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Abstract

Filtral opportunity rankings (FORs) are defined, and their basic properties are studied. A characterization of FORs in terms of their 'essentiality-operators' is provided. It is also shown that a suitably augmented set of FORs admits several 'nice' aggregation rules in an Arrowian setting. JEL Classification Numbers:D71,025.

1 Introduction

In the last few years there has been a remarkable growth of interest in criteria for evaluating social situations in terms of the 'freedom of choice' they afford the relevant agents. Under the most obvious understanding of the problem, all this amounts to looking for a plausible 'opportunity ranking' of the power set $P(X)$ of the set X of basic alternatives, *without relying at all* on information concerning preferences on X. In what follows I shall denote that version of the opportunity-evaluation issue as the *'pure opportunity-ranking'* problem.

In principle, it is widely acknowledged that one should settle for some suitably 'comprehensive' *partial* –as opposed to total– 'opportunity preorder(s)'(see e.g. Puppe(1996) for a recent restatement of this most influential of Amartya Sen's general tenets concerning social-measurement-related issues). However, the early remarkable characterization of the cardinality-based *total* preorder within the class of *all* preorders by means of a simple set of prima facie mild conditions (see Pattanaik,Xu(1990)) has been largely received as a sort of 'impossibility theorem' concerning the 'pure opportunity-ranking' problem. Indeed, it has been repeatedly argued that the Pattanaik-Xu result shows that the evaluation of opportunity sets cannot be sensibly divorced from a comparative evaluation of *single* basic alternatives, i.e. from standard preferences (see e.g.

Sen(1991)). Therefore, the prevailing reaction to the Pattanaik-Xu characterization has arguably resulted in a shift of focus away from the 'pure opportunity-ranking' problem, and –more often than not–from partial preorders as well (with Puppe(1996) and Klemisch-Ahlert(1993) as prominent exceptions). As a matter of fact, most of the subsequent contributions have been centered – with somewhat different emphases– on ways of amalgamating several criteria for evaluating opportunity sets, including total preference preorders on X and other *total* orderings (see among others Klemisch-Ahlert(1993), Bossert,Pattanaik,Xu(1994), Puppe(1995), Dutta,Sen(1996), Bossert(1997), Pattanaik,Xu(1998),Gravel(1998), Nehring,Puppe(1999)).

The present paper is *not* going to dwell on the subtle conceptual issues underlying the foregoing controversy on the proper relationship between 'freedom of choice' and 'preferences' in ranking opportunity sets. Rather, it is assumed here that

i) the 'pure opportunity-ranking' problem is interesting per se, and deserves a separate analysis whatever the comparative weight one is inclined to allocate to 'freedom of choice' as such in the overall evaluation of opportunity sets;

ii) identifying some *partial* –as opposed to *total*– 'opportunity preorder' is to be considered as the primary task involved in our 'pure opportunity- ranking' problem as described above.

Moreover, we also rely on the following motivation. While the whole idea of ranking opportunity sets in terms of 'freedom of choice' arguably requires the notion of 'many degrees' of freedom(s) (and is indeed consistent with the even subtler notion of 'positive' - possibly plural- freedom(s)), there is also a well-established tradition which refers to one (singular) 'freedom' and treats it as a definitely 'crisp'('yes-or-no') concept. Can these two views be somehow reconciled in a formal model?

Working under those assumptions, and relying on such intuitions and motivation, *one* simple class of partial opportunity rankings – hereby called *filtral opportunity rankings (FORs)* – is defined and singled out for analysis in what follows. Briefly, *FORs embody a minimal standard of freedom. Below the standard (a freedom- poverty- line of sorts) the opportunity sets are equated (i.e.indifferent) to a no-freedom situation. Above the standard FORs simply replicate the set -inclusion partial order . The standard of freedom of a FOR consists in an order-filter of* $(P(X), \supseteq)$ (a specialized version of the same notion for *total* opportunity rankings is suggested and briefly discussed in Suppes(1987); however, to the best of my knowledge, Gekker(1999) should be credited for the introduction of general order-filters as standards of freedom in a similar setting).

The present paper is therefore devoted to a quite detailed study of FORs. It turns out that FORs are endowed with a remarkably regular structure which allows an interesting characterization of them in terms of their essentiality-operators as first introduced by Puppe(1996) under a slightly different terminology. Moreover, it is shown that FORs are also amenable to 'nice' aggregation results in an Arrowian setting. The structure of the paper is as follows. In section 2 FORs are introduced, and some of their basic properties–including characterizations– are presented. Section 3 is devoted to the FOR-aggregation problem in a generalized Arrowian setting .

2 Filtral opportunity rankings: basic properties and characterization

Let X be the set of alternatives/opportunities, and $P(X)$ the corresponding set of opportunity sets. It is also assumed that $\#X \geq 3$ in order to avoid trivialities or tedious qualifications. We are concerned here with defining a pure opportunity ranking $\succcurlyeq$ of $P(X)$, namely a binary relation $(P(X), \succcurlyeq)$ that (weakly) extends $(P(X), \supseteq)$ —i.e. $A \supseteq B$ entails $A \succcurlyeq B$— the underlying interpretation being that $A \succcurlyeq B$ means "A embodies more opportunities(or positive freedom) than B ". We also denote as usual by $\succ$ and $\sim$ the asymmetric and symmetric components of $\succcurlyeq$, respectively. In order to accomodate some basic intuitions concerning the very idea of an "opportunity ranking" we shall consider some minimal restrictions on $(P(X), \succcurlyeq)$, namely :

(Preorder (PR)) : $(P(X), \succcurlyeq)$ is transitive and reflexive .

(Freedom Improvability (FI)) : For any $A \neq X$ a $B \subseteq X$ exists such that $A \cup B \succ A$.

(Weak Monotonicity (WM)): For any $A, B \subseteq X$, $A \cup B \succcurlyeq A$.

A weakened version of *(WM)* will also be considered, namely:

(Restricted Weak Monotonicity (RWM)): For any $A, B \subseteq X$, if [either $A = B$ or $A = X$ or else $B = \emptyset$] then $A \succcurlyeq B$.

In this paper, we shall be mainly concerned with those *opportunity rankings* $(P(X), \succcurlyeq)$ which are *preorders* and satisfy both *(FI)* and *(WM)* (or *(RWM)*).

In particular, we are interested here –as mentioned above– in a class of (partial) opportunity rankings that arise in a natural way whenever

a) all the alternatives are (potentially) 'good' -or 'not bad'- : this interpretation of the relevant alternatives can be plausibly supported by referring to an underlying unmodelled set of 'admissible' preferences on X such that any $x \in X$ is a maximum or maximal alternative with respect to some preference in the set.(This suggested motivation is to be distinguished from the approach proposed in Pattanaik,Xu(1998) where the opportunity ranking itself – as opposed to the basic set X –is determined by means of such a set of preferences on X) ;

b) the ultimate significance of the feasible alternatives depends on a threshold effect so that either each alternative is a significant opportunity or none of them is according to the location of the menu w.r.t. the minimum standard (i.e. 'above' or 'below' the standard).

