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On truncation "theorems"

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1. Introduction*

The fact that the internal rate of return is not unique for any investment project, seriously compromises the relation made by Keynes between aggregate investment and interest rate. This problem had been recognised for some time[(1)], and it subsequently provided the major incentive for an attempt to render <u>necessarily</u> unique the internal rate of return by linking it to an economic-technological hypothesis known as projects truncatability. Apart from this historical fact, the results obtained have also found application in other sectors of economic theory[(2)].

The above attempt was initiated in 1959 by C.S. Soper |23|, T.H. Silcock |22|, J.F. Wright |25| and P.H. Karmel |12|. Ten years later it was resumed with new arguments by K.J. Arrow - D. Levhari |2|, followed by (in chronological order) the contributions of C.J. Norstrom |15|, J.S. Flemming - J.F. Wright |6|, J.R. Hicks |11|, D.M. Nuti |16|, C. Filippini - L. Filippini |5|,

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(*) I wish to thank Proff P. Tani and E. Zaghini for stimulating conversation and for their useful comments on the previous drafts of this paper.

(1) See J.D. Pitchford - A.J. Hagger |18|, p. 600; P.H. Karmel |12|, p. 432 and C. Filippini - L. Filippini |5|, pp. 5-6.

(2) See J.R. Hicks |9, 10, 11|, D.M. Nuti |16, 17|, C. Filippini - L. Filippini |5|, E. Burmeister |3|, J. Eatwell |4|, P. Puccinelli |19|, A. Andretta |1|, P. Tani |24|.

J. Eatwell | 4 | , A. Sen | 21 | , S.A. Ross - C.S. Spatt - P.H. Dybvig | 20 | , et al.

These works present two lines of thought characterized by two different approaches. The most representative contribution of the first approach is Karmel's[3], while the most representative contribution of the second approach comes from Arrow and Levhari. In this article we shall therefore refer to these approaches as to Karmel's and to Arrow and Levhari's.

As is well known, both approaches have received criticism of an external nature, with substantial reference to the common assumption of projects truncatability. This criticism will be summed up in our Conclusion, while the fundamental thesis of this article is that neither approach is free from much more serious criticism of an internal nature. We shall show that in fact Karmel's approach rests on a behavioural hypothesis which is quite unjustifiable from the point of view of economic rationality. Arrow's and Levhari's approach, on the other hand, is implicitly based on a redefinition of the internal rate of return which, when rendered fully explicit, appears devoid of any consistent economic significance.

This article has two minor aims besides. The first is to offer a simpler proof of Karmel's theorem, based on a recent generalization

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(3) We reaffirm the opinion previously expressed in S. Gronchi | 8 | that we cannot share the judgment of all those authors | 2, 4, 15, 16, 19 | who assimilate Karmel's (later) contribution to that of Soper. Soper's proposed criterion for optimal truncation is radically different from Karmel's. Moreover, Soper's analysis suffers from several serious errors, largely invalidating his conclusions. These errors are quite correctly and accurately pointed out by Karmel.

of Soper's sufficient condition for the uniqueness of the internal rate of return[4]. The second aim is to rectify some imperfections which, in our opinion, invalidate Nuti's | 16 | proof of Arrow and Levhari's theorem[5] (we shall prove this theorem by 'adapting' Nuti's proof).

2. Definitions and basic concepts

For the purpose of this article, it is useful to accept a very general definition of an investment project. Therefore we define a project as a time profile (vector) of expected net outputs, $(a_0 , . .. , a_n)$, such that $a_0 < 0$.

For any project thus defined we assume the hypothesis of perfect truncatability: we assume that in each period of the physical life of a project a pure and simple operation can be made, both technologically and economically, to 'discard' net outputs expected for all successive periods.

Within class A of all feasible projects, consider class A of projects $(a_0 , \dots , a_n)$, such that $a_i > 0$ for at least one $i > 0$ and such that $a_n \neq 0$. Clearly each project included in

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(4) This generalization is presented in S. Gronchi | 7 | . The proof we offer here of Karmel's theorem is implicit in the arguments developed in S. Gronchi | 8 | .

