1 Introduction

The proposal that demand is governed by utility leads to the question of how to arrive at the utility, from evidence provided by the demand. For while demand is in principle directly observable, utility is not, but is dealt with hypothetically for its part in an explanation of demand.

Pareto (1906) approached the question of utility construction on the basis of a demand function, and remarked it could be done by solving certain partial differential equations. These referred to the inverse of the function. Volterra (1906) pointed out, in a review of Pareto's work, that the existence of a solution was not always assured but required certain "integrability conditions" such as had been provided by a theorem of Frobenius. Thereafter the question became known as the "integrability problem". Antonelli (1886) had provided equivalent conditions, stated in the form of a symmetry. Slutsky (1915) approached the question for a differentiable demand function, with reference to a utility function which is continuously twice differentiable, and brought into view conditions in terms of coefficients formed from derivatives of the demand function, instead of the inverse. Evidently, the question stands perfectly well regardless of invertability. He obtained the conditions by differentiation of first order Lagrange conditions, and arrived at further necessary (in fact more than necessary) second order conditions. The approach gained a currency after its rediscovery by Hicks and Allen (1934). Beside *symmetry* the Slutsky matrix was required to have a *negativity* condition, intermediate between its being non-positive and negative definite and different from both. That this negativity requirement must be in part spurious is demonstrated by "The Case of the Vanishing Slutsky Matrix"[1] featuring a perfectly respectable continuously differentiable demand function that has a utility, but the Slutsky coefficients all vanish identically, as would be satisfactory to the extent of Slutsky's symmetry requirement but altogether

[1] Afriat (1972), *Journal of Economic Theory* 5, 208-23.

impossible for the negativity. McKenzie (1957) identified the Slutsky coefficient matrix with the matrix of second derivatives of a utility-cost function, and so necessarily both symmetric and non-positive definite.[2]

Samuelson (1948) introduced the 'revealed preference' approach to the question for a demand function, taken further by Houthakker (1950).[3] Should the general, and elementary, theorem proved here, where utility enters as an arbitrary order, seem a complete answer to the question they and the others subsequently have dealt with, it could be wondered why their theory involved undesirable additional restrictions and so much extra work, with differential equations, problematic limiting processes, and so forth, which are absent with this new theorem. But this cannot be a complete answer. It could have been a satisfactory, and most fundamental, first answer. But despite the "indifference map" having taken over in the subject as being more essential, they still sought a numerical utility. There is forgetfulness of the original submission of a distinct merit for the revealed preference approach, from its being free of the numerical aspect, that being a "last vestige" of the obsolete measurable utility of classical economics.

Samuelson and Houthakker approach the same question as Slutsky, now without restriction on the utility function, and in place of differentiability of the demand function is the requirement that it be continuous and satisfy a Lipschitz condition in respect to income. They brought into view necessary conditions for a utility, and the issue then is sufficiency.

The argument of Samuelson for the case $n = 2$, adapted for $n > 2$ by Houthakker, has several features requiring comment. It depends on construction by the method with ascending and descending sequences of loci in the commodity space which are assumed to be surfaces, whereas they could be manifolds of lower dimension, even just single points. To avoid dependence on invertability it is necessary to work in the budget space rather than the commodity space, and construct a function in that space which can be shown to be the indirect utility function for a function in the commodity space. The ideas of the Samuelson and Houthakker argument with ascending and descending sequences are adaptable

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[2] Sufficiency is the dedication of my *Demand Functions and the Slutsky Matrix*, Princeton University Press, 1980.

[3] Further attention is given by Afriat (1954, 1962, 1972), Uzawa (1959, 1971), Richter (1966), Hurwicz (1971), Stigum (1973), and Mas-Collel (1976).

for that purpose.

In their argument, sequences are constructed corresponding to a dissection of a line segment in the budget space. As the dissection is refined, points associated with a particular ray form sequences, and their argument depends on the convergence of these, which remained problematic. Here, as a central feature, it is argued that they are bounded and monotonic and therefore convergent. Subsequent attempts at the convergence question are by methods that fall down on boundedness arguments, which can be properly supplied by monotonicity. But with the monotonicity the convergence is settled, and the methods are not needed at all. Also, the case dealt with by those other methods is where the inverse exists and has domain the entire commodity space, whereas here we have no such restrictions.

In the continuation of their argument, the limits of ascending and descending sequences as the dissection is refined indefinitely define two functions which satisfy a differential equation which, because of the Lipschitz condition, has just one solution, so they coincide. Here the Lipschitz condition is given also an earlier role, where it is interpreted as determining a limit on the angle of expansion, which assures that when the dissection is fine enough the associated descending sequence will be well defined.

Even if the construction of the utility function in the commodity space, on lines of Sameulson and Houthakker or as done here, is granted, nothing is accomplished if the investigation rests with it being established that the utility function has a maximum under every budget constaint given by the corresponding quantities determined by the demand function. For the utility function which is constant everywhere and so a maximum everywhere is always such a function. It is essential to show the maximum is an absolute maximum. This comes directly from the smoothness of the indirect utility function in the budget space. Thus again it is seen to be important to proceed in terms of constructions in the budget space rather than the commodity space.

For the sake of its history and its own interest, far from abandonment, the original problematic method is here carried through successfully, depending on peculiar mathematical auxiliaries which are a main contribution of this paper, and working in the dual budget space instead of the commodity space. It is similar with proof of sufficiency of Slutsky conditions which also proceeds from the dual.

Important in Samuelson’s approach is the idea, which amounts to a general “revealed preference” principle, that in any act of choice, that is, picking an element out of some set, if x is chosen while y is some other element in the set at the same time available, then we have the preference of x over y, or it is *revealed.* Treated as generally available, as it seems to have been in some hands, the unrestricted principle amounts to taking choice and preference to be synonyms, or to make any choice a result of *efficiency* in respect to some hypothetical objective, or preference system.

An unrestricted appeal to the revealed preference principle, whereby an efficiency is attributed to elections carried out by voting, leads to the well known “Voting Paradox”. Attention to this topic prepares for further considerations about preferences of groups, in particular about ‘welfare’ in a market economy. Having an election by means of voting is a way a group of individuals, all of whom might have different ideas about what is good but are still committed to act together, go about making a choice, picking one element out of a set of possibilities, or candidates. The winner is not the *best* for the group, merely the elected one. Had there been some available prior definition of best candidate there would have been no need to have an election in the first place. But still we have the Voting Paradox where there is determination to see the winner as best, and some surprise at the result.

At the time I joined the Department of Applied Economics, Cambridge, in September 1953, Houthakker’s famous paper was being circulated; it must have been the first thing I read. I immediately restated his “semi-transitivity”, now known as the “Strong Axiom of Revealed Preference”, as the irreflexivity of a transitive closure—the relation T_f that ocurrs here in Section 3. I heard it declared and am pleased to accept that the transitive closure was first introduced into this subject by Uzawa. I also believe him to be the first to uncover and attempt to remedy deficiencies in the original treatments. I submit this work, going over more years than I care to count, as a continuation of that attempt.