Those requirements – that amount to the introduction of some sort of "freedom-poverty line"–can also be regarded as an attempt to accomodate the widespread treatment of "freedom" as a 'yes-or-no' concept while insisting on the idea of 'many'degrees of freedom as suggested by typical notions of 'positive freedom' . We shall heavily rely on the following notion:

Definition 1 *(*Order filter of a preordered set*) Let* $(Y, \gtrsim)$ *be a non-empty preordered set. An* order filter *of* $(Y, \gtrsim)$ *is a non-empty set* $Z \subseteq Y$ *such that for any* x, y , $x \in Z$ *and* $y \gtrsim x$ *entail* $y \in Z$*. In particular, such an order-filter* Z *is said to be* non-trivial *if* $Z \neq Y$.

The following specialization of the previous definition will also be used in the sequel:

Definition 2 *(*Principal order filter of a preordered set *) Let* $(Y, \gtrsim)$ *be a preordered set.. A* principal order filter *of* $(Y, \gtrsim)$ *is a set* $Z \subseteq Y$ *such that* $Z = \{x \in Y : x \gtrsim y\}$ *for some* $y \in Y$ *. It should be recalled here that– whenever* $(Y, \gtrsim)$ *is a lattice(see Section 3 below for an explicit definition)– a* principal order filter *is also a* latticial filter *i.e. is* $\wedge$*-closed .*

The notion of an order filter enables us to formulate in a natural way a special type of opportunity ranking that embodies requirements a) and b) for opportunity rankings as mentioned above. This is made precise by the following definition :

Definition 3 *(*Filtral opportunity rankings *) A* filtral opportunity ranking (FOR) *is a binary relation* $(P(X), \succcurlyeq)$ *such that for some order filter* F *of* $(P(X), \supseteq)$, $A \succcurlyeq B$ *if and only if either* $A \supseteq B$ *or* $B \notin F$.

Notation 4 *A FOR with order filter* F *will also be denoted by* $(P(X), \succcurlyeq_F)$.

Remark 5 *As mentioned above, a specialized version of FORs was first introduced by Suppes(1987), with a quite similar motivation (see also Gekker(1999)). Suppes is concerned with a* total *opportunity ranking* $(P(X), \succcurlyeq_{C^*})$ *defined as follows : for any* $A, B \subseteq X$, $A \succcurlyeq_{C^*} B$ *iff either* $A \supseteq C^*$ *or* $B \not\supseteq C^*$ *(for some suitable* $C^* \subseteq X$*). Clearly enough,* $\succcurlyeq_{C^*}$ *reduces to the following ranking procedure: a) introduce a* principal *order-filter* F *of* $(P(X), \supseteq)$ *(hence a* latticial *filter as well), b) define a FOR* $(P(X), \succcurlyeq_F)$ *with* F *as reference filter , and c) extend* $(P(X), \succcurlyeq_F)$ *to a* total (dichotomic) *preorder by positing* equivalence *between any* $A, B \in F$ *.*

The following elementary properties hold true of FORs :

Proposition 6 *Let* $(P(X), \succcurlyeq)$ *be a binary relation. Then*

i) $(P(X), \succcurlyeq)$ *is a FOR with order filter* F *if and only if –for any* $A, B \subseteq X$– $A \succcurlyeq B$ *if and only if for any order filter* F' *of* $(P(X), \supseteq)$, $F' \subseteq F$ *and* $B \in F'$ *jointly entail* $A \in F'$.

ii) If $(P(X), \succcurlyeq)$ *is a FOR for some order filter* F *of* $(P(X), \supseteq)$ *then it is a* preorder *that satisfies both* WM *and* FI*. Moreover, for any* preorder $(P(X), \succcurlyeq)$ *that satisfies both* WM *and* FI *a pair of order filters* F, F' *of* $(P(X), \supseteq)$ *exist such that* $\succcurlyeq_F \supseteq \succcurlyeq \supseteq \succcurlyeq_{F'}$.

Proof. i) Let $(P(X), \succcurlyeq)$ be a FOR with order filter F, and $A \succcurlyeq B$. Hence either $A \supseteq B$ or $B \notin F$. Now, take an order filter $F' \subseteq F$. If $A \supseteq B \in F'$ then obviously $A \in F'$, whereas if either $A \supseteq B \notin F'$ or $B \notin F$ there is nothing to prove. Conversely, let $(P(X), \succcurlyeq)$ be such that –for some order filter F of $(P(X), \supseteq)$– $A \succcurlyeq B$ iff $A \in F'$ for any order filter $F' \subseteq F$ with $B \in F'$. Let us then assume that $A \succcurlyeq B$, and $B \in F$. If $B \backslash A \neq \emptyset$ then take the order filter $F' = \{C \subseteq X : C \supseteq B\}$. Clearly, $F' \subseteq F$, $B \in F'$, and $A \notin F'$, a contradiction.

ii) Let $(P(X), \succcurlyeq)$ be a FOR with order filter F. Then, *reflexivity* of $\succcurlyeq$ is trivial. *Transitivity* is easily checked: indeed, take $A \succcurlyeq B$, and $B \succcurlyeq C$. Then, either ($A \supseteq B$ and $B \supseteq C$, whence $A \supseteq C$ follows as an obvious consequence), or ($C \notin F$) , or else ($B \notin F$ and $B \supseteq C$, whence again $C \notin F$): in any case, it follows–by definition of $\succcurlyeq$– that $A \succcurlyeq C$.

Weak Monotonicity of $(P(X), \succcurlyeq)$ is immediate since $A \supseteq B$ iff $A \cup B = A$. In order to check *Freedom Improvability*, take any $A \subset X$. If $A \notin F$, take any $A' \supset A$ such that $A' \in F$, and posit $B = A' \backslash A$: then– clearly enough– $A \cup B \succcurlyeq A$ and not $A \succcurlyeq A \cup B$. If $A \in F$, take any $x \in X \backslash A$. Again, $A \cup \{x\} \succcurlyeq A$, and not $A \succcurlyeq A \cup \{x\}$ by definition of $\succcurlyeq$.

