 \ (i \neq j)$$
$$\Rightarrow e_j = -\sum_{i \neq j} e_i > 0,$$

so we also have

$$p_j = 1 \Rightarrow e_j > 0.$$

With the continuity, it follows from Bolzano's theorem[1] that

$$e_j = 0, 0 < p_j < 1, \text{ for some } p_j,$$

in which case also

$$S_j = 0, 0 < p_j < 1, \text{ for some } p_j.$$

[1] A continuous function defined on an interval takes every value between any two of its values. Hence if it is positive and negative it must also be zero at some point.

Thus, other prices being fixed, any one market for a good can be cleared by setting its price.[2]

4 Existence

For general equilibrium, all markets have to be cleared simultaneously, that is,

$$S = 0, \text{ for some } p > 0.$$

As to be seen, the available assumptions already imply this is possible.

Consider continuous excess supply value functions

$$e_j = p_j S_j(p),$$

defined on the normalized prices simplex Δ such that

$$\sum_j e_j = 0,$$

and

$$p_j = 0 \;\Rightarrow\; e_j < 0,$$

that is, satisfying Walras Law and the Intercept Condition. We will show that $S_j(p) = 0$ for all j, for some $p > o$.

Let

$$C_j = \{p : e_j \geq 0\},$$

so these are closed sets since the functions e_j are continuous, and for any subset V of the goods, or of vertices of the simplex, let

$$C_V = \bigcup_{j \in V} C_j.$$

The face of the simplex on the vertices V is

$$\Delta_V = \{p : j \in \overline{V} \;\Rightarrow p_j = 0\}$$

It will be shown that

$$\Delta_V \subset C_V \text{ for all } V$$

and hence, by the KKM lemma of Appendix II,

$$\bigcap_j C_j \neq O.$$

But any

$$p \in \bigcap_j C_j.$$

is by definition such that

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[2] Prices being significant to the extent of their ratios, one price varying while others remain fixed can mean that, as the one varies in the interval $[0, 1]$, the others vary to preserve normalization while their ratios remain fixed. In other words, there is movement on a line connecting a vertex of the normalized price simplex Δ to a point in the opposite face.

$$e_j(p) \geq 0 \text{ for all } j.$$

But since

$$\textstyle\sum_j e_j = 0,$$

this is equivalent to

$$e_j(p) = 0 \text{ for all } j.$$

By the Intercept Condition this implies $p > 0$ and so is equivalent to

$$S_j(p) = 0 \text{ for all } j,$$

as required.

Thus, for any $p \in \varDelta$,

$p \in \varDelta_V$	$\Leftrightarrow$	$j \in \overline{V} \Rightarrow p_j = 0$	def of left
	$\Rightarrow$	$j \in \overline{V} \Rightarrow e_j \leq 0$	Intercept Condition
	$\Rightarrow$	$\sum_{j \in \overline{V}} e_j \leq 0$	inequality sum
	$\Leftrightarrow$	$\sum_{j \in V} e_j \geq 0$	Walras' Law
	$\Rightarrow$	$e_j \geq 0$ for some $j \in V$	inequality sum
	$\Leftrightarrow$	$p \in C_j$ for some $j \in V$	def of C_j
	$\Leftrightarrow$	$p \in C_V$	def of C_V

$\therefore \varDelta_V \subset C_V$. QED

5 Stability

There is also the question of how the prices may be found, by a process that takes place within the economy arising from reactions to the shortages and surpluses that occur. The imbalances between buyers and sellers produce *market forces* which adjust prices upwards and downwards according to the *Law of Supply and Demand*, and should lead towards proper market prices, these making an *equilibrium* for the adjustment process. Hence there is the question of the *stability* of the system, in regard to a model for such forces.

Such a model is provided by the differential adjustment process $\dot{p} = f(p)$ where

$$f_j(p) = -v_j e_j(p) \ \ (v_j > 0).$$

One further assumption provides for global stability under such an adjustment process, and uniqueness of equilibrium. This is the *Slope Condition*, which requires $S_j(p)$, and equivalently e_j, to be decreasing in p_i $(i \neq j)$. From (ii), with this condition, e_j is increasing in p_j.

Suppose market functions e_j are continuously differentiable and such that

$$e_{ij} = \partial e_i / \partial p_j < 0 \ (j \neq i).$$

This is a consequence of the Slope Condition which, though different, is similar to the already familiar gross-substitutes condition on excess demand.

Now from Walras' Law

$$\sum_i e_i = 0,$$

we have

$$\sum_i e_{ij} = 0,$$

and so also $e_{ii} \geq 0$. Because the e_i are homogeneous of degree 1 they satisfy Euler's identity

$$\sum_j e_{ij}\, p_j = e_i,$$

with the consequence that $e_i < 0$ if $p_i = 0$. Thus, *the Slope Condition implies the Intercept Condition*, and so, by the theorem of the last section, we have the existence of $p^* > 0$ for which

$$e_i(p^*) = 0 \text{ for all } i.$$

We will be able to deduce also the uniqueness of such p^* from the following stability considerations.