2 Demand & Utility

A *demand element* is any $(p, x) \in B \times C$ for which $px > 0$, showing a commodity bundle x demanded at prices p, the *expenditure* being px. Then with

$$u = (px)^{-1} p,$$

which is the associated *budget vector*, (u, x) is such that $ux = 1$, so it is a *normal* demand element, in this case the *normalization* of (p, x).[4] Various conditions involving demand elements can be stated well and more simply in terms of their normalizations.

A *utility order* is any $R \subset C \times C$ which is reflexive and transitive,

$$xRx,\ xRyRz \ \Rightarrow\ xRz,$$

xRy being the statement that x has at least the utility of y.[5]

The *chain-extension* of any relation R is the relation $\overrightarrow{R}$ holding between extremities of R-chains, given by

$$x\overrightarrow{R}z \equiv (\vee\, y \cdots)xRyR \cdots Rz.$$

This is the same as the *transitive closure*, or the smallest transitive relation containing R, being transitive, containing R, and contained in every transitive relation that contains R. The extended condition

$$xRyR \cdots Rz \ \Rightarrow\ xRz$$

is equivalent to transitivity, which therefore is equivalent to the condition

$$\overrightarrow{R} \subset R,$$

for R to be identical with its chain-extension, or transitive closure.

With $\overline{R}$ as the *complement* and R' the *converse*, where

$$x\overline{R}y \equiv\ \sim xRy,\ xR'y \equiv yRx,$$

we have

$$\overline{(R')} = (\overline{R})',$$

so there is no ambiguity in the expression $\overline{R}'$ for the *converse complement*.

The relation of equivalence in R, or the *indifference* relation, is the symmetric part

$$E = R \cap R',$$

an equivalence relation, symmetric, reflexive and transitive, since R is an order. The equivalence classes E_x, which are equally the

[4] In present notation, with Ω as the non-negative numbers, $\mathcal{B} = \Omega_n$ is the *budget space* (non-negative row vectors) and $\mathcal{C} = \Omega^n$ the *commodity space* (column vectors). Then any $p \in \mathcal{B}$, $x \in \mathcal{C}$ determine $px \in \Omega$ for the value of the commodity bundle x at the prices p. Sometimes when dealing with demand functions Ω should be the positive numbers. For syntax, a scalar usually multiplies a row vector on the left, and a column vector on the right.

[5] With a binary relation R, beside the usual $(x, y) \in R$ because R is set, also the statements xRy, $x \in Ry$ or $y \in xR$ are available to assert (x, y) is an element of R, or that x has the relation R to y.

sets xE or the Ex, these being the same from symmetry, are such that $x \in E_x$, so their union is C, and

$$xEy \Leftrightarrow E_x = E_y,\ x\overline{E}y \Leftrightarrow E_x \cap E_y = O,$$

so any pair are either disjoint or identical. Hence they constitute a partition of C, expressing C as a union of disjoint subsets.

The antisymmetric part of R, the *strict preference* relation, is

$$P = R \cap \overline{R}',$$

which is a strict order, irreflexive and transitive[6], since R is an order.

The subrelations E and P form a partition of R,

$$P \cap E = O,\ P \cup E = R.$$

An order R is *complete* if

$$\sim xRy \Rightarrow yRx,$$

so for any pair of elements, if they do not have the relation one way, then they have it the other, or they have it one way or the other and possibly both. That is, $\overline{R}' \subset R$, and equivalently, $P = \overline{R}'$.

A *simple* order R is such that

$$xRyRx \Rightarrow x = y.$$

For an order, this is equivalent to

$$xRyR \cdots Rx \Rightarrow x = y = \cdots$$

Otherwise, this is the condition for any relation R to be *anticyclic*, or for the absence of R-cycles of distinct elements. For a reflexive relation, it is the condition for the transitive closure to be a simple order.

The relations I and D of *identity* and *distinction* are given by

$$xIy \equiv x = y,\ xDy \equiv x \neq y.$$

For any simple order R, the symmetric part is $E = I$, so equivalence in R reduces to identity. In this case the antisymmetric part is identical with the irreflexive part, $P = R \cap D$, from which R is recovered as the reflexive closure, $R = P \cup I$.

A *utility function* is any $\phi: C \to \Omega$. It *represents* the utility

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[6] Beale and Drazin (1956) bring attention to this commonly unnoticed transitivity, basic for the scheme adopted here. Also indifference, sometimes treated as absence of preference and then problematic because without transitivity, is here taken to be a positive condition made from preference comparison both ways, necessarily transitive, as required if we are to have an equivalence relation.

order R for which

$$xRy \equiv \phi(x) \geq \phi(y).$$

In this case, for the symmetric and antisymmetric parts,

$$xEy \Leftrightarrow \phi(x) = \phi(y),\ xPy \Leftrightarrow \phi(x) > \phi(y).$$

Any utility order so representable by a utility function is necessarily complete.

Relations connecting a demand element (p, x) and a utility order R are defined by

$$H' \equiv py \leq px \Rightarrow xRy,$$

which corresponds to the *cost-effectiveness* familiar in cost-benefit analysis, and asserts x is as good as any bundle that costs no more, and

$$H'' \equiv yRx \Rightarrow py \geq px,$$

cost-efficiency, that any bundle as good as x costs as much.

While H' represents *utility maximization*, making x a bundle that has maximum utility for the money spent, H'' represents *cost minimization*, making x have minimum cost for the utility obtained. These are equally compelling, generally independent basic economic principles. A later issue involving stricter conditions concerns whether x is the unique bundle admitted by these conditions. The combination

$$H \equiv H' \wedge H''$$

defines *compatibility* between the demand and the utility. In terms of normalizations, these conditions become

$$uy \leq 1 \Rightarrow xRy,\ yRx \Rightarrow uy \geq 1,$$

respectively.

The condition

$$H^o \equiv py \leq px \wedge yRx \Rightarrow y = x,$$

here put symmetrically, has alternative statements,

$$\text{(i)}\ py \leq px \wedge y \neq x \Rightarrow \sim yRx,$$

which exposes a relationship with H', and

$$\text{(ii)}\ yRx \wedge y \neq x \Rightarrow py > px,$$

with H''.

We also consider

$$H^* \equiv py \leq px \wedge y \neq x \Rightarrow xRy \wedge \sim yRx,$$

which, in the case of R being complete, is equivalent to (i), and so to H^o.

According to H^o, any bundle as good as x and costing no more must be identical with x. Reflexivity of R already allows x itself is such a bundle, so the converse is already present.

The antisymmetric or strict part of R being $P = R \cap \overline{R}'$, in terms of this

$$H^* \equiv py \leq px \wedge y \neq x \;\Rightarrow\; xPy.$$

For the case where R is complete, $\overline{R}' \subset R$ and hence $P = \overline{R}'$, so H^* becomes the same as H^o.

As appears from forms (i) and (ii), when H^o is adjoined to each of H' and H'' we obtain the *strict* versions of these conditions, that require x to be the unique bundle which attains the required maximum utility, and minimum cost.