Conversely, let $(P(X), \succcurlyeq)$ be a preorder that satisfies both *WM* and *FI*. Our first claim is that an order filter F of $(P(X), \supseteq)$ exists such that for any $A, B \subseteq X$, $A \succcurlyeq B$ only if either $A \supseteq B$ or $B \notin F$. Indeed, suppose not. Then, for any order filter F of $(P(X), \supseteq)$ a pair $A = A(F) \subseteq X$, $B = B(F) \subseteq X$ exists such that $[A \succcurlyeq B, A \not\supseteq B$ and $B \in F]$. In particular, consider the 'smallest'order filter $F = \{X\}$. Then, $B(F) = X$, and $A(F) \subset B(F)$, i.e. $A(F) \neq X$. Hence $A(F) \cup C \succ A(F)$ for some $C \subseteq X$ (by *FI*). Now, $B(F) \succcurlyeq A(F) \cup C$ (by *WM*). It follows that $B(F) \succ A(F)$ (because $(P(X), \succcurlyeq)$ is a *preorder*):contradiction. Next, our final claim : an order filter F' of $(P(X), \supseteq)$ exists such that for any $A, B \subseteq X$, $A \succcurlyeq B$ if either $A \supseteq B$ or $B \notin F$. Again, suppose not. Then for any order filter F of $(P(X), \supseteq)$, $A(F), B(F) \subseteq X$ exist such that *not* $A \succcurlyeq B$ and either $A \supseteq B$ or $B \notin F$. Now, *not* $A \succcurlyeq B$ and $A \supseteq B$ are inconsistent statements, by *WM*. Hence, it must be the case that *not* $A \succcurlyeq B$ and $B \notin F$. But then, take $F = P(X)$. Hence $B \in F$ for any $B \subseteq X$, a contradiction again.$\square$

We proceed now to provide a characterization of FORs in terms of their *essentiality operators* – henceforth *E-operators* –as first introduced by Puppe(1996) in a slightly different setting.

Definition 7 *(*E-Operator of an Opportunity Preorder*). Let* $(P(X), \succcurlyeq)$ *be a preorder. Then the* E-operator $E_{\succcurlyeq} : P(X) \to P(X)$ *of* $(P(X), \succcurlyeq)$ *is defined as follows: for any* $A \subseteq X$

$$E_{\succcurlyeq}(A) = \{x \in A : A \succ A \backslash \{x\}\} .$$

The following properties of general operators $H_{\gtrsim} : Z \to Z$ on an arbitrary preordered set $(Z, \gtrsim)$ will be considered :

(Deflation) An operator $H_{\gtrsim} : Z \to Z$ is *deflationary* w.r.t. $(Z, \gtrsim)$ iff $z \gtrsim H_{\gtrsim}(z)$ for any $z \in Z$;

(Monotonicity) An operator $H_{\gtrsim} : Z \rightarrow Z$ is *monotonic* w.r.t. $(Z, \gtrsim)$ iff $H_{\gtrsim}(z) \gtrsim H_{\gtrsim}(y)$ for any $z, y \in Z$ such that $z \gtrsim y$.

(Idempotence) An operator $H_{\gtrsim} : Z \rightarrow Z$ is *idempotent* w.r.t. $(Z, \gtrsim)$ iff $H_{\gtrsim}(H_{\gtrsim}(z)) \sim H_{\gtrsim}(z)$ for any $z \in Z$.

In particular, an operator $H_{\gtrsim}$ on a preordered set $(Z, \gtrsim)$ is said to be a *projection* of $(Z, \gtrsim)$ iff it is both *monotonic* and *idempotent* w.r.t. $(Z, \gtrsim)$, and a *kernel operator* of $(Z, \gtrsim)$ iff it is a *deflationary projection* of $(Z, \gtrsim)$. Moreover, a kernel operator $H_{\gtrsim}$ of a preordered set $(Z, \gtrsim)$ with a maximum element z^* (the '*top*'element of Z) is said to be *normal* iff $H_{\gtrsim}(z^*) \sim z^*$.

(Auto-Filtrality) An operator $H_{\gtrsim} : Z \rightarrow Z$ on a preordered set $(Z, \gtrsim)$ with a minimum z_0 (the '*bottom*' element of Z) is said to be *auto-filtral* iff the set $\{z \in Z : z_0 \neq z = H_{\gtrsim}(z)\}$ of its non-bottom fixed points is an order filter of $(Z, \gtrsim)$.

In the more specialized setting of operators on preordered *power* sets the following properties can also be defined:

(Inclusion-Filtrality) An operator $H_{\succcurlyeq} : P(Y) \rightarrow P(Y)$ on a preordered power set $(P(Y), \succcurlyeq)$ is said to be *inclusion-filtral* iff the set $\{A \subseteq Y : \emptyset \neq A = H_{\succcurlyeq}(A)\}$ of its non-empty fixed points is an order filter of $(P(Y), \supseteq)$.

(Double Filtrality) An operator $H_{\succcurlyeq} : P(Y) \rightarrow P(Y)$ on a preordered power set $(P(Y), \succcurlyeq)$ is said to be *doubly filtral* iff it is both auto-filtral and inclusion-filtral.

(Exact Double Filtrality) An operator $H_{\succcurlyeq} : P(Y) \rightarrow P(Y)$ on a preordered power set $(P(Y), \succcurlyeq)$ is said to be *exactly doubly filtral* iff it is doubly filtral and the identity function Id is an order-isomorphism between

$$(\{A \subseteq Y : \emptyset \neq A = E_{\succcurlyeq}(A)\}, \succcurlyeq) \text{ and } (\{A \subseteq Y : \emptyset \neq A = E_{\succcurlyeq}(A)\}, \supseteq).$$

(Meet-Additivity) An operator $H_{\succcurlyeq} : P(Y) \rightarrow P(Y)$ on a preordered power set $(P(Y), \succcurlyeq)$ is *meet-additive* iff for any $A, B \subseteq Y$: $H_{\succcurlyeq}(A \cap B) \sim H_{\succcurlyeq}(A) \cap H_{\succcurlyeq}(B)$.

(Universal Local-Flatness (ULF)) An operator $H_{\succcurlyeq} : P(Y) \rightarrow P(Y)$ on a preordered power set $(P(Y), \succcurlyeq)$ is *universally locally-flat* iff for any $A \subseteq Y$, $H_{\succcurlyeq}(A) \in \{A, \emptyset\}$.

A meet-additive normal kernel operator $H_{\succcurlyeq}$ of a preordered power set $(P(Y), \succcurlyeq)$ may also aptly said to be a *topological kernel operator* (by way of analogy with the special case $(P(Y), \supseteq)$, where such an $H_{\succcurlyeq} = H_{\supseteq}$ does indeed induce a *topology* on Y whose open sets are precisely the fixed points of $H_{\supseteq}$). Moreover, the following simple fact is to be emphasized:

Claim 8 *Let $H_{\succcurlyeq}$ be an* auto-filtral *operator on a preordered power set (or opportunity ranking) $(P(Y), \succcurlyeq)$ which satisfies* WM. *Then $H_{\succcurlyeq}$ is* inclusion-filtral *as well (hence* doubly filtral*).*

Proof. Let $A, B \subseteq X$ be such that $\emptyset \neq A = H_{\succcurlyeq}(A)$, and $B \supseteq A$. Then $B \succcurlyeq A$ by *WM*. Since $H_{\succcurlyeq}$ is *auto-filtral* it follows that $B = H_{\succcurlyeq}(B)$. □