(5) Traces of Nuti's proof can be found in J.F. Wright | 25 | . Besides Nuti and the authors themselves, Arrow and Levhari's theorem has been proved by C. Filippini - L. Filippini | 5 | . It has also been proved by J.S. Flemming - J.F. Wright | 6 | , by A. Sen | 21 | and by S.A. Ross - C.S. Spatt - P.H. Dybvig | 20 | as a corollary of more general theorems.

class $\underline{A}$ - A is a replica of a project included in A , with the sole redundant addition of one or more net outputs equal to zero, otherwise it is a project whose net outputs are all non-positive. For these reasons, class A can be defined as the class of economically relevant projects. This article deals only with this class of projects. Yet, not all truncated projects one can obtain from a project included in A necessarily belong to A : on the contrary, some of these belong in general to class $\underline{A}$ - A . It follows that if we restrict the definition of 'project' to include only those projects contained in A , not all truncations that can be obtained from these, may be termed projects. This explains the generality of the definition we must accept.

For each possible project we finally define as internal rate of return an interest rate which makes the present value function (defined within the economically significant range $(-1 , +\infty)$) equal to zero. Of course, there are, at most, n internal rates of return associated with a project $(a_0 , \ldots , a_n)$.

3. Karmel's approach

Karmel's aim is to use the hypothesis of truncatability to render the internal rate of return unique for any project contained in class A of economically relevant projects. To achieve this aim Karmel 'integrates' the technological-economic hypothesis of truncatability with a behavioural hypothesis concerning the criterion by which the investor chooses the economic life of a project. The hypotehesis in question may be formulated thus: given that for each (hypothetical) duration of a project there may exist zero, one or more internal

rates of return, having calculated them all, the investor chooses that particular duration with which the maximum internal rate of return among those calculated is associated. This hypothesis ultimately 'produces' the uniqueness of the internal rate of return for any project contained in A since it can be proved that no other internal rate of return is associated with that duration of a project with which the maximum internal rate of return is associated.

Let us now analyse in detail Karmel's contribution. Given a project $A := (a_0 , \ldots , a_n)$ contained in class A of economically relevant projects, consider the vectors:

$$\text{(I)} \qquad A_i := (a_0 , \ldots , a_i) \qquad (i = 0 , \ldots , n) \quad .$$

In accordance with our accepted definition of a project, vectors (I) are projects (even if they do not all necessarily belong to class A to which the given project belongs).

Let

$$R_i \qquad (i = 0 , \ldots , n)$$

be the sets of internal rates of return associated with projects (I) and let:

$$R := \bigcup_{i=0}^{n} R_i \quad .$$

Since $A \in \mathbf{A}$, therefore:

$$\text{(III)} \qquad R \neq \phi \quad .$$

If r is the interest rate and $A_i(r)$ the present value of project A_i, then clearly:

$$\text{(IV)} \quad \lim_{r \to +\infty} A_i(r) = a_0 < 0 \qquad (i = 0 , \dots , n) \quad .$$

On the other hand, the fact that $A \in \mathbf{A}$ implies that $a_i > 0$ for at least one i. It follows that, for the same i:

$$\lim_{r \to -1} A_i(r) = +\infty \quad ,$$

such that $A_i(r)$ equals zero at least once in the interval $(-1, +\infty)$, and therefore $R_i \neq \phi$.

Since (III) holds true, consider the maximum internal rate of return in R: let it be r_*.

The hypothesis of truncatability confronts the investor with the problem of choice between projects (I) and Karmel assumes that this choice is made in favour of a[6] project with which r_* is associated.

The consequences of this behavioural hypothesis on the uniqueness of the internal rate of return associated with the truncated project, are studied by Karmel in collaboration with the mathematician B.C. Rennie. In the Appendix to Karmel's article, Rennie proves the following theorem.

<u>Theorem 1</u>. If A_k is one of projects (I) with which r_* is associated, then $R_k = \{r_*\}$ (i.e. r_* is the only internal rate of return associated with A_k).

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(6) Nothing excludes more projects (I) with which r_* is associated, which explains the use of the indefinite article.

