QUADERNI

Università degli Studi di Siena
DIPARTIMENTO DI ECONOMIA POLITICA

Sydney Afriat

The Connection between Demand and Utility

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1 Demand and utility

A *demand element* is any $(p, x) \in B \times C$ for which $px > 0$, showing a commodity bundle x demanded at prices p, the *expenditure* being px. Then with

$$u = (px)^{-1}p,$$

which is the associated *budget vector*, (u, x) is such that $ux = 1$, so it is a *normal* demand element, in this case the *normalization* of (p, x).[1] Various conditions involving demand elements can be stated well and more simply in terms of their normalizations.

A *utility order* is any $R \subset C \times C$ which is reflexive and transitive,

$$xRx,\ xRyRz \ \Rightarrow\ xRz,$$

xRy being the statement that x has at least the utility of y.[2]

The *chain-extension* of any relation R is the relation $\overrightarrow{R}$ holding between extremities of R-chains, given by

$$x\overrightarrow{R}z \equiv (\vee\, y \cdots)xRyR \cdots Rz.$$

This is the same as the *transitive closure*, or the smallest transitive relation containing R, being transitive, containing R, and contained in every transitive relation that contains R. The extended condition

$$xRyR \cdots Rz \ \Rightarrow\ xRz$$

is equivalent to transitivity, which therefore is equivalent to the condition

$$\overrightarrow{R} \subset R,$$

for R to be identical with its chain-extension, or transitive closure.

With $\overline{R}$ as the *complement* and R' the *converse*, where

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[1] In present notation, with Ω as the non-negative numbers, $B = \Omega_n$ is the *budget space* (non-negative row vectors) and $C = \Omega^n$ the *commodity space* (column vectors). Then any $p \in B$, $x \in C$ determine $px \in \Omega$ for the value of the commodity bundle x at the prices p. Sometimes when dealing with demand functions Ω should be the positive numbers. As a syntax rule, a scalar multiplies a row vector on the left and a column vector on the right.

[2] With a binary relation R, beside the usual $(x, y) \in R$ because R is set, also the statements xRy, $x \in Ry$ or $y \in xR$ are available to assert (x, y) is an element of R, or that x has the relation R to y.

$$x\overline{R}y \equiv \ \sim xRy, \ xR'y \equiv yRx,$$

we have

$$\overline{(R')} = (\overline{R})',$$

so there is no ambiguity in the expression $\overline{R}'$ for the *converse complement*.

The relation of equivalence in R, or the *indifference* relation, is the symmetric part

$$E = R \cap R',$$

an equivalence relation, symmetric, reflexive and transitive, since R is an order. The equivalence classes E_x, which are equally the sets xE or the Ex, these being the same from symmetry, are such that $x \in E_x$, so their union is C, and

$$xEy \ \Leftrightarrow \ E_x = E_y, \ x\overline{E}y \ \Leftrightarrow \ E_x \cap E_y = O,$$

so any pair are either disjoint or identical. Hence they constitute a partition of C, expressing C as a union of disjoint subsets.

The antisymmetric part of R, the *strict preference* relation, is

$$P = R \cap \overline{R}',$$

which is a strict order, irreflexive and transitive[3], since R is an order.

The subrelations E and P form a partition of R,

$$P \cap E = O, \ P \cup E = R.$$

An order R is *complete* if

$$\sim xRy \ \Rightarrow \ yRx,$$

so for any pair of elements, if they do not have the relation one way, then they have it the other, or they have it one way or the other and possibly both. That is, $\overline{R}' \subset R$, and equivalently, $P = \overline{R}'$.

A *simple* order R is such that

$$xRyRx \ \Rightarrow \ x = y.$$

For an order, this is equivalent to

$$xRyR \cdots Rx \ \Rightarrow \ x = y = \cdots$$

Otherwise, this is the condition for any relation R to be *anticyclic*, or for the absence of R-cycles of distinct elements. For a reflexive

[3] Beale and Drazin (1956) recognize this transitivity which is basic for the scheme adopted here. Also indifference, sometimes treated as absence of preference and then problematic because without transitivity, is here taken to be a positive condition made from comparison both ways, necessarily transitive, as required if we are to have an equivalence relation.

relation, it is the condition for the transitive closure to be a simple order.

The relations I and D of *identity* and *distinction* are given by

$$xIy \equiv x = y, \; xDy \equiv x \neq y.$$

For any simple order R, the symmetric part is $E = I$, so equivalence in R reduces to identity. In this case the antisymmetric part is identical with the irreflexive part, $P = R \cap D$, from which R is recovered as the reflexive closure, $R = P \cup I$.