Remark 9 *In what follows we shall mostly consider opportunity rankings having an auto-filtral E-operator. The most obvious family of examples of opportunity rankings having an E-operator that is* not *auto-filtral consists of opportunity* *rankings induced by a linear ordering of X, through maximization of the latter on $P(X)$. Another example of an opportunity ranking with a* non-auto-filtral *E-operator is provided by the total opportunity preorder $(P(X), \succcurlyeq_{C^*})$ already encountered before : indeed, it is easily checked that the set of non-empty fixed points of $E_{\succcurlyeq_{C^*}}$ reduces to $\{C^*\}$, not an order-filter of $(P(X), \succcurlyeq_{C^*})$ whenever $C^* \neq X$.*

The following lemmata will prove to be most useful in the sequel :

Lemma 10 *Let $(P(X), \succcurlyeq)$ be an opportunity* preorder *that satisfies* WM*, and such that $E_{\succcurlyeq}$ is* auto-filtral *and satisfies* ULF*. Then, $E_{\succcurlyeq}$ is an* order-homomorphism *of $(P(X), \succcurlyeq)$ i.e. $A \succcurlyeq B$ implies $E_{\succcurlyeq}(A) \succcurlyeq E_{\succcurlyeq}(B)$ for any $A, B \subseteq X$. Moreover, if $E_{\succcurlyeq}$ is* exactly doubly filtral *as well, then $E_{\succcurlyeq}$ is an* order-embedding *of $(P(X), \succcurlyeq)$ i.e. $A \succcurlyeq B$ if and only if $E_{\succcurlyeq}(A) \succcurlyeq E_{\succcurlyeq}(B)$ for any $A, B \subseteq X$.*

Proof. Let $(P(X), \succcurlyeq)$ be an opportunity preorder satisfying *WM* and such that $E_{\succcurlyeq}$ is both *auto-filtral* and *ULF.* Now, consider $A, B \subseteq X$ such that $A \succcurlyeq B$. We claim that $E_{\succcurlyeq}(A) \succcurlyeq E_{\succcurlyeq}(B)$. If $A = \emptyset$,then $\emptyset \succcurlyeq B \succcurlyeq E_{\succcurlyeq}(B)$ (by *WM*) hence $\emptyset \succcurlyeq E_{\succcurlyeq}(B)$, by transitivity. It follows that $E_{\succcurlyeq}(A) \succcurlyeq E_{\succcurlyeq}(B)$, by *WM* and transitivity again. Similarly, if $B = \emptyset$ then by definition $E_{\succcurlyeq}(B) = \emptyset$, whence $E_{\succcurlyeq}(A) \succcurlyeq E_{\succcurlyeq}(B)$ by *WM.* Thus, we may safely assume that $A \neq \emptyset, B \neq \emptyset$. It follows from the *ULF* property of $E_{\succcurlyeq}$ that one of the following four cases obtains : (a) $E_{\succcurlyeq}(A) = A$, $E_{\succcurlyeq}(B) = B$; (b)$E_{\succcurlyeq}(A) = A$, $E_{\succcurlyeq}(B) = \emptyset$; (c) $E_{\succcurlyeq}(A) = E_{\succcurlyeq}(B) = \emptyset$; (d) $E_{\succcurlyeq}(A) = \emptyset$, $E_{\succcurlyeq}(B) = B$. Under case (a) $E_{\succcurlyeq}(A) \succcurlyeq E_{\succcurlyeq}(B)$ follows trivially from our hypothesis that $A \succcurlyeq B$. Under cases (b) and (c) $E_{\succcurlyeq}(A) \succcurlyeq E_{\succcurlyeq}(B)$ follows immediately from *WM.* Under case (d) we may infer that $B \in F_{\succcurlyeq} \{A \subseteq X : A \neq \emptyset,\ A = E_{\succcurlyeq}(A)\}$–an *order-filter* of $(P(X), \succcurlyeq)$ by *auto-filtrality* of $E_{\succcurlyeq}$. But then $A \succcurlyeq B$ entails $A \in F_{\succcurlyeq}$ i.e. $A = E_{\succcurlyeq}(A)$, whence $E_{\succcurlyeq}(A) = A \succcurlyeq B = E_{\succcurlyeq}(B)$. This completes the proof of the first part of our lemma.

Now, suppose that $E_{\succcurlyeq}$ is also *exactly doubly filtral,* and consider $A, B \subseteq X$ such that $E_{\succcurlyeq}(A) \succcurlyeq E_{\succcurlyeq}(B)$. Since $E_{\succcurlyeq}$ satisfies the *ULF* property one of the cases (a),(b),(c),(d) as listed above must occur. We have to show that $A \succcurlyeq B$ is also the case. This follows trivially under case (a), and immediately from *WM* and transitivity under case (d). Let us then consider cases (b) and (c), where $E_{\succcurlyeq}(B) = \emptyset$. Of course, $B \notin F_{\succcurlyeq}$ as defined above. If $B = \emptyset$ the thesis is immediate again by *WM.* Thus, we may assume w.l.o.g. that $B \neq \emptyset$. It follows from *ULF* and *WM* that $B \sim B \setminus \{x\}$ for any $x \in B$. Since $F_{\succcurlyeq}$ is an order-filter of $(P(X), \succcurlyeq)$ and $B \notin F_{\succcurlyeq}$, it follows from *exact double-filtrality* of $E_{\succcurlyeq}$ that $B \setminus \{x\} \notin F_{\succcurlyeq}$ as well, hence $E_{\succcurlyeq}(B \setminus \{x\}) = \emptyset$ (by the *ULF* property). It is then immediate to establish by a simple induction argument –and by transitivity– that $B \sim \emptyset$, whence $A \succcurlyeq B$. □

Lemma 11 *Let $(P(X), \succcurlyeq_F)$ be a FOR with order filter F. Then, i) the E-operator $E_{\succcurlyeq_F}$ satisfies* ULF *. In particular, for any non-empty $A \subseteq X$: $E_{\succcurlyeq_F}(A) = A$ iff $A \in F$ (or equivalently $E_{\succcurlyeq_F}(A) = \emptyset$ iff $A \notin F$) . Moreover, ii) $E_{\succcurlyeq_F}$ is* exactly doubly-filtral , *and iii)* $E_{\succcurlyeq_F}$ *is an* order-embedding *of* $(P(X), \succcurlyeq_F)$ *i.e. for any* $A, B \subseteq X$, $A \succcurlyeq_F B$ *iff* $E_{\succcurlyeq_F}(A) \succcurlyeq_F E_{\succcurlyeq_F}(B)$.