While the conjunction of H' and H'' provides compatibility H, we have the conjunction of the strict versions $H^o \wedge H'$ and $H^o \wedge H''$ to define *strict compatibility.* Since

$$(H^o \wedge H') \wedge (H^o \wedge H'') \;\Leftrightarrow\; H^o \wedge (H' \wedge H''),$$

this condition is also $H^o \wedge H$.

Theorem Strict compatibility, simultaneously requiring strict cost-effectiveness and strict cost-efficiency, is obtained by the condition H^*, which implies H and is equivalent to H^o if R is complete.

It is immediate that

$$H^* \;\Leftrightarrow\; H^o \wedge H',$$

and also

$$H^o \;\Rightarrow\; H''.$$

Therefore, with $H \equiv H' \wedge H''$, we have

$$H^* \Leftrightarrow\; H^o \wedge H,$$

as required. Consequently also $H^* \;\Rightarrow\; H$. The last part has already been remarked.

3 Demand functions

With prices p, and an amount M of money to be spent on some bundle of goods x, there is the budget constraint $px = M$. Given a function $x = F(p, M)$ that determines the unique maximum of a function ϕ under any budget constraint, F is a *demand function* which *has ϕ as a utility function*, or is *derived* from ϕ.

Now with any given function F it may be asked whether it is such a function so associated with a utility function. The function F first must have the properties

$$pF(p,M) = M,\ F(p,M) = F(M^{-1}p,1),$$

usually associated with demand functions. Then a function ϕ is sought for which, for all p and M, $x = F(p,M)$ is the unique maximum of ϕ under the constraint $px \leq M$, that is,

$$py \leq M,\ y \neq x\ \Rightarrow\ \phi(y) < \phi(x).$$

Introducing the *budget vector* $u = M^{-1}p$, the budget constraint $px = M$ is stated $ux = 1$. The *standard* demand function F determines the *normal* demand function *f*, its *normalization*, given by

$$f(u) = F(u,1),$$

with the property

$$uf(u) = 1,$$

from which it is recovered as

$$F(p,M) = f(M^{-1}p).$$

Instead of the usual standard form F, it fits what follows to deal with a demand function in the normal form *f*. Then a function ϕ is sought for which, for all u, $x = f(u)$ is the unique maximum of ϕ under the constraint $ux = 1$, that is,

$$uy \leq 1,\ y \neq x\ \Rightarrow\ \phi(y) < \phi(x).$$

The *expansion path* for any prices p is described by $x = F(p,M)$ as expenditure M varies while p remains fixed. If M is altered to ρM by a factor ρ, x becomes $x_\rho = F(p,\rho M)$, the budget vector $u = M^{-1}p$ is altered to the point $\rho^{-1}u$ on the ray through u, we have $x_\rho = f(\rho^{-1}u)$, and $ux_\rho = \rho$.

Expansion paths are therefore images in the commodity space C of rays in the budget space B. With a given path $\mathcal{U} \subset B$, we also consider its projecting cone $\mathfrak{U}$, consisting of the rays through its points, and projections of $\mathcal{U}$, which are paths in the same cone (in particular projections which are integral paths, to be dealt with).

It is simpler and has other advantage to deal with the question about F through its normalization *f*. For similar reasons a utility order R can take the place of the utility function. If it is the order represented by the function, it provides all that is important about the function. But it is most natural to have an arbitrary order in view, free of such representation.

A demand function f is *compatible* with a utility R if every demand element (u,x) which, being such that $x = f(u)$, so it belongs to f, is compatible with R. With $H_f(R)$ denoting this condition, H_f asserts the existence of a such a compatible R, or the *consistency* of f. Similarly $H^*_f(R)$ can assert *strict compatibility*, and H^*_f the *strict consistency* of f.

For $H^*_f(R)$ we have that for all u, and $x = f(u)$,

$$uy \leq 1 \wedge y \neq x \;\Rightarrow\; xRy \wedge \sim yRx.$$

Therefore, for any cyclic sequence $u_0, u_1, \ldots, u_m, u_0, \ldots,$

$$\begin{gathered} u_0x_1 \leq 1 \wedge u_1x_2 \leq 1 \wedge \cdots \wedge u_{m-1}x_m \leq 1 \\ \Downarrow \\ x_0Rx_1 \ldots Rx_m \\ \Downarrow \\ x_0Rx_m. \end{gathered}$$

But also

$$u_mx_0 \leq 1 \wedge x_0 \neq x_m \;\Rightarrow\; \sim x_0Rx_m.$$

Therefore

$$\begin{gathered} u_0x_1 \leq 1 \wedge u_1x_2 \leq 1 \wedge \cdots \wedge u_mx_0 \leq 1 \\ \Downarrow \\ x_0 = x_m. \end{gathered}$$

This condition on f, to be denoted K^*_f, has been seen to be a consequence of the strict consistency of f,

$$H^*_f \;\Rightarrow\; K^*_f.$$

From the cyclic symmetry, it is equivalent to the *strict cyclical consistency* condition

$$\begin{gathered} u_0x_1 \leq 1 \wedge u_1x_2 \leq 1 \wedge \cdots \wedge u_mx_0 \leq 1 \\ \Downarrow \\ x_0 = x_1 = \cdots = x_m. \end{gathered}$$

Then it is also equivalent to

$$\begin{gathered} u_0x_1 \leq 1 \wedge u_1x_2 \leq 1 \wedge \cdots \wedge u_{m-1}x_m \leq 1 \\ \wedge \\ x_0 \neq x_1 \vee x_1 \neq x_2 \vee \cdots \vee x_{m-1} \neq x_m \\ \Downarrow \\ u_mx_0 > 1. \end{gathered}$$

and to

$$\begin{gathered} u_0x_1 \leq 1 \wedge u_1x_2 \leq 1 \wedge \cdots \wedge u_{m-1}x_m \leq 1 \\ \wedge \\ x_m \neq x_0 \end{gathered}$$

$$\Downarrow$$
$$u_m x_0 > 1.$$

This last form shows the condition obtained by Houthakker (1950), elaborating the ‘revealed preference’ method of Samuelson (1948). A part of it is that

$$u_0 x_1 \leq 1 \wedge x_1 \neq x_0 \;\Rightarrow\; u_1 x_0 > 1,$$

which is Samuelson’s condition.[7]

Let $R_f(u)$, the *directly revealed preference* relation of *f* associated with the budget *u*, be defined by

$$xR_f(u)y \equiv x = f(u) \wedge \mathrm{uy} \leq 1,$$

and let R_f, the *revealed preference* relation of *f*, be the transitive closure of the union of these,

$$R_f = \vec{\bigcup_u} R_f(u).$$

This is reflexive[8] because the $R_f(u)$ are reflexive, and transitive by construction as a transitive closure, so it is an order.

