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A Representation Theorem for Choices
under Risk and Uncertainty

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I. INTRODUCTION

Expected utility theory combines linearity in probabilities and a utility function, which is either concave or convex if a decision-maker is risk averse or seeking. However, maximization of the expected utility as a criterion of choice among alternatives involving risk fails to explain the existence of both insurance and lotteries. In two seminal papers Friedman and Savage (1948) and Markowitz (1952) proposed a S-shaped utility function for rationalizing the purchase of both insurance and lottery tickets that contradict the behavior stated in the standard von Neumann and Morgenstern theory of expected utility. By the S-shaped utility function an individual can simultaneously exhibit both risk seeking and risk aversion. A lot of models have been introduced to explain the S-shaped utility function amongst others Kahneman and Tverrsky (1979), Luce and Fishburn (1991), Tversky and Kahneman (1992), Wakker and Tversky (1993), Tversky and Wakker (1995).

The paper generalizes an approach introduced by Wakker and Tversky (1993) based on Choquet expected utility. The starting point is a decision making process involving “Knightian” uncertainty about consequences. Since an outcome depends on which state of the world occurs, I shall treat events and outcomes as synonymous. Instead of considering acts, namely functions from states of the world to consequences, I assume that the objects of choice are prospects (gambles) defined with respect to a reference point, like in prospect theory. This paper argues that the decision-maker exhibits quite different attitudes with respect to outcomes involved in a decision process. I distinguish among three classes of outcomes: a broad group of outcomes with both high and reliable probabilities and two groups of either catastrophic, or very attractive, outcomes with both low and unreliable probabilities.

Standard representation theorems used to formalize the utility of an action may not be fully satisfactory, since they could misrepresent catastrophic and attractive events, and “the observed asymmetry is far too extreme to be explained by income effects or by decreasing risk aversion” (Khaneman and Tversky, 1992, p.298). It has been shown formally that under uncertainty the framework based on expected utility maximization is insensitive to small probability outcomes, meanwhile “... experimental evidence shows that humans treat choices under uncertainty somewhat differently from what the Von-Neumann Morgernstern axioms would predict” (Chichilnisky, 1996) moreover, it has been agreed that “people are limited in their ability to comprehend and evaluate probabilities of extreme outcomes, highly unlikely events may be ignored or else over estimated and difference between high probability and certainty is either neglected or exaggerated” (Kahneman and Tversky, 1979, p.282).

This paper derives a representation theorem, which is based on the distinction between risky and uncertain outcomes. The representation theorem is based on Choquet expected utility and it is supposed to describe how the decision-maker orders prospects.

II. HARD UNCERTAINTY AND CAPACITY

Like in prospect theory, it is assumed that the decision-maker takes into account changes in wealth or welfare, rather than wealth and welfare. Since I consider real outcomes, for the sake of simplicity, I assume that either the current income or wealth (*status quo*) are good proxies to define the reference point. Stated the reference point, I define relative outcomes as gains and losses and distinguish among three classes of outcomes: customary, catastrophic and windfall. By customary outcomes I mean here the outcomes not 'very far' from the reference point. The customary outcomes (familiar world) are related to the subset of events the probability of which is considered by the decision-maker to be 'sufficiently high' and reliable, on the basis of her/his own personalized life experience. By catastrophic and windfall outcomes (unfamiliar world) I represent outcomes related to events the probability of which is perceived to be 'low' and not fully reliable.[3]

The decision-maker is assumed to face hard or Knightian uncertainty, which is represented by means of a non-additive measure, or capacity, on the set of all events. Hard uncertainty can arise from attitude towards ambiguous events, e.g. Ellsberg paradox, omitted states of the world, and misspecification of the space state.