Proof. i) To begin with, $E_{\succcurlyeq_F}(\emptyset) = \emptyset$ follows trivially from the definition of $E_{\succcurlyeq_F}$.Let $A \subseteq X$ be such that $A \neq \emptyset$, and $A \in F$. Then, for any $x \in A$, *not* $A \setminus \{x\} \succcurlyeq_F A$, while–by *WM*– $A \succcurlyeq_F A \setminus \{x\}$. Hence, by definition of $E_{\succcurlyeq_F}$, $E_{\succcurlyeq_F}(A) = A$. By contrast, if $A \notin F$ then –by definition of $\succcurlyeq_F$– $A \sim_F A \setminus \{x\}$ for any $x \in A$, whence $E_{\succcurlyeq_F}(A) = \emptyset$.

ii) Let $A, B \subseteq X$ be such that $A \neq \emptyset$, $E_{\succcurlyeq_F}(A) = A$, and $B \succcurlyeq_F A$. We claim that $E_{\succcurlyeq_F}(B) = B$. For, suppose not . Then, by part i) of the present proof , $A \in F$, and $E_{\succcurlyeq_F}(B) = \emptyset$. Moreover, $B \succcurlyeq_F A$ entails $\emptyset = E_{\succcurlyeq_F}(B) \succcurlyeq_F E_{\succcurlyeq_F}(A) = A$ since $E_{\succcurlyeq_F}$is an order-embedding of $(P(X), \succcurlyeq)$, by part ii) above. Since $A \in F$ it follows that $\emptyset = A$, by definition of $\succcurlyeq_F$: contradiction. Thus, $E_{\succcurlyeq_F}$is *auto-filtral.* Then, *inclusion-filtrality* (hence *double filtrality*) of $E_{\succcurlyeq_F}$follows from Weak Monotonicity of $\succcurlyeq$, in view of Claim 8 above. It remains to be checked that $(\{A \subseteq X : \emptyset \neq A = E_{\succcurlyeq_F}(A)\}, \succcurlyeq_F)$ and $(\{A \subseteq X : \emptyset \neq A = E_{\succcurlyeq_F}(A)\}, \supseteq)$ are order-isomorphic. To see this, consider any $A, B \subseteq X$ such that $E_{\succcurlyeq_F}(A) = A \neq \emptyset$, $E_{\succcurlyeq_F}(B) = B \neq \emptyset$, and $A \succcurlyeq_F B$. Then, by definition of $\succcurlyeq_F$, either $A \supseteq B$ or $B \notin F$. However, $B \notin F$ implies $E_{\succcurlyeq_F}(B) = \emptyset$, contradicting our choice of B. Therefore it must be the case that $A \supseteq B$. Conversely, let $A, B \subseteq X$ be such that $E_{\succcurlyeq_F}(A) = A \neq \emptyset$, $E_{\succcurlyeq_F}(B) = B \neq \emptyset$, and $A \supseteq B$. Then $A \succcurlyeq_F B$, by Weak Monotonicity of $\succcurlyeq_F$. It follows that Id is in fact an order-isomorphism, and $E_{\succcurlyeq_F}$is therefore *exactly doubly-filtral* as required.

iii) An immediate corollary of parts i)-ii) of the present proof as combined with Lemma 10 above . □

Furthermore, it turns out that the *E-operator* of a FOR is remarkably regular, as testified by the following proposition:

Proposition 12 *Let* $(P(X), \succcurlyeq_F)$ *be a FOR with order filter* F*. Then its E-operator* $E_{\succcurlyeq_F}$*is a* normal kernel operator *of* $(P(X), \succcurlyeq_F)$*. Moreover, if* F *is a non-trivial* principal *order filter then its E-operator* $E_{\succcurlyeq_F}$*is a* topological kernel operator *of* $(P(X), \succcurlyeq_F)$.

Proof. First, observe that $E_{\succcurlyeq_F}$is *deflationary* since $\succcurlyeq_F$ is in fact a preorder and satisfies Weak Monotonicity, by Proposition 6 ii) above (indeed, for any $A \subseteq X$, $A \supseteq E_{\succcurlyeq_F}(A)$ by definition, hence–by WM of $\succcurlyeq_F$– $A \succcurlyeq_F E_{\succcurlyeq_F}(A)$).

Clearly enough, Lemma 11 above, part iii) implies that $E_{\succcurlyeq_F}$is indeed *monotonic w.r.t.* $(P(X), \succcurlyeq_F)$.

Now, assume $B \notin F$. In this case we also know from Lemma 11 i) that $E_{\succcurlyeq_F}(B) = \emptyset$. Therefore $E_{\succcurlyeq_F}(A) \supseteq E_{\succcurlyeq_F}(B)$, hence again Weak Monotonicity of $\succcurlyeq_F$entails $E_{\succcurlyeq_F}(A) \succcurlyeq_F E_{\succcurlyeq_F}(B)$.

Idempotence of $E_{\succcurlyeq_F}$is proved as follows. For any $A \subseteq X$, consider $E_{\succcurlyeq_F}(E_{\succcurlyeq_F}(A))$. We know from Lemma 11 i) that $A \in F$ entails $E_{\succcurlyeq_F}(A) = A$ hence $E_{\succcurlyeq_F}(E_{\succcurlyeq_F}(A)) = E_{\succcurlyeq_F}(A)$, whereas $A \notin F$ entails $E_{\succcurlyeq_F}(A) = \emptyset$ hence $E_{\succcurlyeq_F}(E_{\succcurlyeq_F}(A)) = E_{\succcurlyeq_F}(\emptyset) = \emptyset = E_{\succcurlyeq_F}(A)$ again .

Furthermore, take any $x \in X$. We know from Proposition 6 ii) above that $\succcurlyeq_F$satisfies Freedom Improvability: it follows that $X \succ_F X \setminus \{x\}$. Therefore $E_{\succcurlyeq_F}(X) = X$, hence a fortiori $E_{\succcurlyeq_F}(X) \sim_F X$ i.e. $E_{\succcurlyeq_F}$is *normal* .

Finally, let $F = \{A \subseteq X : A \supseteq A^*\}$ for some $A^* \subseteq X$, $A \neq \emptyset$. Then, for any $A, A' \in F$, $A \cap A' \in F$ as well (i.e. F is indeed *principal*, hence a *latticial* filter of $(P(X), \supseteq)$). Now, take any $A, B \subseteq X$: by Lemma 11 i) if $\{A, B\} \nsubseteq F$ then $\emptyset \in \{E_{\succcurlyeq_F}(A), E_{\succcurlyeq_F}(B)\}$. Therefore $E_{\geq_F}(A) \cap E_{\succcurlyeq_F}(B) = \emptyset \subseteq E_{\succcurlyeq_F}(A \cap B)$. Otherwise, i.e. if $\{A, B\} \subseteq F$, we may conclude from the foregoing observation that $A \cap B \in F$, hence–by Lemma 11 i) again– $E_{\succcurlyeq_F}(A \cap B) = A \cap B \supseteq E_{\succcurlyeq_F}(A) \cap E_{\succcurlyeq_F}(B)$. In both cases $E_{\succcurlyeq_F}(A \cap B) \succcurlyeq_F E_{\succcurlyeq_F}(A) \cap E_{\succcurlyeq_F}(B)$ follows from Weak Monotonicity of $\succcurlyeq_F$. Furthermore, Monotonicity of $E_{\succcurlyeq_F}$w.r.t. $(P(X), \succcurlyeq_F)$ (as established above in the previous Lemma, part iii)) and Weak Monotonicity of $\succcurlyeq_F$–as combined with *ULF* and *exact double-filtrality* jointly entail $E_{\succcurlyeq_F}(A) \cap E_{\succcurlyeq_F}(B) \succcurlyeq_F E_{\succcurlyeq_F}(A \cap B)$. Thus, $E_{\succcurlyeq_F}$is indeed *meet-additive*, which is our thesis.□