Consider now the differential price adjustment system

$$\dot{p}_i = \ - v_i e_i(p),$$

where the *reaction coefficients*, or velocities, are any constants $v_i > 0$. Since replacing p_i by p_i / v_i corresponds to a change of the arbitrary physical units, we can suppose the change already made, so in effect the coefficients v_i become all equal to 1 and the system becomes simply

$$\dot{p} = -e(p).$$

Given any initial $p(0)$ in the interior of Δ, this has a unique solution $p(t)$ $(t \geq 0)$, and the Intercept Condition assures us that this remains in the interior of Δ. We want to show that

$$p(t) \to p^* \ (t \to \infty).$$

In other words, p^* is a globally stable equilibrium, in the differential adjustment process, and hence also the unique equilibrium.

For a small time interval τ the adjustment process is approximated by the finite difference system

$$p' - p = -\tau e(p),$$

where p becomes p' after time τ. Thus we have $p' = f(p)$ where

$$f(p) = p - \tau e(p),$$

so $e(p^*) = 0$ is equivalent to $f(p^*) = p^*$.

For the derivatives of f_i we have

$$f_{ii} = 1 - \tau e_{ii},\, f_{ij} = -\tau e_{ij} > 0 \; (j \neq i),$$

and

$$\textstyle\sum_i f_{ij} = 1 - \tau \sum_i e_{ij} = 1.$$

Thus the derivative matrix g of f is a positive row distribution matrix provided

$$\tau < 1/e_{ii}.$$

From continuity of the derivatives and compactness of Δ, τ can be made small enough to make this so for all p. Then with g positive and continuous, and Δ compact, the elements of g have a lower bound $\mu > 0$.

Now consider the r-fold iterated image

$$f^{(r)}(p) = f(f(\ldots f(p)\ldots)),$$

that is,

$$f^{(0)}(p) = p,\;\; f^{(r)}(p) = f\big(f^{(r-1)}(p)\big) \;\; (r = 1, 2, \ldots).$$

The derivative matrix, by the chain rule, is

$$g^{(r)} = g^1 \ldots g^r,$$

where g^s is g evaluated at $f^{(s-1)}(p)$. Then, by the Theorem, Appendix III, Corollary, for all $p, q \in \Delta$,

$$\big|(q-p)g^{(r)}\big| < |q-p|(1-\mu)^r.$$

But by the Theorem of the Mean,

$$f^{(r)}(q) - f^{(r)}(p) = (q-p)g^{(r)}(z),$$

where $z \in \; < p, q >$. We can now conclude that, for any $\epsilon > 0$, there exists s such that, for all $p, q \in \Delta$,

$$\big|f^{(r)}(q) - f^{(r)}(p)\big| < \epsilon$$

for all $r > s$, and hence that $f^{(r)}(p)$ $(r \to \infty)$ converges to a constant function, with the single value p^* since in any case this must be one of its values. In other words, p^* is a stable equilibrium in the finite adjustment system $p' = f(p)$.

The differential system and the finite difference systems have the same equilibria. The finite difference systems are stable and approximate the differential system for small τ. It follows that the differential system is also stable.

Appendices

I Sperner's Lemma

Let S be a simplex and V the set of its vertices, these being $n+1$ in number if S has dimension n, making it an n-simplex. Taking barycentric coordinates with S as simplex of reference, any point $x \in S$ has coordinates x_i $(i = 0, 1, \ldots, n)$ where

$$x_i \geq 0,\ \Sigma\, x_i = 1.$$

Thus *vertex* j of S has coordinates

$$x_i = 0\ (i \neq j),\, x_j = 1.$$

A *face* of S is a simplex S' whose vertices are a subset $V' \subset V$ of the vertices of S. The vertices of S are the faces of dimension 0, and S itself is the only face of dimension n. The 1-faces or *edges* are the line segments joining pairs of vertices.

A *simplicial dissection* of S is a collection T of simplices covering S any pair of which are either disjoint or have a common face for their intersection. The vertices of T include all the vertices of its simplices, and so also all the vertices of S. A dissection T of S provides a dissection also of the faces of S, whose simplices are faces of the simplices of T. By dissecting the simplices of a dissection T of S we have a further dissection of S, whose vertices contain the vertices of T as a subset.

The *barycentric subdivision* of a simplex S is the particular simplicial dissection $B(S)$ whose vertices are the vertices of S together with the bisectors of the edges. Barycentric subdivision can also apply to any dissection. By n-times repeated subdivision, we have the nth barycentric subdivision $B(\ldots B(S)\ldots)$ of S, whose simplices have edges which are $\frac{1}{2^n}$ of the edges of S.