The decision-maker has well-defined risk and uncertainty attitude, but unlike the standard framework, the capacity is strictly non-additive on events related to catastrophic losses and windfall gains and linear on events related to customary outcomes. As a result, the decision-maker gives a larger decision weight to events closer to his experienced ordinary world than extreme, very far, events. The decision-maker perceives to face genuine uncertainty with respect to catastrophic losses and windfall gains, but she/he is uncertainty neutral across the customary outcomes.[4]

Let $\Omega=\{\omega_1,\ldots,\omega_n\}$ be a non empty finite set of states of the world and let $S=2^{\Omega}$ be the set of all events. The decision-maker chooses a prospect in the set F. In particular a prospect $f\in F$ is a function assigning a consequence to each state, $f:\Omega\rightarrow C$, the set of consequences. A function $\mu:S\rightarrow R$ is a capacity, or a non-additive measure, if it has the following characteristics:

- $\mu(\emptyset)=0$, $\mu(\Omega)=1$ (it is normalized);
- $\forall A,B\in S$ such that $A\subset B$, $\mu(A)\leq\mu(B)$ (it is monotone)

[3] Reliable or unreliable mean that probabilities are sure or unsure in the Savage's sense, that is they express a general conception of ambiguity.
[4] It has been reported some evidence that attitude toward ambiguity are roughly uncorrelated with attitudes toward risk.

A capacity is convex (concave) if for all $A,B\in S$ such that $A\cup B\in S$ and $A\cap B\in S$ $\mu(A\cup B)+\mu(A\cap B)\geq (\leq)\ \mu(A)+\mu(B)$. It is superadditive (subadditive) if $\mu(A\cup B)\geq(\leq)\ \mu(A)+\mu(B)$ for all $A,B\in S$ such that $A\cup B\in S$ and $A\cap B=\varnothing$. A capacity is additive or a probability if for all $A,B\in S$ such that $A\cup B\in S$, and $A\cap B=\varnothing$, $\mu(A\cup B)=\mu(A)+\mu(B)$.

Given an utility function u, such that $u{:}S\rightarrow R$, a capacity μ on S and a set of comonotonic prospects $X\subseteq F$, such that $x,\ y\in X$ are comonotonic[5] if and only if there are no $\omega_1,\omega_2\in\Omega$, so that $x(\omega_1)>x(\omega_2)$ and $y(\omega_1)<y(\omega_2)$, the Choquet integral permits the evaluation of the Choquet expected utility[6].

In such a way, a preference order on the set of the decision maker's feasible prospects is represented by a utility function, unique up to a positive linear transformation.

The decision-maker expresses hard uncertainty aversion (preference) if she/he "assigns larger probabilities to states when they are unfavorable, than when they are favorable" (Wakker 1990), that is if his non-additive measure is convex (concave).[7] Hence, the convexity (concavity) of the capacity, that implies superadditivity (subadditivity) of the Choquet integral, captures the decision-maker's attitude towards hard uncertainty.[8] As a consequence a pessimist (hard uncertainty averse) decision-maker over-weights the worst outcome, on the contrary an optimist (hard uncertainty seeking) decision-maker over-weights the best consequence.

Given a preference relation $\geq$ on the set of prospects, the quintuple $\{\Omega,S,X,C,\mu\}$ is a decision making process under uncertainty and the outcomes across all states are defined as a prospect.

The decision-maker orders each prospect from the worst outcome to the best one. Catastrophic losses are represented by very large loss outcomes and windfall gains are expressed by very large gain outcomes. Customary consequences are represented by positive and negative outcomes. Hence, the decision-maker decomposes the set of all consequences into three mutually exclusive and exhaustive subsets. I assume that on the subset of catastrophic losses the decision-

[5] Roughly speaking, two actions $x,y\in X$ are comonotonic if they induce the same favorable state ordering and the same permutation. The above statement means that none of prospects in X order states, with respect to consequences, in a contradictory way.
[6] The proper integral for the non additive measures is the Choquet integral, originally due to Choquet (1954) and discussed in Schmeidler (1986).
[7] Schmeidler (1989) points out that smoothing or averaging utility distribution and the convexity of preference ordering are equivalent to the convexity of the capacity. In fact decision-maker is uncertainty averse if for any three actions $x,y,z\in X$ and $\alpha\in[0,1]$, such that $x\geq z$ and $y\geq z$, then $\alpha x+(1-\alpha)y\geq z$.
[8] See Chateauneuf 1991.

maker is optimistic, on the second subset of customary outcomes the decision-maker is neutral with regard to uncertainty, and on the subset of windfall gains the decision-maker is pessimistic.[9]

III. A REPRESENTATION THEOREM

Before formulating the representation theorem I now introduce some necessary axioms.