Remark 13 *The foregoing result has been included here in order to stress one interesting similarity between FORs (especially FORs with a principal filter) and another much studied opportunity ranking motivated by 'freedom of choice', namely the cardinality-based one* $(P(X), \geq_\#)$. *Indeed, it is easily checked that for any* $A \subseteq X$, $E_{\geq_\#}(A) = A$, *i.e.* $E_{\geq_\#} = Id$. *It follows that* $E_{\geq_\#}$ *also satisfies* ULF, *and is a topological kernel operator (as well as a topological closure operator): clearly enough, the resulting topology is the* discrete *one.*

It turns out that the *ULF* property of FORs is a crucial part of a simple characterization of the latter, as stated in the following proposition:

Proposition 14 *Let* $(P(X), \succcurlyeq)$ *be an opportunity ranking. Then* $(P(X), \succcurlyeq)$ *is a FOR with order filter* $F \in \mathcal{F}(P(X), \supseteq)$ *if and only if* $(P(X), \succcurlyeq)$ *satisfies* PR,WM *and is such that* $E_{\succcurlyeq}$ *is* exactly doubly-filtral *and satisfies* ULF .

Proof. It follows from Proposition 6 ii) and Lemma 11 above that a FOR $(P(X), \succcurlyeq_F)$ satisfies *PR* and *WM,* and $E_{\succcurlyeq_F}$satisfies *ULF,* is *exactly doubly-filtral* and is an *order-embedding* of $(P(X), \succcurlyeq_F)$. To prove the converse implication, take an arbitrary opportunity ranking $(P(X), \succcurlyeq)$ that satisfies the foregoing properties. We have to prove that $(P(X), \succcurlyeq)$ is a FOR with order filter F, for some $F \in \mathcal{F}(P(X), \supseteq)$. Indeed, suppose not. Then, for *every* order filter $F \in \mathcal{F}(P(X), \supseteq)$ it must be the case that $\succcurlyeq \neq \succcurlyeq_F$ i.e. equivalently either (a) a pair $(A(F), B(F)) \in P(X) \times P(X)$ exists such that $A(F) \nsupseteq B(F)$, $B(F) \in F$, *and* $A(F) \succcurlyeq B(F)$, or (b) a pair $(A(F), B(F)) \in P(X) \times P(X)$ exists such that $B(F) \notin F$ and *not* $A(F) \succcurlyeq B(F)$ (the case with $A(F) \supseteq B(F)$ and *not* $A(F) \succcurlyeq B(F)$ can be ruled out immediately since by hypothesis $(P(X), \succcurlyeq)$ satisfies WM).

Let us first consider possibility (a), and take $F = \{A \subseteq X : \emptyset \neq A = E_{\succcurlyeq}(A)\}$: *exact double filtrality* of $E_{\succcurlyeq}$ entails that indeed $F \in \mathcal{F}(P(X), \succcurlyeq) \cap \mathcal{F}(P(X), \supseteq)$, and the identity function $Id : F \to F$ is an order-isomorphism between $(F, \succcurlyeq)$ and $(F, \supseteq)$. But then, $A(F) \succcurlyeq B(F)$ and $B(F) \in F$ jointly entail $A(F) \in F$: hence *exact double filtrality* entails $A(F) \supseteq B(F)$, contradicting (a).

Let us then consider possibility (b), and choose again

$$F = \{A \subseteq X : \emptyset \neq A = E_{\succcurlyeq}(A)\}.$$

Now, take any $B \subseteq X$ such that $B \notin F$. Then, by definition of F and the *ULF* property of $E_{\succcurlyeq}$, it must be the case that $E_{\succcurlyeq}(B) = \emptyset$. Thus, for any $A \subseteq X$, $E_{\succcurlyeq}(A) \supseteq E_{\succcurlyeq}(B)$, whence $E_{\succcurlyeq}(A) \succcurlyeq E_{\succcurlyeq}(B)$ (by *WM*). Since $E_{\succcurlyeq}$ is by hypothesis an *order-embedding* of $(P(X), \succcurlyeq)$, it follows that $A \succcurlyeq B$, contradicting (b) as well. Thus, we have singled out an order filter $F \in \mathcal{F}(P(X), \supseteq)$ such that $(P(X), \succcurlyeq) = (P(X), \succcurlyeq_F)$, and our thesis is therefore established. □

Remark 15 *In order to check general independence of the foregoing conditions, consider the following examples with* $\#X \geq 4$*:*

i) $(P(X), \geq_{\#})$ *is a preorder that satisfies* WM*, and* $E_{\geq_{\#}}$ *is* ULF (*and an* order embedding of $(P(X), \geq_{\#})$) *but is* not exactly doubly filtral ;

ii) $(P(X), \succcurlyeq_{\perp})$ *(where* $\succcurlyeq_{\perp} = \{(A, B) \in P(X) \times P(X) : \text{either } A = X \text{ or } A = B\}$*) is a (trivial) preorder that does* not *satisfy* WM*, and such that* $E_{\succcurlyeq_{\perp}}$ *is* ULF, exactly doubly filtral and an order-embedding of $(P(X), \succcurlyeq_{\perp})$ *(because* $E_{\succcurlyeq_{\perp}}(A) = \emptyset$ *for any* $A \neq X$*, and* $E_{\succcurlyeq_{\perp}}(X) = X$ *) ;*

iii) $(P(X), \succcurlyeq_F^{+})$ *, where* $F = \{A \subseteq X : A \supseteq \{x, y\}\}$ *and* $\succcurlyeq_F^{+} = \succcurlyeq_F \cup \{(\{x\}, \{y\}), (\{y\}, \{z\})\}$ *for some* $x, y, z \in X$ *such that* $\#\{x, y, z\} = 3$ *: here, by definition,* $E_{\succcurlyeq_F^{+}} = E_{\succcurlyeq_F}$*, while* $\succcurlyeq_F^{+}$ *satisfies* WM, but is not transitive*;*

iv) $(P(X), \succcurlyeq(\geq))$*, where*

$$\succcurlyeq(\geq) = \left\{ \begin{array}{l} (A, B) \in P(X) \times P(X) : A = X, \text{ or } A \supseteq B, \text{ or } B \neq X \text{ and} \\ \max_{\geq} A \geq \max_{\geq} B \text{ , or } A = \emptyset \text{ and } B = \{x\} \text{ for some } x \in X \end{array} \right\}$$

and $(X, \geq)$ *is a* linear order *(i.e. an antisymmetric,total, transitive binary relation), is a preorder that satisfies* WM *(by definition), while* $E_{\succcurlyeq(\geq)}$ *is* exactly doubly filtral *(because* $E_{\succcurlyeq(\geq)}(A) = A$ *iff* $A = X$*) and an* order-embedding, *but does not satisfy* ULF .