Another expression for K_f^*, proceeding from the original statement, is that, for $x = f(u)$,

$$uy \;\leq 1 \wedge yR_f x \;\Rightarrow\; y = x.$$

Since $uy \leq 1 \;\Rightarrow\; xR_f y$, this is equivalent to

$$uy \leq 1 \wedge y \neq x \;\Rightarrow\; xR_f y \wedge \sim yR_f x,$$

that is, $\mathrm{H}_f^*(\mathrm{R}_f)$, so we have

$$H_f^* \;\Rightarrow\; K_f^* \;\Rightarrow\; H_f^*(R_f) \;\Rightarrow\; H_f^*,$$

and hence:

Theorem 3 $\qquad H_f^* \;\Leftrightarrow\; H_f^*(R_f) \;\Leftrightarrow\; K_f^*.$

In other words, *a demand function is strictly consistent*, or strictly compatible with some utility order[9], *if and only if it is strictly*

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[7] Samuelson dealt with the two-commodity case for which his and Houthakker's condition are equivalent, as proved by Rose (1958) and Afriat (1965).

[8] Rather, it is reflexive just at points in the range of the demand function. Without altering anything important but to give respect to the definition of an order, it could be made reflexive simply by taking its reflexive closure, or union with ‘=’.

[9] The present theorem has no special requirements at all about the utility order, or about the demand function. Samuelson and Houthakker sought a continuous numerical utility, involving auxilliary assumptions about the demand function, and a differential equation method. The following, still without a published report, asks less for the demand function, and for the

compatible with its own revealed preference order, and this is if and only if the strict cyclical consistency condition holds—that is, Houthakker's condition, often referred to as the "Strong Axiom of Revealed Preference".

The *strict revealed preference* relation of f is the strict or antisymmetric part of R_f,

$$P_f = R_f \cap \overline{R}'_f \,,$$

and the *revealed indifference* relation is the symmetric part

$$E_f = R_f \cap R'_f \,.$$

The *directly revealed strict preference* relation is the irreflexive part of R_f,

$$S_f = R_f \cap D,$$

so this is irreflexive by construction, though not transitive. Its transitive closure,

$$T_f = \overrightarrow{S}_f,$$

is the *revealed strict preference* relation, transitive by construction, not necessarily irreflexive.

Other expressions for the Houthakker condition K^*_f are

(i) $E_f = I$

(ii) $P_f = S_f$

(iii) $P_f = T_f$

(iv) S_f is transitive

(v) T_f is irreflexive

(vi) $S_f = T_f$

With revealed preferences there can be none of the "violation of transitivity" sometimes entertained, and no inconsistencies obtained from them alone. They are transitive by construction, and any contradictions come only when they are taken together with the less well-noticed *revealed non-preferences*. With Samuelson for instance these are provided by

$$py \leq px \wedge y \neq x \;\; \Rightarrow \;\; \sim yRx,$$

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utility: For a demand function f to have a lower semicontinuous numerical utility, it is necessary and sufficient that Houthakker's condition holds, and that the sets $f^{-1}(x)$ be closed.

as part of the strict compatibility H^*, or instead there are fewer coming from

$$py < px \;\Rightarrow\; \sim yRx,$$

which is the H'' part of the weaker compatibility condition H.

4 Obliquity

Theorem 4.1 If f is continuous, then for all $u, v > o$ there exists a $\rho > 0$ such that

$$uf(\rho^{-1}v) = 1.$$

Generally, such ρ need not be unique, but uniqueness will be assured later under a further condition.

For any $u, v > o$ there exist $\lambda, \mu > 0$ such that

$$\lambda^{-1}vx = 1 \;\Rightarrow\; ux < 1,\; \mu^{-1}vx = 1 \;\Rightarrow\; ux > 1,$$

for all x. Thus, take

$$\begin{aligned}\lambda &< \min\{vx{:}\, ux \geq 1\}\\ &= \min\{\textstyle\sum(v_i/u_i)u_ix_i{:}\,\sum u_ix_i \geq 1\}\\ &= \min_i v_i/u_i,\end{aligned}$$

and μ similarly.

Since $wf(w)$=1 for all w, we therefore have

$$uf(\lambda^{-1}v) < 1,\; uf(\mu^{-1}v) > 1.$$

Now with continuity of f, by Bolzano's theorem, there exists a ρ lying between λ and μ which is as required. QED

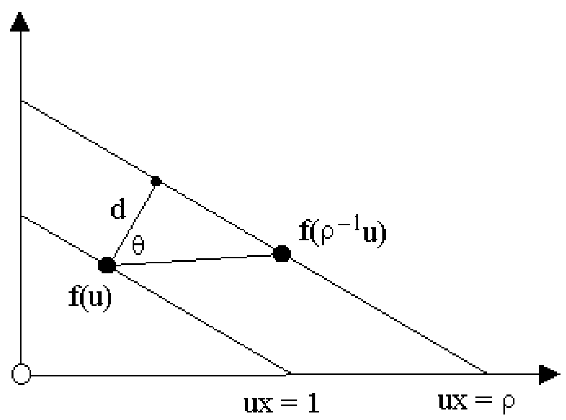

Figure 1

For any $u \geq o$ and $\rho, \sigma > 0$ consider the loci $ux = \rho, ux = \sigma$. They are parallel hyperplanes separated by a perpendicular displacement

$$d = u'(uu')^{-1}(\rho - \sigma),$$

corresponding to a minimum distance

$$|d| = |u|^{-1}|\rho - \sigma|.$$

With $\sigma = 1$, this is illustrated in Figure 1.

Let x_ρ, x_σ denote any points on these loci. The displacement $x_\rho - x_\sigma$ between them makes an angle θ with the perpendicular displacement d given by

$$\sec\theta = |x_\rho - x_\sigma| / |d| = |x_\rho - x_\sigma| \; |u| / |\rho - \sigma|.$$

This applies in particular with

$$x_\rho = f(\rho^{-1}u), \; x_\sigma = f(\sigma^{-1}u),$$

since then

$$ux_\rho = \rho, \; ux_\sigma = \sigma.$$

The thus determined $\theta = \theta_f(u; \rho, \sigma)$ defines the *obliquity* in the expansion from $\rho^{-1}u$ to $\sigma^{-1}u$.

Now with

$$\sec\theta_f(u, \rho) = |f(\rho^{-1}u) - f(u)| \, |u| / (\rho - 1),$$

so that

$$\theta_f(u; \rho, \sigma) = \theta_f(\sigma^{-1}u, \rho),$$

let

$$\theta_f(u) = \lim_{\rho \to +1} \sup \theta_f(u, \rho)$$

define the *limit obliquity* for expansion with *f* from any $u > o$, and

$$\theta_f = \sup_u \theta_f(u)$$

gives this with reference usually to a closed region of u that excludes the origin. Then

$$\theta_f < \pi/2$$

is the *bounded obliquity* condition, for expansion with f from points in the region.

The angle $\angle u, v$ between a pair of budget vectors u, v is given by

$$cos^2\angle u, v = uv'(vv')^{-1}vu'(uu')^{-1}.$$

Theorem 4.2 If

$$\angle u, v < \pi/2 - \theta_f,$$

then

$$uf(\rho^{-1}v) = 1$$

for at most one ρ.