- A1. The preference relation $\geq$ is a complete and transitive, that is a weak order, on the set of prospects;
- A2. The preference relation $\geq$ is continuous ;
- A3. Prospects in X satisfy comonotonic tradeoff consistency.

A.1 and A.2 are usual axioms. A.3, which is due to Wakker (1989), argues that:
comonotonic tradeoff consistency is satisfied if there are no outcomes $a,b,a',b' \in C$ such that $ab \geq a'b'$ and $ab \leq a'b'$ hold.[10]

A capacity μ is associated to each event $A \in S$. For each $x \in X$, consequences are ordered from the worst outcome to the best one and the set of consequences is partitioned in three closed intervals, such that $[x_1^-, x_i^-]$ is the subset of catastrophic losses, $[x_{i+1}^-, x_j^+]$ is the subset of customary outcomes both losses and gains, and $[x_{j+1}^+, x_n^+]$ is the subset of windfall gains. Since the decision-maker is optimistic on catastrophic losses, neutral on customary outcomes and pessimistic on windfall gains, the capacity μ is non-additive on S, but it is locally concave on the subset of catastrophic losses, it is additive, that is μ is a probability, on customary consequences and locally convex on the subset of windfall gains.

THEOREM. Let u be a utility function, then for every $x,y \in X$,

$$x \geq y \Leftrightarrow U(x) \geq U(y) \qquad (1)$$

where U is defined as

$$U(x) \equiv \int u(x^-(.))d\mu^- + \int u(x(.))d\mu + \int u(x^+(.))d\mu^+ \qquad (2)$$

The integrals on the right hand side are Choquet integrals with respect to a capacity.

Being $\mu^-(A)=1-\mu^+(A^C)$ the conjugate or dual of μ, then μ^- is subadditive (superadditive) if μ is superadditive (subadditive) and $\mu^- = \mu^+$ if μ is additive (probability), by asymmetry of Choquet integral

$$\int u(x^-(.))d\mu^- = -\int u(x^+(.))d\mu^+ \qquad (3)$$

[9] It is reported that an agent has a different behavior with respect to low probability bets with gain payoffs and low probability bets with loss payoffs. See Heath and Tversky (1991).
[10] Tradeoff consistency may be considered an expression of the comonotonic independence in a preference ordering.

and U can be written as[11]

$$U(x) \equiv -\int u(x^{+}(.))d\mu^{+} + \int u(x(.))d\mu + \int u(x^{+}(.))d\mu^{+} \equiv (-\int u(x^{+}(.))d\mu^{+}) + \int u(x(.))d\mu + \int u(x^{+}(.))d\mu^{+} \quad (4)$$

or $\quad U(x) \equiv \int u(x(.))d\mu$ (Choquet integral with respect to μ).

Yet the decision-maker gives a quite different decision weight to ordinary and extreme outcomes and let $\beta \in [0,1]$ be the confidence index, that is the subjective confidence in the customary outcomes, and $(1-\beta)$ the residual confidence in extreme outcomes, then the expected value of x can be written as

$$U(x) \equiv \beta \int u(x(.))d\mu + (1-\beta)\,[\int u(x^{+}(.))d\mu^{+} + (-\int u(x^{+}(.))d\mu^{+})] \quad (5)$$

When the subjective confidence overlaps the decision weights defined by Choquet integral, the decision-maker is supposed to obtain the standard Choquet Expected Utility

IV. CONCLUSIONS

This representation theorem is supposed to express simultaneously the decision-maker attitude with respect to both risk and uncertainty by employing a unique capacity. Moreover the representation theorem is supposed to overcame ‘insensitivity to small probability events’ emphasizing the reliability of ordinary consequences in a prospect by Choquet permutation of outcomes and confidence index. The representation theorem is also compatible with conclusions defined by Chichilnisky (1996).

[11] Proof of the theorem with both an additive measure and a non-additive measure is in Wakker and Tversky 1993.

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