<u>Proof</u>. Instead of Rennie's cumbersome proof, let us offer a simpler one, based on the sufficient condition for the uniqueness of the internal rate of return proved in the Appendix. Given this condition[7], to prove the theorem it is sufficient to prove that:

$$\sum_{j=0}^{i} a_j \, (1+r_*)^{i-j} \leq 0 \qquad (i = 0 , \dots , k-1) \quad .$$

Let us suppose that this were not true, and that therefore for at least one i:

$$\text{(V)} \quad \sum_{j=0}^{i} a_j \, (1+r_*)^{i-j} > 0 \quad .$$

Since $r_* > -1$ (such that $1+r_* > 0$), (V) implies:

$$\text{(VI)} \quad A_i(r_*) > 0 \quad .$$

Since (IV) holds true, (VI) implies, in its turn, that $A_i(r)$ changes signs between the extremes of the interval $(r_* , +\infty)$ and in consequence will admit at least one root greater than r_*. But this contradicts the fact that r_* is maximum within set R. It follows that no $i = 0 , \dots , k-1$ can yield (V). <u>Q.E.D.</u>.

The central position held by Theorem 1 in Karmel's analysis is quite evident. In effect it states that the aim of rendering the internal rate of return unique for any project contained in $\mathbf{A}$ is <u>spontaneously</u> achieved by the investor, assuming (as it is natural)

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(7) The condition in question would not apply if $k = 0$. But the eventuality is precluded by the fact that $A_0(r) = a_0$, hence $R_0 = \phi$.

that he simply requires maximum convenience. We shall verify (Section 5) whether or not the rational modalities of such a requirement are really those claimed by Karmel.

4. Arrow and Levhari's approach

Like Karmel, Arrow and Levhari also try to use the hypothesis of the truncatability of projects to render the internal rate of return unique for any project contained in class **A** of economically relevant projects.

We hold that their approach implicitly contains a true and proper redefinition of the internal rate of return; on the 'excellence' of this redefinition we shall pass judgment in the following section.

The way in which we shall present the contribution of Arrow and Levhari is in line with our persuasion; hence the above redefinition will be explicitly recognized.

In the first place, given a project

$$A := (a_0 , \dots , a_n) ,$$

and also its truncations

$$A_i := (a_0 , \dots , a_i) \qquad (i = 0 , \dots , n)$$

and the respective present values:

$$\text{(VII)} \quad A_i(r) \qquad (i = 0 , \dots , n) ,$$

we define as maximum present value of A the following function of r :

$$\text{(VIII)} \quad A(r) := \text{Max} \left\{ A_i(r) : i = 0 , \dots , n \right\} .$$

Since functions (VII) are defined in the domain $(-1 , +\infty)$, function (VIII) must be understood as defined in the same domain.

We shall now prove the following fundamental theorem (known as truncation theorem).

Theorem 2. Given a project $A \in$ **A** , its maximum present value $A(r)$ is a continuous function such that:

$$\lim_{r \to -1} A(r) = +\infty$$

and that:

$$\lim_{r \to +\infty} A(r) = a_0 ;$$

moreover $A(r)$ is monotonically decreasing, yet there can exist a value $\bar{r}$ of r such that $A(r) = a_0$ for $r \in [\bar{r} , +\infty)$.

Proof. Put $A := (a_0 , \dots , a_n)$, let us begin with some observations of a preliminary nature. The first observation is that, considering polynomials:

$$\text{(IX)} \quad A_i(y) := \sum_{j=0}^{i} a_j y^j \qquad (i = 0 , \dots , n) ,$$

each function (VII) assumes at (any) point r the value which the corresponding polynomial (IX) assumes at point $y := (1 + r)^{-1}$. In consequence, considering function:

$$\text{(X)} \quad A(y) := \text{Max} \left\{ A_i(y) : i = 0 , \dots , n \right\} ,$$

function (VIII) assumes at (any) point r the value which function (X) assumes at point $y := (1+r)^{-1}$. The second observation is as follows. Let I be the set of instants i's such that $a_i \neq 0$ and let I' be the set of all other instants[8]. From polynomials (IX) we extract:

$$\text{(XI)} \quad A_i(y) \qquad (i \in I) \quad .$$

Quite clearly for each $i' \in I'$ there exists an $i \in I$ such that:

$$\text{(XII)} \quad A_{i'}(y) = A_i(y) \quad .$$

It follows that, considering function:

$$B(y) := \text{Max}\,\{A_i(y) : i \in I\} \, ,$$

we have:

$$A(y) = B(y) \quad .$$

Since number $y := (1+r)^{-1}$ decreases continuously from $+\infty$ to 0 while number r increases from -1 to $+\infty$, we deduce from the above two observations that, to prove the theorem, it is sufficient to prove the following two points:

(i) function $B(y)$ is continuous in the interval $(0, +\infty)$ and moreover:

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(8) Since $A \in \mathbf{A}$, therefore $a_n \neq 0$ (besides $a_0 \neq 0$), necessarily $0 \in I$ and $n \in I$.

$$\text{(XIII)} \quad \lim_{y \to +\infty} B(y) = +\infty$$

$$\text{(XIV)} \quad \lim_{y \to 0} B(y) = a_0 \quad ;$$

(ii) function $B(y)$ is monotonically increasing within the interval $(0, +\infty)$, yet there can exist a positive value $\bar{y}$ of y such that $B(y) = a_0$ for $y \in (0, \bar{y}]$.

Let us first prove point (i). The continuity of $B(y)$ is self-evident, $B(y)$ being an outer envelope of continuous functions. To demonstrate (XIII), it is sufficient to note that, since $A \in \mathbf{A}$, there exists at least one $i \in I$ such that $a_i > 0$, with the consequence that for this same i :

$$\lim_{y \to +\infty} A_i(y) = +\infty \quad .$$

To demonstrate (XIV), it is sufficient to note (even more simply) that for each i :

$$\lim_{y \to 0} A_i(y) = a_0 \quad .$$

We shall now prove point (ii). To this end, we shall prove two lemmas.

<u>Lemma 1</u>: the positive values of y are finite in number such that:

$$\text{(XV)} \quad B(y) = A_i(y)$$

for more than one $i \in I$. To prove the lemma, form all possible pairs of polynomials (XI) and for each pair consider the difference between the polynomials which belong to the pair. Note that the pairs, and therefore the respective differences, are finite in number. Note also that each difference is in turn a polynomial, and in consequence it admits a finite number of positive roots (at most equal to its degree). It is thus possible to affirm the following: the positive values of y are finite in number, such that at least one difference is null. Let p be this number. At this point, to prove the lemma, it is sufficient to note that the positive values of y, such that (XV) is true for more than one $i \in I$, cannot be more than p (see Fig 1 which shows a case in which polynomials (XI) are three and $p = 4$, yet y has only two positive values such that (XV) is true for more than one i).

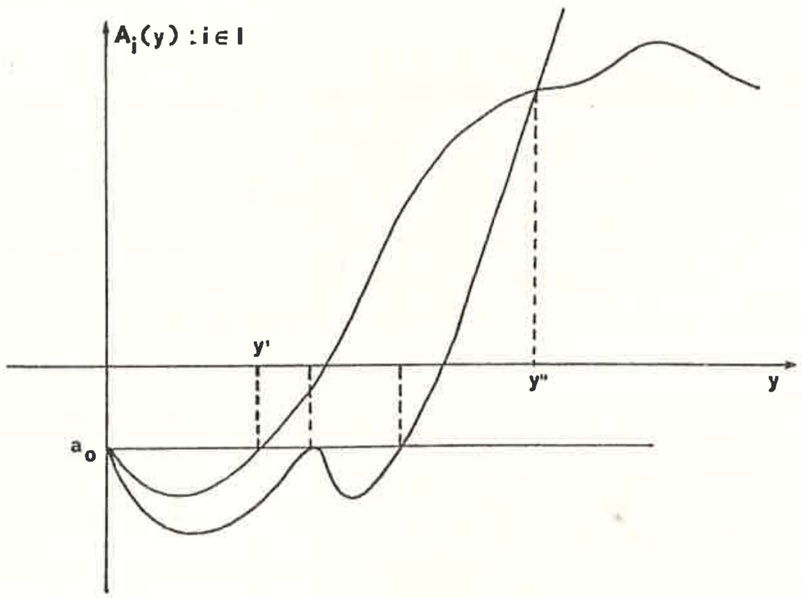

Fig 1

Lemma 2. If $B'(y)$ is the derivative (if it exists) of $B(y)$, for a positive value of y not included among those considered in Lemma 1[9], we must have either:

$$B(y) = a_0$$
$$B'(y) = 0 \quad ,$$

or:

$$B(y) > a_0$$
$$B'(y) > 0$$

(no other possibility is allowed). To prove the lemma, let $k \in I$ be such that:

$$\text{(XVI)} \quad B(y) = A_k(y) \quad .$$

In the hypotheses of the lemma[10], clearly:

$$\text{(XVII)} \quad B'(y) = A'_k(y) \quad ,$$

where $A'_k(y)$ represents the derivative of $A_k(y)$. We distinguish the following two cases. If $k = 0$, then (since $A_0(y) = a_0$) it follows from (XVI) and (XVII) that $B(y) = a_0$ and $B'(y) = 0$. If $k \geq 1$, howerer, then $B(y) \ a_0$. First of all the definition of $B(y)$ precludes that $B(y) > a_0$. Neither can $B(y) = a_0$ since from (XVI) it would follow that $A_k(y) = a_0$ and therefore $A_k(y) = A_0(y)$, contra-

(9) Until proved otherwise, values of y included in those considered in Lemma 1, where $B(y)$ is differentiable, cannot be excluded: the fact that (XV) holds true for $q \in I$ and $t \in I$ does not imply that the polynomials $A_q(y)$ and $A_t(y)$ intersect: thus they may 'simply' be tangents.

(10) Referring back to Fig 1, the hypotheses of the lemma are that we are within one of the following intervals of abscissa: (0, y'), (y', y"), (y", +∞).

ry to the hypotheses. It only remains to prove that $B'(y) > 0$. Since $k \geq 1$, consider the differences:

$$A_k(y) - A_i(y) \qquad (i \in I \; , \; i < k) \; .$$

In the hypotheses of the lemma, necessarily:

$$\text{(XVIII)} \qquad A_k(y) - A_i(y) > 0 \qquad (i \in I \; , \; i < k) \; .$$

On account of (XII), (XVIII) imply:

$$\text{(XIX)} \qquad A_k(y) - A_i(y) > 0 \qquad (i = 0 \; , \; \dots \; , \; k-1) \; .$$

Summing up inequalities (XIX), we get:

$$\text{(XX)} \qquad \sum_{i=1}^{k} i \, a_i \, y^i > 0 \; .$$

In the hypotheses of the lemma (which include $y > 0$), (XX) implies:

$$\text{(XXI)} \qquad \sum_{i=1}^{k} i \, a_i \, y^{i-1} > 0 \; .$$

It is now sufficient to note that:

$$A'_k(y) = \sum_{i=1}^{k} i \, a_i \, y^{i-1} \; ,$$

to conclude, given (XVII), that $B'(y) > 0$. The lemma is thus proved.

It is quite clear that Lemmas 1 and 2 imply point (ii), considering the continuity of $B(y)$. Q.E.D..

The crucial point of Arrow and Levhari's approach is the following redefinition of the internal rate of return: given a project A, its internal rate of return is defined as a value of the interest rate which makes the maximum present value of A equal to zero.

An immediate corollary of Theorem 2 is that the thus-defined internal rate of return exists and is unique for any project $A \in \mathbf{A}$.

5. Critical Remarks

As we have seen, Karmel's contribution on the one hand, and Arrow and Levhari's on the other, offer two different solutions to the same problem: how to use the hypothesis of truncatability to render the internal rate of return unique for any economically relevant project, that is for any project contained in class **A** of projects $(a_0 \; , \; \dots \; , \; a_n)$ such that $a_i > 0$ for at least one $i > 0$ and such that $a_n \neq 0$.