A *refinement* of a simple order R is another S which properly contains it,

$$R \subset S, \; R \neq S.$$

A *total* order is one which is simple and complete.

Theorem 1.1 Any simple order is either a total order, or it has a refinement, and not both.

Suppose R is a simple order, but not complete, so for some x, y

$$x \neq y, \; \sim xRy, \; \sim yRx,$$

and let Q be R with the element (x, y) adjoined. Then Q, reflexive since R is reflexive, is also anticyclic. For any Q-cycle must be either an R-cycle, impossible since R is anticyclic, or it contains the link (x, y) together with an R-chain from y to x, which by transitivity of R implies yRx, contrary to hypothesis and so again impossible. Since Q therefore is reflexive and anticyclic, its transitive closure is a simple order. It properly contains R, since from $R \subset Q$ with the transitivity of R we have

$$R = \overrightarrow{R} \subset \overrightarrow{Q},$$

and also $x\overrightarrow{Q}y$ while $\sim xRy$, showing that $R \neq \overrightarrow{Q}$. Hence it is a refinement of R.

Now suppose T is a total order, simple and complete. Then it cannot have any refinement. For were R one, we would have $T \subset R$, and for some x, y

$$x \neq y, xRy, \; \sim xTy.$$

But then because T is complete, yTx, and then also yRx, and then, since R is simple, $x = y$, so there is a contradiction.

Corollary In a finite set, any simple order is either a total order, or it has a total order refinement.

For, because the set is finite, so the set of orders is also, any chain of refinements of a given simple order must terminate. This can only be in a simple order without refinement and hence, by the Theorem, in a total order, refining the original.

Theorem 1.2 Any simple order is either a total order, or it has a total order refinement.

This restates the theorem of Szpilrajn (1930). Above is a proof for the finite case. The general proof depends on the axiom: every chain in a set of sets has a maximal refinement. A *chain* in a set S of sets is a subset $C \subset S$ such that, for any $A, B \in C,$ either $A \subset B$ or $B \subset A$. It has another chain D as a *refinement* if $C \subset D$. A *maximal* chain is one without refinements.

If a given simple order R is a total order, then it has no refinement, and otherwise it does, by Theorem 1.1. In that case let S consist of R and its refinements. By the axiom, the chain in S which consists of R alone has a maximal refinement M. Then the union $T = \cup M$ of the elements of M is a refinement of $R,$ and it must be a total order. For otherwise, by Theorem 1.1, it would have a refinement $T^+,$ and then the set M^+ obtained by adjoining T^+ to M would be a chain in S that refines M, contradicting that M is maximal.

For any order R, the quotient relation $\mathcal{R} = R/E$, in the quotient space $\mathcal{C} = C/E$ whose points are the equivalence classes of E, is a simple order, such that

$$E_x \mathcal{R} E_y \ \Leftrightarrow\ xRy,$$

such $\mathcal{R}$ being defined because

$$xEx' \wedge yEy' \,.\Rightarrow.\, xRy \ \Leftrightarrow\ x'Ry'.$$[4]

[4] The ‘indifference map’ introduced by Pareto, which represents utility free of the numerical aspect, brings attention to the utility order as constructed from a simple order of indifference classes. It really is a preference map even though it may have been called an indifference map because, from monotonicity considerations, when the classes are given there can be no doubt at all about their order.

Theorem 1.3 Any order is contained in some complete order with the same equivalence relation.

Apply Theorem 1.2 to the quotient relation $\mathcal{R}$.

Corollary For any order R there exists a complete order R^+ such that

$$R \subset R^+,\ P \subset P^+.$$

For

$$R \subset R^+ . \Rightarrow . \, P \subset P^+ \Leftrightarrow\ E = E^+.$$

A *utility function* is any $\phi\colon C \to \Omega$. It *represents* the utility order R for which

$$xRy \equiv \phi(x) \geq \phi(y).$$

In this case, for the symmetric and antisymmetric parts,

$$xEy \Leftrightarrow \phi(x) = \phi(y),\ xPy \Leftrightarrow \phi(x) > \phi(y).$$

Any utility order so representable by a utility function is necessarily complete.

2 Demand–utility compatibility

Relations connecting a demand element (p, x) and a utility order R are defined by

$$H' \equiv py \leq px \ \Rightarrow\ xRy,$$

which corresponds to the *cost-effectiveness* familiar in cost-benefit analysis, and asserts x is as good as any bundle that costs no more, and

$$H'' \equiv yRx \ \Rightarrow\ py \geq px,$$

cost-efficiency, that any bundle as good as x costs as much.