Let

$$y_\rho = f(\rho^{-1}v),\ y_\sigma = f(\sigma^{-1}v).$$

It will be shown that if

$$uy_\rho = 1,\ uy_\sigma = 1,$$

then

$$\angle u, v \geq \pi/2 - \theta_f,$$

so the theorem will be proved.

Let $\mathcal{V}_\sigma$ be the hyperplane $vx = \sigma$. It contains y_σ since $\sigma^{-1}vy_\sigma = 1$. Let p be the foot of the perpendicular from y_ρ to $\mathcal{V}_\sigma$. Then

$$\angle py_\rho y_\sigma \leq \theta_f, \qquad 4.1$$

by definition of θ_f.

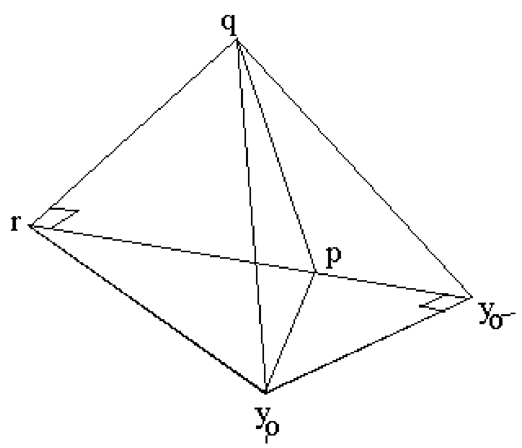

Figure 2

Now y_ρ, y_σ lie on the hyperplane $\mathcal{U}$ with equation $ux = 1$, by hypothesis. Let the perpendicular at y_ρ to $\mathcal{U}$ cut $\mathcal{V}_\sigma$ in q, and let r be the foot of the perpendicular from q to the line py_σ. Then

$$\angle u, v = \angle py_\sigma q. \qquad 4.2$$

Consider the tetrahedron y_ρ, y_σ, q, r and let Δ_x refer to the face opposite a vertex x, or the plane through it. It is going to be shown that

$$y_\rho y_\sigma \perp y_\rho r.$$

Because $y_\rho p \perp \mathcal{V}_\sigma$ and $\Delta_{y_\rho} \subset \mathcal{V}_\sigma$, it follows that $y_\rho p \perp \Delta_{y_\rho}$ and hence that $\Delta_q \perp \Delta_{y_\rho}$, and this with $qr \perp y_\sigma r$, given by construction, implies $qr \perp \Delta_q$, which implies $qr \perp y_\rho y_\sigma$. Now $y_\rho y_\sigma \perp y_\rho q$ which is given, together with $qr \perp y_\rho y_\sigma$ just concluded, implies $\mathrm{y}_\rho \mathrm{y}_\sigma \perp \Delta_{y_\sigma}$ which implies $\mathrm{y}_\rho \mathrm{y}_\sigma \perp \mathrm{y}_\rho \mathrm{r}$, as required.

This shows that

$$\angle py_\rho r = \pi/2 - \angle py_\rho y_\sigma,$$

which with 4.2 gives

$$\angle py_\rho r \geq \pi/2 - \theta_f. \qquad 4.3$$

It will now be shown that

$$\angle py_\rho q \geq \angle py_\rho r \qquad 4.4$$

so with 4.2 and 4.3 the theorem will be proved.

Since $y_\rho p \perp \Delta_{y_\rho}$, as already remarked, it follows that $y_\rho p \perp pq$ and $y_\rho p \perp pr$, so that

$$\tan \angle py_\rho q = pq/py_\rho,\ \tan \angle py_\rho r = pr/py_\rho.$$

But because $qr \perp y_\sigma r$, by construction, it follows that $pq \geq pr$, and hence that

$$\tan \angle py_\rho q \geq \tan \angle py_\rho r,$$

equivalently (iv). QED

Corollary If f is continuous, then for all $u, v > o$ there exists a $\rho > 0$ such that

$$uf(\rho^{-1}v) = 1.$$

If $\theta_f < \pi/2$ and

$$\angle u, v < \pi/2 - \theta_f,$$

then such ρ is unique, and $uf(\rho^{-1}v)$ is strictly increasing in ρ.

5 Ascent and descent

Theorem 5 Subject to Samuelson's axiom

$$uf(v) \leq 1 \wedge vf(u) \leq 1 \ \Rightarrow\ f(u) = f(v),$$

if
$$vf(u) = 1,\ wf(v) = 1,\ v = \lambda u + \mu w,\ \lambda \geq 0, \mu > 0,$$
then
$$wf(u) \geq 1,\ wf(u) = 1\ \Leftrightarrow\ f(u) = f(v).$$

If $f(u) = f(v)$ there is nothing more to prove. Otherwise, by the axiom,
$$uf(v) > 1,\ vf(w) \geq 1.$$
Then
$$1 = vf(v) = (\lambda u + \mu w)f(v) = \lambda uf(v) + \mu wf(v) > \lambda + \mu.$$
Suppose now, if possible, that $wf(u) \leq 1$. Then
$$1 = vf(u) = (\lambda u + \mu w)f(u) = \lambda + \mu wf(u) \leq \lambda + \mu,$$
so there is a contradiction. Hence $wf(u) > 1,$ and the theorem is proved.

Corollary (i) Subject to Samuelson's axiom, if
$$\alpha^{-1}vf(u) = 1,\ \beta^{-1}wf(\alpha^{-1}v) = 1,\ \gamma^{-1}wf(u) = 1,$$
where
$$v = \lambda u + \mu w,\ \lambda \geq 0, \mu > 0,$$
then
$$\beta \leq \gamma,\ \beta = \gamma\ \Leftrightarrow\ f(\alpha^{-1}v) = f(u).$$

Corollary (ii) Subject to Samuelson's axiom, if
$$uf(\alpha^{-1}v) = 1,\ \alpha^{-1}vf(\beta^{-1}w) = 1,\ uf(\gamma^{-1}w) = 1,$$
where
$$v = \lambda u + \mu w,\ \lambda > 0, \mu \geq 0,$$
then
$$\beta \geq \gamma,\ \beta = \gamma\ \Leftrightarrow\ f(\alpha^{-1}v) = f(\beta^{-1}w).$$

Corollary (i) follows directly from the theorem, and Corollary (ii) follows from Corollary (i) together with the theorem.

Any u, v have *ascent* and *descent coefficients* α and δ, for which
$$\alpha^{-1}vf(u) = 1,\ uf(\delta^{-1}v) = 1.$$

With $T_f \subset B \times B$ defined by

$$uT_f v \equiv uf(v) \leq 1,$$

reflexive since $uf(u) = 1$, f being a demand function, and with

$$S_f = \overrightarrow{T_f}\ ,$$

the transitive closure, reflexive and transitive, and so an order, evidently then

$$\rho \geq \alpha \ \Rightarrow\ \rho^{-1} v S_f u\,,\ \ \rho \leq \delta \ \Rightarrow\ u S_f \rho^{-1} v,$$

the latter implication holding provided $uf(\delta^{-1}v)$ is increasing in δ. In this way ascent goes from u to $\alpha^{-1}v$ in the order S_f, and descent goes from u to $\delta^{-1}v$.