The *carrier face* of any point of S is the face of lowest dimension which contains it. Thus, $V_x = \{i\colon\, x_i > 0\}$ is the set of vertices of the carrier face S_x of x. The vertices of S are their own carrier faces. Also, any face of a simplex of a dissection has a carrier face, which is the face of lowest dimension of the base simplex which carries all its vertices.

A *Sperner label* for a point of S can be any vertex of its carrier face. Thus, L is a Sperner label for x if $L \in V_x$, that is, if $x_L > 0$. In particular, the only possible Sperner label for a vertex of S is the vertex itself. A *Sperner labelling* for a simplicial dissection T of S is a function L which assigns a Sperner label $L(x)$ to every vertex x of T. Then a $Sperner$ *simplex* is a simplex of T whose vertex labels describe all the vertices of S.

Theorem (*Sperner*) For any simplex S and any simplicial dissection T of S, and any Sperner labelling of the vertices of T, the number of Sperner simplices is odd, and consequently there exists at least one.

First we will see what it means for a 1-simplex S, with vertices 0, 1. This is a line segment with 0, 1 as endpoints, say on the left and right.

```
o--------------------------------------o
0                                      1
```

1-Simplex S

For a simplicial dissection of T of S the vertices are a set of points of S including 0 and 1. Neighbouring ones join in segments, which are the simplices of the dissection T. These cover S, and any pair either are disjoint or have a face, in this case an endpoint, in common.

```
o-------o------o---------o---------o----o
0                                       1
```

Dissection T of S

S itself is the carrier face of every vertex of T except its own vertices, which carry themselves. Therefore with a Sperner labelling they carry either 0 or 1 as labels, while the vertices of S label themselves.

```
o-------o------o---------o---------o----o
0       0      1         1         0    1
```

Sperner labelling L of T

A Sperner simplex is a segment whose vertex labels include both 0 and 1. It is clear that one must exist in this special case. Scanning the vertices of T from the left, we start with a 0. Succeeding ones carry labels 0 or 1, but the last has the label 1. The first 1 that occurs produces the first Sperner simplex. Since there is a vertex which carries the label 1, the last being a case, so that a 1 must be encountered sooner or later, at least one Sperner simplex exists. The argument of the theorem is that the number of Sperner simplices is odd, so there must be at least one. We can see that in this case. In scanning the vertices from the left, every alternation of labels produces a Sperner simplex. Starting from label 0, any number of alternations that ends with a 1 must be odd, and we do end with a 1 on the right. Therefore the number of Sperner simplices is odd.

For the general proof of the theorem, the method is by induction on the simplex dimension n. The case $n = 0$ is true, vacuously. We have, moreover, just given an independent proof for $n = 1$. Suppose now that the theorem is true for dimension $n - 1$, and consider a simplex S of dimension n. Let T be a simplicial dissection of S, and let L be a Sperner labelling of T. These induce the same on every face of S, to which the inductive hypothesis applies. Therefore every face of S contains a simplex, a face of some simplex of T, whose vertex labels exactly describe its vertices; moreover, there is an odd number of such simplices in any face. The proof of the theorem has two parts and we use this consequence of the inductive hypothesis in the second.

Consider the class C of faces of T with vertex labels $1, \dots, n$. Any one is a face of a simplex of T whose remaining vertex label is either 0 or different from 0 and so among $1, \dots, n$. Let N be the number of simplices of T with labels $0, 1, \dots, n$. These are the Sperner simplices. Also let N' be the number of simplices of T with labels $1, \dots, n$ and another repeating one of these. Each of the N has one face of class C and each of the N' has two. Thus the number of occurrence of a simplex of class C as a face of some simplex of T is $N + 2N'$.

Now further, each C-simplex is counted once if it is in a face of S, since then it is a face of just one simplex of T, and otherwise twice, since then it is a face of two simplices of T. But a C-simplex in the face $1, \dots, n$ of S is a Sperner simplex for that face, and by the inductive hypothesis the number of these is odd. Also, a C-simplex cannot lie in any other face, by the Sperner labelling rule. Thus an odd number M of faces of T have been counted once in the total $N + 2N'$ and the remaining ones, say M', have been counted twice. Thus we have $N + 2N' = M + 2M'$, where M is odd. It follows that N is odd. QED

II The KKM Lemma

Lemma If U is compact and closed subsets $F_i \subset U$ are such that $\bigcap_i F_i = O$, then there exists $\epsilon > 0$ such that, for any X, if $X \cap F_i \neq O$ for all i then X has diameter at least ϵ.