While H' represents *utility maximization*, making x a bundle that has maximum utility for the money spent, H'' represents *cost minimization*, making x have minimum cost for the utility obtained. These are equally compelling, generally independent basic economic principles. A later issue involving stricter conditions concerns whether x is the unique bundle admitted by these conditions.

The combination

$$H \equiv H' \wedge H''$$

defines *compatibility* between the demand and the utility.

In terms of normalizations, these conditions become

$$uy \leq 1 \;\Rightarrow\; xRy,\;\; yRx \;\Rightarrow\; uy \geq 1,$$

respectively.

Introducing relations $W, V, I \subset B \times C$ by

$$uWy \equiv uy \leq 1,\;\; uVy \equiv uy < 1,\;\; uIy \equiv uy = 1,$$

by which any commodity bundle y is *within*, *under* or *on* a budget u, the conditions are stated

$$uW \subset xR,\;\; Rx \subset u\overline{V}.$$

The following has reference to a given utility order R, and demand element (p, x) with normalization (u, x).

Theorem 2.1 If R is complete and xR is closed, then $H'' \Rightarrow H'$.

From the hypothesis, together with H$''$, we have to deduce H$'$. We have

$$\begin{aligned} uVy \;&\Rightarrow\; \sim yRx && \because H'' \text{ contrapositive} \\ &\Rightarrow\; xRy && \because \text{completeness} \end{aligned}$$

$$\therefore uV \subset xR.$$

But xR is closed, and uW is the closure of uV. Hence

$$uW \subset xR,$$

which is H'. QED

We have *oversatiation* at a point y if there is a bundle z which is less but as good, that is,

$$z < y \;\wedge zRy.$$

The denial of such a possibility, or *insatiability*, requires

$$z < y \;\Rightarrow\; \sim zRy.$$

Theorem 2.2 If R is insatiable, then $H' \;\Rightarrow\; H''$.

With denial of H'', and assumption of H', we will find the insatiability hypothesis contradicted.

From the denial, $zRx \wedge uz < 1$ for some z. Then $uy \leq 1$ for some $y > z$. Then by H', from $uy \leq 1$ we have xRy, which with zRx and transitivity gives zRy, and this together with $z < y$ contradicts the hypothesis. QED

A function $\phi(x)$ is *non-decreasing*, *semi-increasing*, *or increasing* according to the conditions

$$x \leq y \Rightarrow \phi(x) \leq \phi(y),$$
$$x < y \Rightarrow \phi(x) < \phi(y),$$
$$x \lesssim y \Rightarrow \phi(x) < \phi(y).$$[5]

The three different conditions are increasingly restrictive for a continuous function. For a differentiable function ϕ with gradient g they require

$$g \geq o,\ g \gtrsim o,\ g > o.$$

For an example of the intermediate case, the Leontief type function

$$\phi(x) = \max\{t : at \leq x\},$$

for any $a \in C$, is semi-increasing but not increasing.

Representation of utility by a semi-increasing function assures insatiability.

Theorem 2.3 If the utility is representable by a continuous semi-increasing function, then $H' \Leftrightarrow H''$.

With the representation by a continuous function, we have the completeness and closure which provides $H'' \Rightarrow H'$ by Theorem 2.1. If also the function is semi-increasing, the insatiability condition in Theorem 2.2 is obtained so that $H' \Rightarrow H''$. QED

Consequently, for the conjunction H required by compatibility, under such usual conditions we have both $H \Leftrightarrow H'$ and $H \Leftrightarrow H''$. Or in place of the traditional H' of demand analysis, for cost effectiveness or utility maximization, the H'' for cost-efficiency or cost minimization can serve just as well, and permits another development.

3 Strict compatibility

The condition

$$H^o \equiv py \leq px \wedge yRx \Rightarrow y = x,$$

[5] For notation, $x \leq y \equiv x_i \leq y_i$ for all i, $x \lesssim y \equiv x \leq y \wedge x \neq y$, $\mathrm{x} < \mathrm{y} \equiv \mathrm{x}_i < \mathrm{y}_i$ for all i.

here put symmetrically, has alternative statements,

$$\text{(i)}\ \ py\ \leq px \wedge y \neq x\ \Rightarrow\ \ \sim yRx,$$

which exposes a relationship with H', and

$$\text{(ii)}\ \ yRx \wedge y \neq x\ \Rightarrow\ py > px,$$

with H''.