6 Paths

A *path* $\mathcal{U} \subset B$ is described by $u = u(t)$ $(0 \leq t \leq 1)$ for which $\dot{u} = du/dt$ exists and is continuous, and $\dot{u} \nparallel u$.[10] It is taken directed between its extremities, from $u_0 = u(0)$ to $u_1 = u(1)$. It is an *integral path*, for a given demand function f, if it satisfies[11]

$$\dot{u} f(u) = 0.$$

With given $\mathcal{U}$, any continuously differentiable function $\rho = \rho(t)$ determines a further path $\mathcal{V}$, described by $v = \rho^{-1}u$, which derives from the other as its *projection*. The given path and its projection have the same initial point u_0 provided $\rho(0) = 1$.

To be considered now how the given path can be projected into an integral path with the same initial point, and later through some other point.

From $v = \rho^{-1}u$ it follows that

$$\dot{v} = \rho^{-1}(\dot{u} - \dot{\rho}\rho^{-1}u),$$

and therefore, since $\rho^{-1}uf(\rho^{-1}u) = 1$, f being a demand function, that

$$\dot{v} f(v) = \rho^{-1}(\dot{u} f(\rho^{-1}u) - \dot{\rho}),$$

so $\dot{v} f(v) = 0$ is equivalent to

$$\dot{\rho} = \dot{u} f(\rho^{-1}u).$$

A solution of this differential equation for ρ with initial condition $\rho(0) = 1$ will provide a projection $\mathcal{V}$ of $\mathcal{U}$ which is an

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[10] Here $x \parallel y$ means vectors x, y have elements in the same ratio, or $y = xt$ for $t \neq 0$, and $x \nparallel y$ is the denial.

[11] An extended definition of path is the broken path, or path-chain, a series of paths where successors are joined. It is of use later, though it could also be made the reference for some of what follows now.

integral path with the same initial point. $\mathcal{U}$ is itself an integral path if and only if $\rho(t) = 1$ for all t is already a solution.

Theorem 6 If f is a continuous demand function with bounded expansion obliquity then any path in the budget space has a unique projection which is an integral path with the same initial point.

Introducing
$$F(\rho, t) = \dot{u}(t)f(\rho^{-1}u(t)),$$
the above differential equation is
$$\dot{\rho} = F(\rho, t), \text{ with } \rho(0) = 1.$$
With $\dot{u}$ and f continuous, F is continuous in t and ρ. Then with F Lipschitz in ρ, the existence and uniqueness of a solution is assured. The Lipschitz condition requires that, for some L,
$$|F(\rho, t) - F(\sigma, t)| \leq L\,|\rho - \sigma|,$$
for all ρ, σ and t.

If f has bounded expansion obliquity so that $\sec\theta_f < \infty$, for a closed region of B that excludes the origin, then from
$$|\,f(\rho^{-1}u) - f(\sigma^{-1}u)| \leq |\rho - \sigma|\ \sec\theta_f / |u|\,,$$
in section 4, follows
$$|\,f(\rho^{-1}u) - f(\sigma^{-1}u)| \leq K\,|\rho - \sigma|\,,$$
for all u on a path in the region, and $\rho, \sigma > 0$, where
$$K = \sec\theta_f / \min|u|.$$
With $\dot{u}$ continuous on $<0, 1>$, and therefore bounded, say $|\,\dot{u}\,| \leq D$, it follows that the Lipschitz condition holds with $L = KD$, so the theorem is proved.

Corollary Under the same hypothesis, any path has a unique projection which is an integral path through a given projection of one of its points.

7 Relations

With $\rho_s^{-1}u(s)$ $(0 \leq s \leq 1, \rho_s > 0)$ as a given projection of a point, the differential equation for ρ has a unique solution with $\rho(s) = \rho_s$. Then it determines a value $\rho_t = \rho(t)$ for any t. This value is now a function of s, ρ_s, t and as such may be represented as $\rho_t = \mathcal{T}_{ts}(\rho_s)$, where, from the construction, $\rho_s = \mathcal{T}_{ss}(\rho_s)$.

Then also $\rho_s = \mathcal{T}_{st}(\rho_t)$, since starting now instead with the condition $\rho(t) = \rho_t$, the same differential equation solution is obtained as before with $\rho(s) = \rho_s$, since for this it happens that $\rho_t = \rho(t)$.

By similar argument, if also $\rho_s = \mathcal{T}_{sr}(\rho_r)$, then it can be concluded that $\rho_t = \mathcal{T}_{tr}(\rho_r)$. Hence we have

$$\mathcal{T}_{ts}\mathcal{T}_{sr} = \mathcal{T}_{tr}.$$

With the correspondence

$$(\mathrm{s}, \rho_s) \leftrightarrow \rho_s^{-1}u(s),$$

a binary relation $\mathcal{T}$ between points on rays of the cone projecting the path $\mathcal{U}$ is defined by

$$(s, \rho_s)\mathcal{T}(t, \rho_t) \equiv \rho_t = \mathcal{T}_{ts}(\rho_s).$$

From the observations just made this is reflexive, symmetric and transitive, and so an equivalence. The cone projecting the path $\mathcal{U}$ is now partitioned into equivalence classes, each described by an integral path projection of $\mathcal{U}$, and with a single representative on each ray.

This relation between points in rays projecting the path generally depends on the connecting path. Independence from the path is a condition of importance, as will appear.

8 Sequences

Let $\mathcal{U}$ now be an integral path, so $\dot{u}f(u) = 0$. A dissection of the unit interval $I = \langle 0, 1\rangle$ is any $T = (t_0, \ldots, t_k)$ with[12]

$$0 = t_0 < t_1 < \cdots < t_k = 1.$$

The intervals are (t_{i-1}, t_i) $(i = 1, \ldots, k)$, with minimum length

$$\tau = \min_i(t_i - t_{i-1})$$

which defines the norm of T.

Let the rationals in I be given some enumeration, excluding $0, 1$. Let T_k, to be distinguished as the k-th rational dissection, be the dissection obtained by introducing the first k in this enumeration, so

$$T_{k-1} \subset T_k\ ,\ \tau_{k-1} \geq \tau_k\ ,$$

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[12] In dealing with a broken path, or path-chain, the initial T should be the dissection that breaks it into segments that make up the chain.

τ_k being the norm of T_k. Since the rationals are dense,

$$\tau_k \to 0 \ (k \to \infty).$$

With T now as any dissection of I, let

$$u_i = u(t_i), \ \dot{u}_i = (t_i - t_{i-1})^{-1}(u_i - u_{i-1}),$$

so $\dot{u}_i$ is near to $\dot{u}(t_i)$ when τ is small, since $\dot{u}$ exists and is continuous.