To make the conceptual difference between the two solutions clearer, consider Fig 2 in which are represented set **A** and its subset B of projects to which a unique internal rate of return is attached (the set $\underline{A}$ of all feasible projects is also represented).

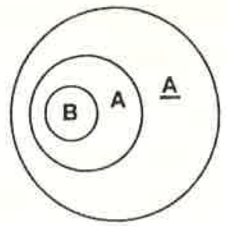

Fig 2

Karmel's solution consists in adopting a behavioural hypothesis in which every project contained in A (which may or may not have, a unique internal rate of return) is reduced to one of its truncations contained in B (thus of course provided with a unique internal rate of return).

Arrow and Levhari's solution, on the contrary, consists in a 'drastic' redefinition of the internal rate of return, designed in such a way that the internal rate of return be unique for any project contained in A.

Here we expound our strongly critical view of both proposed solutions, beginning with Karmel's.

The behavioural hypothesis of Karmel, at first sight convincing, is actually unacceptable. We have seen that such a hypothesis consists in a criterion for optimal truncation. Examined closely, this criterion results from the application of the following three rules valid for the ranking of any pair of possible durations:

(i) if each duration of the pair yields a unique internal rate of return, then that duration which has the greater internal rate of return is preferable;

(ii) if at least one duration of the pair yields more than one internal rate of return, then ranking must be made according to the greatest rate associated with each duration (ignoring all the others);

(iii) if one duration of the pair yields no internal rate of return, while the other duration yields at least one, then the second is preferable.

Only if these three rules are applied, it is possibile to conclude (with Karmel) that the duration with which the maximum internal rate of return (r_*) is associated, is preferable to any other duration. In other words, the above three rules and Karmel's criterion are 'the same thing', the latter being a 'more synthetic version' of the former.

Unfortunately, rules (ii) and (iii) are anything but applicable. In the first place, when more internal rates of return are associated with a project, each of these rates represents a 'suspect' measure of project profitability: why should the highest rate inspire more 'confidence' than the others (rule (ii))[11]? Secondly, how can one duration to which no internal rate of return is associated be compared with another (rule (iii))? Evidently these questions remain unanswered, hence the criterion of choice by which Karmel's investor 'transforms' a project (included in A) into one of its truncations (included in B) is not economically correct.

Consider now Arrow and Levhari's solution. Since it consists in a 'redefinition' of the internal rate of return, it raises the question as to what new economic significance this might have with respect to the usual definition. Clearly the validity of their entire proposed approach rests upon the answer to this question.

In our opinion it is impossible to attribute any appropriate economic significance to the new definition. Arrow and Levhari's internal rate of return is simply a critical value of the rate of interest, up to which it is possible to truncate the project in such a way that the present value of the truncated project is positive. In this sense, Arrow and Levhari's internal rate of return is an

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(11) Is not Karmel's ultimate aim perhaps to eliminate the ambiguity inherent in each non-unique internal rate of return? Coherence would then prohibit the use of 'means' inconsistent with the 'end'.

upper bound to those market rates of interest for which the present value criterion suggests acceptance of the project (optimally truncated). But this meaning is not what might be expected from something which is called internal rate of return!

We would point out that the new definition of internal rate of return is so-to-speak, 'strongly implicit' in the literature. Thus, Arrow and Levhari, in the presentation of their contribution, write: "In the following we prove that if, with a given constant rate of discount, we choose the truncation period so as to maximise the present value of the project, then the internal rate of return of the truncated project in unique. More fully, we prove that if the life of the project is optimally chosen, then the maximised present value of the project is a monotonic decreasing function of the rate of interest"[12]. And Hicks (in terminology somewhat different from ours, yet easy to grasp): "The yield, of a given process (...) is unique so long as we keep the condition that the process is to be carried on for the optimal duration"[13]. Finally Nuti writes (his terms production flow and process are synonymous with project and his symbol V stands for maximum present value): "The truncation theorem states that, if it is possible to terminate a production flow before the end of its physical lifetime at no extra cost, then maximisation of present value ensures that the present value of the net production flow is a monotonically decreasing function of the discount rate. A corollary

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(12) K.J. Arrow - D. Levhari| 2 | , p. 560.