No claim of elegance or special prominence is made concerning the foregoing characterization of FORs . What *is* claimed is that characterizations of this sort –i.e. in terms of E-operators– may be much helpful for improving our understanding of the main structural differences between alternative opportunity rankings.

3 Filtral opportunity rankings : aggregation

Under the most obvious interpretation, FORs can be regarded as the individual evaluations–or opinions– on alternative opportunity sets which are entertained by agents of a certain society (even a Minskian "society of mind" might perhaps be included among the possibly relevant interpretations). A natural question then immediately arises: can FORs be aggregated according to some 'nice' general rule, and–if so– how?

Aggregation problems for opportunity rankings have been previously addressed by several authors. Indeed, Suppes(1987) includes a short discussion on the aggregation of *total* opportunity rankings as defined on *different* sets of alternatives, and admitting representations by numerical *ratio* scales. Dutta,Sen(1996) consider ways of aggregating the cardinality-based and the indirect-utility-maximization *total* opportunity preorders, and tackle the issue by reducing it to a two-agent Arrow-like aggregation problem on a *restricted* domain of profiles. In what follows we address the problem within a generalized ordinal Arrowian setting.

To begin with, we recall here the notion of an aggregation rule in an ordinal Arrowian framework .

Let X be the non-empty *finite* set of basic alternatives, N be the non-empty finite set of agents, $n = \#N$, $PR(P(X))$ the set of all preorders on $P(X)$, and $\mathcal{F}^*(P(X))$ the set of all filtral opportunity rankings (we already know from our first proposition above that $\mathcal{F}^*(P(X)) \subseteq PR(P(X))$). We also posit $\succcurlyeq_0 = \{(A, B) : A = X, \text{ or } B = \emptyset, \text{ or } A = B \}$, and denote by $\mathcal{F}^*_0(P(X))$ the $\succcurlyeq_0$-*augmented set of filtral opportunity rankings* i.e. $\mathcal{F}^*_0(P(X)) = \mathcal{F}^*(P(X)) \cup \{\succcurlyeq_0\}$ (it can be easily checked that $\succcurlyeq_0$is indeed a *preorder* that safisfies both *FI and RWM* , but not *WM :* hence, obviously $\succcurlyeq_0 \notin \mathcal{F}^*(P(X))$). For any $P \subseteq PR(P(X))$ a $N-$profile on P is a function $\pi : N \to P$. The set of all $N-$profiles on P is denoted by P^N. For any profile $\pi \in [PR(P(X))]^N$, and any pair $(A, B) \in P(X) \times P(X)$ we posit $N(\pi, (A, B)) = \{i \in N : (A, B) \in \pi(i)\}$ (where $\pi(i) = \succcurlyeq_i$). Similarly, for any $\pi \in [PR(P(X))]^N$ and any $\succcurlyeq \in PR(P(X))$ we also posit $N(\pi, \succcurlyeq) = \{i \in N : \text{for any } A, B \subseteq X, \text{ if } A \succcurlyeq B \text{ then } (A, B) \in \pi(i)\}$, and $N^*(\pi, \succcurlyeq) = \{i \in N : \text{for any } A, B \subseteq X, \text{ if } (A, B) \in \pi(i) \text{ then } A \succcurlyeq B\}$.

An *aggregation rule* on $[PR(P(X))]^N$ is a function $f : D^N \to D'$ with $D, D' \subseteq PR(P(X))$ (the intended interpretation being that $(A, B) \in f(\pi)$ whenever A embodies more freedom than B, according to 'aggregate' opportunity ranking $f(\pi)$). The following well-known properties of aggregation rules are also to be recalled :

(*Independence*) for any $A, B \subseteq X$, and for all $\pi, \pi' \in D^N$, if $N(\pi, (A, B)) = N(\pi', (A, B))$ then $(A, B) \in f(\pi)$ iff $(A, B) \in f(\pi')$.

(*Unanimity,* (or *Pareto Efficiency*)) for any $A, B \subseteq X$, and any $\pi \in D^N$, if $N(\pi, (A, B)) = N$ then $(A, B) \in f(\pi)$.

(*Anonymity*) for any permutation (i.e. bijection) $\sigma : N \to N$, and any $\pi \in D^N$, $f(\pi_\sigma) = f(\pi)$ (where for any $i \in N : \pi_\sigma(i) = \pi(\sigma(i))$).

Some other important –if less standard–properties of aggregation rules will also be used in the sequel (see e.g. Monjardet(1990) for a presentation of such properties in a more general setting, and with a slightly different terminology):

(*Isotony*) : for any $\pi, \pi' \in D^N$, if for every $i \in N$, and for any $A, B \subseteq X$, $(A, B) \in \pi(i)$ entails $(A, B) \in \pi'(i)$ then for any $A, B \subseteq X$, $(A, B) \in f(\pi)$ entails $(A, B) \in f(\pi')$.

(*Idempotence*) : for any $\pi \in D^N$, if a $\succcurlyeq \in D$ exists such that $\pi(i) = \succcurlyeq$ for every $i \in N$, then $f(\pi) = \succcurlyeq$.

(*Bi-Idempotence*) : for any $\pi \in D^N$, if a pair $\{\succcurlyeq, \succcurlyeq'\} \subseteq D$ exists such that $\pi(i) \in \{\succcurlyeq, \succcurlyeq'\}$ for every $i \in N$, then $f(\pi) \in \{\succcurlyeq, \succcurlyeq'\}$.

(*General Neutral N-Monotonicity*): for any $\succcurlyeq, \succcurlyeq' \in PR(P(X))$and $\pi, \pi' \in$

$[PR(P(X)]^N$, if $N(\pi, \succcurlyeq)) \subseteq N(\pi', \succcurlyeq')$ then $[((A,B) \in f(\pi)$ for any $A, B \subseteq X$ such that $A \succcurlyeq B$) entails $((C,D) \in f(\pi')$ for any $C, D \subseteq X$ such that $C \succcurlyeq' D)]$.