Let τ be so small that

$$\angle u_i, u_{i-1} < \pi/2 - \theta_f.$$

Then unique α_i, δ_i are defined for which $\alpha_0 = 1, \delta_0 = 1$ and

$$\alpha_i^{-1} u_i f(\alpha_{i-1}^{-1} u_{i-1}) = 1, \ \ \delta_{i-1}^{-1} u_{i-1} f(\delta_i^{-1} u_i) = 1. \qquad 8.1$$

Let

$$\dot{\alpha}_i = (t_i - t_{i-1})^{-1}(\alpha_i - \alpha_{i-1}), \ \dot{\delta}_i = (t_i - t_{i-1})^{-1}(\delta_i - \delta_{i-1}).$$

Then, since $uf(u) = 1$, 8.1 is equivalent to

$$\dot{\alpha}_i = \dot{u}_i f(\alpha_{i-1}^{-1} u_{i-1}), \ \dot{\delta}_i = \dot{u}_i f(\delta_i^{-1} u_i), \qquad 8.2$$

together with $\alpha_0 = 1, \delta_0 = 1$.

Thus, given a dissection T with small τ, functions $\alpha(t), \delta(t)$ have been defined for $t \in T$ with values

$$\alpha(t_i) = \alpha_i, \ \delta(t_i) = \delta_i.$$

For any other dissection T' with norm τ', which is a refinement of T, so $T' \supset T$ and $\tau' \leq \tau$, the corresponding functions $\alpha'(t), \delta'(t)$ defined for $t \in T'$ are also defined for $t \in T$. Moreover, again since $T' \supset T$, by Theorem 4.2, Corollaries, we have

$$\delta(t) \leq \delta'(t), \ \alpha'(t) \leq \alpha(t), \qquad 8.3$$

for all $t \in T$.

Now take T to be the k-th rational dissection T_k with k large so that τ_k is small and the functions are well defined. Also, given any rationals $s, t \in I$, we have $s, t \in T$ when k is sufficiently large.

By construction of the functions α, δ and definition of the relation S_f,

$$\alpha_s^{-1} u_s \, S_f \, u_0 \, S_f \, \delta_t^{-1} u_t,$$

whence by transitivity,

$$\alpha_s^{-1} u_s \, S_f \, \delta_t^{-1} u_t.$$

It follows from here, with Houthakker's axiom, that

$$\delta_t^{-1} u_t f(\alpha_s^{-1} u_s) > 1, \tag{8.4}$$

unless

$$f(\delta_t^{-1} u_t) = f(\alpha_s^{-1} u_s).$$

In particular with $s = t$, because $uf(u) = 1$, it appears that

$$\delta_t \leq \alpha_t. \tag{8.5}$$

From 8.3, as k increases δ_t is non-decreasing and α_t non-increasing. But from 8.5 both are bounded. Since any bounded monotone sequence is convergent, it follows that both are convergent, say

$$\delta_t \to \overline{\delta}_t,\ \alpha_t \to \overline{\alpha}_t\ \ (k \to \infty), \tag{8.6}$$

and moreover,

$$\delta_t \leq \overline{\delta}_t \leq \overline{\alpha}_t \leq \alpha_t. \tag{8.7}$$

These limits determine functions $\overline{\alpha}(t), \overline{\delta}(t)$ defined on all rational points.

For rational t, and k so large that $t \in T_k$, let t' be the predecessor of t in $T = T_k$, so $t' \to t$ $(k \to \infty)$. Let $\Delta t = t - t'$, so $\Delta t \to 0$ $(k \to \infty)$, and let

$$\begin{aligned}\Delta\alpha(t) &= \alpha(t) - \alpha(t'),\\ \Delta\delta(t) &= \delta(t) - \delta(t'),\\ \Delta u(t) &= u(t) - u(t').\end{aligned}$$

Then 8.2 shows that

$$\Delta\alpha(t) = \Delta u(t) f(\alpha(t')^{-1} u(t')),\ \Delta\delta(t) = \Delta u(t) f(\delta(t)^{-1} u(t)).$$

But, because $u(t)$ is continuously differentiable,

$$\Delta u(t)/\Delta t \to \dot{u}(t)\ \ (k \to \infty).$$

Then, because $u(t)$ and $f(u)$ are continuous and $\alpha \to \overline{\alpha},\ \delta \to \overline{\delta}$ it follows that

$$\lim_{k \to \infty} \Delta\alpha(t)/\Delta t, \quad \lim_{k \to \infty} \Delta\delta(t)/\Delta t$$

exist and are given by

$$\dot{u} f(\overline{\alpha}^{-1} u),\ \dot{u} f(\overline{\delta}^{-1} u)$$

evaluated at t.

It is concluded from this that $\overline{\alpha}, \overline{\delta}$ defined at all rational points have continuous extensions defined at all points, and then that these extensions are continuously differentiable and satisfy the differential equation

$$\dot{\rho} = \dot{u} f(\rho^{-1} u),$$

with the condition $\rho(0) = 1$. But this differential equation with this condition has just one solution, which since $\mathcal{U}$ is an integral path must coincide with $\rho(t) = 1$, for all t. It follows that

$$\overline{\alpha}(t) = 1,\ \overline{\delta}(t) = 1 \qquad 8.8$$

for all t. This with 8.7 enables 8.5 to be replaced by

$$\delta_t \leq 1 \leq \alpha_t.$$

Letting $k \to \infty$ in 8.4, in view of 8.6 and 8.8, it appears that

$$u(t)f(u(s)) \geq 1. \qquad 8.9$$

9 Surfaces

For a demand function f under the current assumptions, consider the relation $I_f \subset B \times B$ of integral path connection, where uI_fv mean u, v are connected by an integral path. Then for

$$E_f = \overrightarrow{I_f}\ ,$$

the transitive closure, identical with the chain extension, uE_fv means u, v are connected by a chain of integral paths, or broken path where the segments are integral paths.

Theorem 9.1 Subject to Houthakker's axiom,

$$uE_f\rho^{-1}u \ \Rightarrow\ \rho = 1.$$

Let $\mathcal{U}$ be the path making the connection, and suppose, if possible, that $\rho > 1$.

With any dissection of $\mathcal{U}$ there corresponds a descending sequence that starts from u and terminates in some $\sigma^{-1}u$, in which case $uS_f\sigma^{-1}u$. If the dissection is fine enough this sequence is close to $\mathcal{U}$, in particular $\sigma^{-1}u$ is close to $\rho^{-1}u$, to make $\sigma > 1$.

Now with $\sigma > 1$ we have $\sigma^{-1}uf(u) = \sigma^{-1} < 1$, and hence $\sigma^{-1}uS_fu$, by definition of S_f. But also $f(u) \neq f(\sigma^{-1}u)$ and so, by Houthakker's axiom, also $\sim uS_f\sigma^{-1}u$, making a contradiction, so $\rho > 1$ is impossible. Similarly $\rho < 1$ is impossible. QED

Houthakker contributed the essential idea of this argument, which applies to the case $n > 2$. It is dispensible in the case $n = 2$, treated by Samuelson, when he introduced the method with ascending and descending sequences approaching an integral curve. However, both work with utility surfaces in the commodity space without regard for the possibility that these could be degenerate, even just single points. The discussion here takes place instead in the budget space.