(13) J.R. Hicks| 11 | , p. 22.

of this is that V = 0 for no more than one (if any) discount rate, i. e. if the internal rate of return of the process exists it is always unique"[14]. Elsewhere Nuti writes (now using the term consumption sequence): "If there is more than one sign inversion in the consumption sequence (...), there may be multiple internal rates of return (...) for which the present value of the sequence is zero. If, however, the consumption sequence can be 'truncated' (...) at any date (...), the present value of the sequence is a monotonically decreasing function of the discount rate, and therefore the internal rate of return of the sequence (...) is unique"[15].

As can be seen, the authors make the uniqueness of the internal rate of return derive from a 'process of maximization of the present value'. There is no doubt that so-doing implicitly changes the definition of internal rate of return, but this change is not clearly recognized, still less is it economically justified-which would be equivalent to explaining the economic significance of the new definition.

6. The Numerical Equivalence of Both Approaches

Despite their conceptual difference, Karmel's approach and that of Arrow and Levhari both lead to 'numerically' coincident results. The subject has been treated by C.J. Norstrom| 15 | , who proves the following theorem.

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(14) D.M. Nuti| 16 |, p. 487. Note that he uses 'if' in the last proposition of the passage quoted because the preceding proposition is intended to refer to the entire class A of feasible projects.

(15) D.M. Nuti | 17 | , pp. 347-348.

Theorem 3. The internal rate of return (as usually defined) associated with a project $A \in \mathbf{A}$ optimally truncated according to Karmel's criterion, is equal to the internal rate of return which is associated with the same project A where Arrow and Levhari's definition is accepted.

Proof. Unlike Norstrom, let us prove the proposition in the following simple manner. We recall that the internal rate of return, r_*, associated with project A optimally truncated according to Karmel's criterion, is maximum whithin set R of internal rates of return associated with all possible durations of A. Let r^* represent the internal rate of return associated with project A where Arrow and Levhari's definition is accepted: quite clearly, $r^* \in R$, which precludes the possibility that $r_* < r^*$. To prove that the contrary possibility, $r_* > r^*$, is also precluded, note first that in r_* (as in every other rate in R) at least one among the present values of all possible durations of A, will equal zero. It follows that the maximum present value of A is non-negative. In symbols:

$$\text{(XXII)} \quad A(r_*) \geq 0 \quad .$$

Since, by definition of r^*:

$$\text{(XXIII)} \quad A(r^*) = 0 \quad ,$$

if $r_* > r^*$ (see Fig 3), from (XXII) and (XXIII) it would follow that

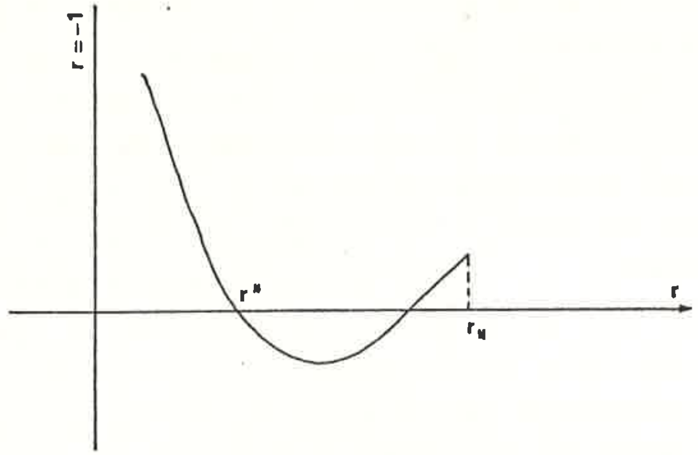

Fig 3

$A(r)$ would not obey the dictates of Theorem 2. Hence $r_* = r^*$. _Q.E.D._.

7. Conclusion

We have shown that attempts to use the hypothesis of truncability of projects to render unique the internal rate of return cannot be considered successful. Karmel's behavioural hypothesis cannot be justified from the point of view of economic rationality, while the redefinition of internal rate of return implicitly accepted by Arrow and Levhari has no consistent economic significance.