(*Dual General Neutral N-Monotonicity*): for any $\succcurlyeq, \succcurlyeq' \in PR(P(X))$, and $\pi, \pi' \in [PR(P(X))]^N$, if $N^*(\pi, \succcurlyeq) \subseteq N^*(\pi', \succcurlyeq')$ then [($(A,B) \in f(\pi)$ entails $A \succcurlyeq B$ for any $A, B \subseteq X$) entails $((C,D) \in f(\pi')$ entails $C \succcurlyeq' D$ for any $C, D \subseteq X$)].

It turns out that FORs–as opposed to, say, total preorders or linear orders–admit 'nice' i.e.*independent,Pareto efficient and anonymous* aggregation rules. In order to prove this fact, a few definitions are to be recalled. A *lattice* is a partially ordered set $(L, \geq)$ i.e. a transitive, reflexive, antisymmetric binary relation such that for any $x, y \in L$, both a *lowest upper bound* $x \vee y$ and a *greatest lower bound* $x \wedge y$ exist for the pair $\{x, y\}$. A lattice is *distributive* if for any $x, y, z \in L$: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (or equivalently $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$). The following lemma is the crucial step for proving the positive result on aggregation rules alluded to above.

Lemma 16 $(\digamma_0^*(P(X)), \supseteq)$ *is a* distributive lattice *isomorphic to the lattice* $(\digamma(P(X)), \subseteq)$ *of order filters of* $(P(X), \supseteq)$ *as ordered by set-inclusion.*

Proof. To begin with, it is well-known that $(\digamma(P(X)), \subseteq)$ is a distributive lattice (see e.g. Anderson(1987), lemma 13.1.4). Hence, proving our lemma amounts to exhibiting a latticial isomorphism $h : (\digamma(P(X)), \subseteq) \rightarrow (\digamma_0^*(P(X)), \supseteq)$ or equivalently a bijection h mapping the set $\digamma(P(X))$ of all order filters of $P(X)$ onto $\digamma_0^*(P(X))$ and such that, for any $F, F' \in \digamma(P(X))$, $F \subseteq F'$ if and only if $h(F) \supseteq h(F')$.

Thus, we define a function $h : \digamma(P(X)) \rightarrow \digamma_0^*(P(X))$ by positing, for any $F \in \digamma(P(X)) \setminus \{P(X)\}$,

$h(F) = \succcurlyeq_F = \{(A,B) : A, B \subseteq X,$ and either $A \supseteq B$ or $B \notin F\}$,

and $h(P(X)) = \succcurlyeq_0$. Clearly enough, $\succcurlyeq_F$is defined in an unambiguous way, hence h is well-defined as a function. If $F, F' \in \digamma(P(X))$ and $F \neq F'$, then w.l.o.g. $A \in F \backslash F'$ for some $A \subset X$. If $A = \emptyset$,then $F = P(X) \neq F'$ hence by definition $h(F) = \succcurlyeq_0 \neq h(F') = \succcurlyeq_{F'} \in \digamma^*(P(X))$. If $A \neq \emptyset$, then by definition $\emptyset \succcurlyeq_{F'} A$ and *not* $\emptyset \succcurlyeq_F A$, whence $h(F) \neq h(F')$. Moreover, if $\succcurlyeq \in \digamma^*(P(X))$ then by definition $\succcurlyeq = \succcurlyeq_F$i.e. $\succcurlyeq = h(F)$ for some $F \in \digamma^*(P(X))$. It follows that h is indeed a bijection .

Next, consider $F, F' \in \digamma(P(X))$ such that $F \subseteq F'$. Two cases must be distinguished, namely $F' \neq P(X)$ and $F' = P(X)$. If $F' \neq P(X)$ then by definition $F \subseteq F'$ entails $h(F') = \{(A,B) : A, B \subseteq X,$ and either $A \supseteq B$ or $B \notin F'\} \subseteq \{(A,B) : A, B \subseteq X$ and either $A \supseteq B$ or $B \notin F\} = h(F)$, and vice versa.

If $F' = P(X)$, then

$h(F') = \succcurlyeq_0 = \{(A,B) : A, B \subseteq X,$ and $A = X,$ or $B = \emptyset$, or $A = B\} \subseteq$

$\subseteq \{(A,B) : A, B \subseteq X$ and either $A \supseteq B$ or $B \notin F\} = h(F)$

while $h(F) \supseteq h(P(X)) = \succcurlyeq_0$ obviously entails $F \subseteq P(X)$. □

Proposition 17 *Let* $f : [\digamma_0^*(P(X))]^N \rightarrow \digamma_0^*(P(X))$ *be an aggregation rule for the* $\succcurlyeq_0$ *−augmented set of FORs. Then i) f satisfies* Anonymity, Independence,Unanimity, Isotony and Bi-Idempotence *iff –for any* $\pi \in [\digamma_0^*(P(X))]^N$–

$f(\pi) = \bigcup_{\{S\subseteq N:\#S\geq\lfloor n/2\rfloor+1\}} \bigcap_{i\in S} \pi(i)$, *i.e.f is the* Simple Majority Rule *; ii) f satisfies* Anonymity, General Neutral N-Monotonicity and Idempotence *iff –for any* $\pi \in [F_0^*(P(X))]^N$*–* $f(\pi) = \bigcap_{i\in N} \pi(i)$ *, i.e. f is the* Unanimity Rule*; iii) f satisfies* Anonymity, Dual General Neutral N-Monotonicity, and Idempotence *iff –for any* $\pi \in [F_0^*(P(X))]^N$*–* $f(\pi) = \bigcup_{i\in N} \pi(i)$ *, i.e. f is the* Acceptance Rule.

Proof. This proposition is a straightforward corollary of Lemma 16 as combined with some known results on lattice-polynomial aggregation functions as presented in Monjardet (1990). In particular, i) follows from our Lemma and Corollary 7.4 in Monjardet(1990) which characterizes the majority aggregation rule on distributive lattices; ii) follows from our Lemma and Corollary 1 as stated in Barthélemy, Leclerc, Monjardet (1984); iii) follows from (latticial) dualization of the argument implicit in ii) . □

It should be remarked that the foregoing Proposition amounts to a strongly positive result on Arrowian-style aggregation (it should also be emphasized here that the set of indirect-preference-maximizing total preorders arising from a linear order on X is order-isomorphic to the set of all linear orders on X and therefore–in view of Arrow's Impossibility Theorem– *not* amenable to a 'nice' aggregation process, while the cardinality-based total preorder on $P(X)$ is *uniquely* defined hence not amenable to *any non-trivial* aggregation).

Indeed, as mentioned above, we may use the label 'nice' – consistently with standard Arrowian aggregation theory–for aggregation rules satisfying *Anonymity, Independence, and Unanimity (i.e. Pareto Efficiency).* Then, it is immediately checked by direct inspection that *the Simple Majority Rule, the Unanimity Rule, and the Acceptance Rule as defined above are indeed 'nice'.* Our previous claim to the effect that the set of FORs (as properly augmented) allows for 'nice' solutions of the classic aggregation problem is therefore established.

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