Corollary (i) Under the same hypothesis, the relation E_f is reflexive, symmetric and transitive, and so an equivalence.

From here together with the theorem, the classes in this equivalence have a unique representative on any ray. The relation being continuous, this shows

Corollary (ii) The classes of the equivalence E_f are surfaces cutting each ray just once.

The surface $E_f u$ through any point u can be constructed by taking any path $\mathcal{U}$ through u and projecting it into an integral path with u as initial point. This integral path is then a path in the surface. Every element du of such a path, and so of the surface, then satifies $duf(u) = 0$, so we have an integral surface of this differential equation, now to be distinguished as an *integral surface* of the demand function *f*.

Corollary (iii) The classes of E_f are integral surfaces of the differential equation $duf(u) = 0$.

The budget space is now partitioned into classes, provided by the integral surfaces. It is an ordered partition, since the surfaces are ordered by their intercepts on any given ray, the order being independent of the ray chosen.

The surfaces can then be represented as level surfaces of a function ψ, with value $e = \psi(u)$ determined by the point $e^{-1}b$ where the surface through u cuts the ray through b. By taking a linear path $u(t) = u + t(b - u)$, or any other with u, b as initial and final points, the solution of

$$\dot{\rho} = \dot{u}f(\rho^{-1}u), \text{with } \rho(0) = 1,$$

determines $\psi(u) = \rho(1)$.

Theorem 9.2 $\qquad uE_f v \Rightarrow uf(v) \geq 1.$

If uE_fv then u, v are connected by an integral path, say $w(t)$ with $w(0) = u$, $w(1) = v$. But then, as already concluded,

$$w(s)f(w(t)) \geq 1$$

for all s and t, in particular for $s = 0,\ t = 1$.

An $x \in \Omega^n$ is a *support* of a set $S \subset \Omega_n$ at a point $u \in S$ if

$$ux = 1,\ v \in S \Rightarrow vx \geq 1.$$

Thus S^+ can denote the set of supports of S.

A set S is *orthogenous* if

$$u \in S\ \wedge u \leq v \Rightarrow v \in S,$$

and *orthoconvex* if also it is convex. The *orthoconvex closure* of any set S is the smallest orthoconvex set containing it.

For any set S, the set of supports S^+ is closed orthoconvex, and S^{++} is the closed orthconvex closure of S.

Theorem 9.3 The integral surfaces are orthoconvex.

That is, boundaries of orthoconvex regions. For any u, consider the region of w for which

$$uE_fv \Rightarrow wf(v) \geq 1.$$

This is convex, and by the previous propositions it contains the integral surface E_fu through u. But since $vf(v) = 1$ it contains E_fu, described by all v such that uE_fv, on its boundary. Obviously every ray cuts this boundary just once, and since every ray cuts E_fu just once it follows that E_fu coincides with this boundary. Also, for all u, $f(u)$ is a support of E_fu at u.

Theorem 9.4 The integral surfaces are smooth.

This is because an integral surface is convex and every continuously differentiable path projects into it, that is, cuts rays which cut it, in a continuously differentiable path. Equivalently, again because the surfaces convex, each surface has a unique support at every point.

Theorem 9.5

$$uE_fv\,.\Rightarrow.\,uf(v) = 1 \Leftrightarrow f(u) = f(v).$$

For if $uE_f v$ then u, v lie on an integral surface I, and $f(u)$,$f(v)$ are unique supports to I at u, v. But I being convex, $uf(v) = 1$ means $f(v)$ is a support to I also at u, which is impossible unless $f(u) = f(v)$.

10 Utility

The integral surfaces cut every ray just once, and are ordered by the order of the points in which they cut any one ray, which is the same for all rays. Since there is just one through every point in the budget space, they constitute a partition of that space. Thus a completely ordered partition of the budget space is obtained defining a complete order B_f in the budget space $\mathcal{B}$, in which E_f is the relation of equivalence. The sets $B_f u$, uB_f contain all sequences ascending and descending from u, which shows that $S_f \subset B_f$. Moreover, they are identical with the limits of points in such sequences, by the argument in Section 7, so they are identical with the closures of $S_f u, uS_f$. The budget space $\mathcal{B}$ being connected and separable, Debreu's Theorem now in any case assures the existence of a continuous function $\psi : \mathcal{B} \to \Omega$ such that

$$uB_f v \Leftrightarrow \psi(u) \geq \psi(v).$$

Taking any fixed $e \in \mathcal{B}$, for any $u \in \mathcal{B}$ there is unique $t \geq 0$ for which $uE_f te$. Define $\psi(u) = t$. Then $\psi(u)$ is such a function. Evidently then the relation B_f is in fact closed, and is the closure of S_f.

Now introduce a relation $C_f \subset \mathcal{C} \times \mathcal{C}$ in the commodity space $\mathcal{C}$ with the definition

$$xC_f y \equiv (\vee\, vy \leq 1)(\wedge\, ux \leq 1)uB_f v.$$

Also introduce a function $\phi : \mathcal{C} \to \Omega$ in the commodity space by

$$\phi(x) = \min\{\psi(u) : ux \leq 1\}.$$

Then because of continuity and orthoconvexity of level sets[13],

$$xC_f y \Leftrightarrow \phi(x) \geq \phi(y),$$

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[13] Concerning the direct and indirect utility functions and order relations, defined in the commodity and budget spaces, their necessary properties and the relation between them, an account is in my *Demand Functions and the Slutsky Matrix*, Princeton University Press, 1980, Chapter IX, Section 1; also in *Logic of Choice and Economic Theory*, Clarendon Press, Oxford, 1987, Part II, Chapter 5, pp 156-72.

and B_f, ψ are recovered from C_f, ϕ by

$$uB_f v \equiv (\vee\, ux \leq 1)(\wedge\, vy \leq 1) xC_f y,$$
$$\psi(u) = \max\{\phi(x) : ux \leq 1\}.$$

Moreover, because the sets uB_f are smooth-orthoconvex, the sets $C_f x$ are round-orthoconvex, that is, any support has a unique contact. But the unique support to uB_f at u is $f(u)$. It follows that u is a support to $C_f f(u)$ at $f(u)$ which has contact only at $f(u)$. This shows that C_f is a relation, and $\phi(x)$ a function, which validates the condition H_f^*.

Now for the main conclusion:

Theorem 10 If a demand function f is continuous and $\theta_f < \pi/2$ then Houthakker's condition K_f^* implies f has a utility ϕ where

$$\phi(x) = \min\{\psi(u) : ux \leq 1\},$$

and for all $u > 0$, $\psi(u) = \rho(1)$ where, with b fixed and arbitrary, and $u(t) = u + t(b - u)$, $\rho(t)$ is a solution of

$$\dot{\rho} = \dot{u} f(\rho^{-1} u), \text{with } \rho(0) = 1.$$

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