In addition to this, there are standard criticisms which refer to the technological-economic plausibility of the hypothesis of projects truncatability. The hypothesis that in any period of the physical life of a project it is possibile to discard the 'tail' and keep (unaltered) the 'head', is a strong hypothesis, not only because it presupposes that there are no obligations to continue the project[16], but also (perhaps above all) because it implies scrap values equal to zero.

In reality, it can be shown that both the approaches, of Karmel and of Arrow and Levhari, preserve their formal validity even where scrap values are non-negative[17]; however there are clearly no valid reasons for maintaining that this must necessarily be the case. A truly general approach should ignore the sign of the scrap values, and so allow the final net output of a project truncated at time i to be simply different form a_i (and not necessarily equal or greater)[18].

Apart from this, how to determine the optimum life of a project is still an economically relevant problem; as relevant as that of arranging a unique internal rate of return for any feasible project. Nevertheless 'mixing' the two problems is certainly no way of solving them.

How to determine the optimum life of a project is easily solved as soon as the present value is taken as the criterion. Trying to solve this problem by taking the internal rate of return as the criterion primarily requires the successful arrangement of a unique (and economically significant) internal rate of return for every feasible project

(16) See T.H. Silcock | 22 |, pp. 818-819, C. Filippini - L. Filippini | 5 |, pp. 14-15, E. Burmeister | 3 |, pp. 419-420, A. Sen | 21 |, p. 341.

(17) See Rennie's Appendix to Karmel's article and pp. 563-564 of Arrow and Levhari's article. See also J.S. Flemming - J.F. Wright | 6 |, pp. 259-260.

(18) Some of these difficulties are resolved by A. Sen | 21 |.

(and hence for every possible duration of each project). In this direction, in our opinion, more work is needed.

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APPENDIX

Theorem. Given a project $A := (a_0, \dots, a_n)$ and an internal rate of return r_1 attached to it, r_1 is unique if the following inequalities are satisfied:

$$\sum_{j=0}^{i} a_j (1+r_1)^{i-j} \leq 0 \qquad (i = 0, \dots, n-1) .$$

Proof. Put $x_1 := 1+r_1$, from the definition of internal rate of return it follows that to prove the theorem it is sufficient to prove that the following polynomial in x:

$$\text{(XXIV)} \qquad \sum_{j=0}^{n} a_j x^{n-j}$$

has no positive roots other than x_1 if the following inequalities are satisfied:

$$\text{(XXV)} \qquad \sum_{j=0}^{i} a_j x_1^{i-j} \leq 0 \qquad (i=0, \dots, n-1) .$$

As it is well Known[19], a polynomial exists (and is unique):

$$\text{(XXVI)} \qquad \sum_{i=0}^{n-1} q_i x^{n-i-1}$$

such that:

(19) See, for example, A. Kurosh | 14 |, pp. 148-152.

$$\sum_{j=0}^{n} a_j x^{n-j} = (x-x_1) \sum_{i=0}^{n-1} q_i x^{n-i-1} .$$

Therefore the other positive roots of polynomial (XXIV) (if any) are the same as the positive roots of polynomial (XXVI). It is likewise known that:

$$\text{(XXVII)} \qquad q_i = \sum_{j=0}^{i} a_j x_1^{i-j} \qquad (i=0, \dots, n-1) .$$

Which shows that, if inequalities (XXV) are satisfied, all the coefficients of polynomial (XXVI) are non-positive. Therefore, Descartes' rule of signs[20] guarantees that polynomial (XXVI) has no positive root. Q.E.D..

(20) See, for example, A. Kurosh | 14 |, pp. 263-267.

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A B S T R A C T

Both the approach based on the maximization of the internal rate of return and the approach based on the maximization of the present value, are criticized. The first approach rests on a criterion for optimal truncation which is shown to be unjastifiable from the point of view of economic rationality. The second approach is shown to be implicitly based on a redefinition of the internal rate of return which, when rendered fully explicit, appears devoid of any consistent economic significance.

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