The set $\bigotimes_i F_i$ is compact, and the function

$$f(x) = \min_{ij} |x_i - x_j| \quad (x_i \in F_i)$$

defined on it is continuous and so attains a minimum ϵ, where $\epsilon > 0$ since $\bigcap_i F_i = O$. Hence for any a_i, $a_i \in X \cap F_i$ for all i implies that the diameter of X is

$$\sup \{|x-y| : x, y \in X\} \geq f(a) \geq \epsilon,$$

so it is at least ϵ. QED

Theorem (*Knaster-Kuratowski-Mazurkiewicz*) If S is a simplex, S_v the face on any subset v of vertices, F_i a closed subset associated with any vertex i, and $F_v = \bigcup_{i \in v} F_i$, and if $S_v \subset F_v$ for all v, then $\bigcap_i F_i \neq O$.

Suppose if possible that $\bigcap_i F_i = O$ and let $\epsilon > 0$ be as in the Lemma. Consider a simplicial subdivision T of S where the simplices have diameter $< \epsilon$. For any vertex t of T let v be the set of vertices of the carrier face of S, so that $t \in S_v \subset F_v$ and hence $t \in F_i$ for some i. Let $L(t) = i$. Then L is a Sperner labelling of the vertices of T. Hence by Sperner's Lemma there exists a simplex X of T such that, for all i, $L(t) = i$ for some vertex t of X, so $X \cap F_i \neq O$ for all i. But by the lemma this is impossible since X has diameter less than ϵ. Hence the hypothesis is impossible. QED

III Distribution matrices

A *distribution* on n objects is a set of n numbers which are non-negative and sum to 1. A *distribution vector* is any vector whose elements form a distribution. Introducing I to denote a column vector of any order with elements all 1, the condition for a row vector p to be a distribution vector can be stated

$$p \geq o,\ pI = 1.$$

Then

$$\Delta = \{p : p \geq o, pI = 1\}$$

is the $(n-1)$-simplex of distribution vectors of order n.

A row (or column) *distribution matrix* is any matrix whose rows (or columns) are each given by a distribution vector. Any product ab of distribution matrices a, b is again a distribution matrix. The condition for a matrix a to be a distribution matrix can be stated

$$a \geq o, aI = I,$$

the orders of the Is here being different, unless a is square, in which case a can be regarded as a mapping

$$a : \Delta \to \Delta$$

of the simplex Δ into itself where any point $p \in \Delta$ has image $pa \in \Delta$. For if $q = pa$ then we have $q \geq o$ because $p \geq o,\ a \geq o$. Also, because $pI = 1,\ aI = I$, we have

$$qI = (pa)I = p(aI) = pI = 1,$$

so $qI = 1$. Hence also $q \in \Delta$.

Let Λ be the set of displacements in Δ, so

$$\Lambda = \{q - p : p, q \in \Delta\},$$

and

$$v \in \Lambda \Leftrightarrow vI = 0.$$

For any $v \in \Lambda$, let

$$|v| = \textstyle\sum_i |v_i|,$$

so

$$|v| \geq 0, |v| = 0 \Leftrightarrow v = o.$$

If a is a distribution matrix, and $v \in \Lambda$, then also $va \in \Lambda$. For from $aI = I$ we have

$$(va)I = v(aI) = vI.$$

So from $vI = 0$, also $(va)I = 0$, and so we can consider also $|va|$.

Theorem If a is a distribution matrix and a_k is the smallest element in column k, and if v is any vector for which $vI = 0$, then

$$|va| \leq |v|(1 - a_k).$$

It can be taken that

$$\textstyle\sum_i v_i a_{ik} \geq 0, \tag{i}$$

since otherwise v can be replaced by $-v$. Now we have

$$a \geq o, \tag{ii}$$

$$aI = I \text{ and definitions of } |v| \text{ and } a_k, \tag{iii}$$

$$vI = 0, \tag{iv}$$

so that

$$
\begin{aligned}
|va| &= \textstyle\sum_{j\neq k}\left|\sum_i v_i a_{ik}\right| + \sum_i v_i a_{ik} && \because \text{(i)}\\
&\le \textstyle\sum_{j\neq k}\sum_i |v_i| a_{ij} + \sum_i v_i a_{ik} && \because \text{(ii)}\\
&= \textstyle\sum_j\sum_i |v_i| a_{ij} - \sum_i (|v_i| - v_i) a_{ik} && \\
&\le |v| - \textstyle\sum_i (|v_i| - v_i) a_k && \because \text{(iii)}\\
&= |v|(1 - a_k) && \because \text{(iv)}
\end{aligned}
$$

$$\therefore \quad |va| \le |v|(1 - a_k) \quad \text{QED.}$$

Corollary If also $g^1, \ldots, g^r$ are distribution matrices and μ is a lower bound for their elements, then

$$|vg^1 \ldots g^r| \le |v|(1-\mu)^r.$$

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