We also consider

$$H^* \equiv py \leq px \wedge y \neq x\ \Rightarrow\ xRy \wedge\ \sim yRx,$$

which, in the case of R being complete, is equivalent to (i), and so to H^o.

According to H^o, any bundle as good as x and costing no more must be identical with x. Reflexivity of R already allows x itself is such a bundle, so the converse is already present.

The antisymmetric or strict part of R being $P = R \cap \overline{R}'$, in terms of this

$$H^* \equiv py \leq px \wedge y \neq x\ \Rightarrow\ xPy.$$

For the case where R is complete, $\overline{R}' \subset R$ and hence $P = \overline{R}'$, so H^* becomes the same as H^o.

As appears from forms (i) and (ii), when H^o is adjoined to each of H' and H'' we obtain the *strict* versions of these conditions, that require x to be the unique bundle which attains the required maximum utility, and minimum cost.

While the conjunction of H' and H'' provides compatibility H, we have the conjunction of the strict versions $H^o \wedge H'$ and $H^o \wedge H''$ to define *strict compatibility*. Since

$$(H^o \wedge H') \wedge (H^o \wedge H'')\ \Leftrightarrow\ H^o \wedge (H' \wedge H''),$$

this condition is also $H^o \wedge H$.

Theorem 3.1 Strict compatibility, simultaneously requiring strict cost-effectiveness and strict cost-efficiency, is obtained by the condition H^*, which implies H, and is equivalent to H^o if R is complete.

It is immediate that

$$H^*\ \Leftrightarrow\ H^o \wedge H',$$

and also

$$H^o\ \Rightarrow\ H''.$$

Therefore, with $H \equiv H' \wedge H''$, we have

$$H^* \Leftrightarrow H^o \wedge H,$$

as required. Consequently also $H^* \Rightarrow H$. The last part has already been remarked.

4 Utility-cost

The *utility-cost function* based on R is

$$\rho(p, x) = \min\{py : yRx\},$$

which, when it is defined, tells the cost at prices p of attaining the utility of a commodity bundle x.[6] According to this definition, for all p,

$$\rho(p, x) \leq py \text{ for all } y \in Rx,$$

and

$$\rho(p, x) = pz \text{ for some } z \in Rx.$$

Uniqueness for such z, provided by strict compatibility, will be significant when we come to consider the differentiability of ρ as a function of p.

Generally we have the cost *limit function*

$$\rho(p, x) = \inf\{py : yRx\},$$

which is always defined, and coincides with the cost function when this is defined. When it is, as is assured by the cost-efficiency part of the compatibility condition, we can say ρ is *attained*. For this definition of ρ,

$$\rho(p, x) \leq py \text{ for all } y \in Rx,$$

and for any $t > \rho(p, x)$,

$$py < t \text{ for some } y \in Rx.$$

In other words, ρ is a greatest lower bound of the values $\{py\}_{yRx}$, being a lower bound while any value greater is not.

From the form of its definition as lower limit of a family $\{py$

[6] This form of definition which has reference to an order, any reflexive transitive binary relation R in the commodity space, serves to emphasize the absence of prior assumptions, even that utility has a numerical representation. It is important in other connections, as when $\rho(p, x)$ takes special forms which cannot be stated with the usual definition which involves a numerical utility level, for instance when $\rho(p, x) = \theta(p)\phi(x)$, which makes expansion paths to be rays and is significant in dealing with price indices, or $\rho(p, x) = \theta(p)\phi(x) + \nu(p)$, which makes them general lines, as required for the "New Formula" of Wald (1939), account of which is in my 1987 book.

$\}_{yRx}$ of homogeneous linear functions py, with gradients $y \in Rx$, $\rho(p, x)$ is concave conical[7] in p.

Theorem 4.1 For any x, the function $\rho(p, x)$ is attained for $p > 0$, if the sets Rx are closed.

Since xRx, we also have

$$\rho(p, x) \ = \ \min\{py : py \leq px, yRx\},$$

and, for $p > 0$, the budget set

$$\{py : py \leq px\}$$

is compact, and so is its intersection with Rx if this is closed. Hence, by the theorem of Weierstrass, the minimum of py in this intersection is attained.

Theorem 4.2

(i) The inequality

$$\rho(p, x) \leq px$$

holds for all p and x.

(ii) The equality

$$\rho(p, x) = px$$

is equivalent to the condition H″.

Proof (i):

From the definition of ρ,

$$yRx \ \Rightarrow \ \rho(p, x) \leq py.$$

Therefore, since xRx, we have the inequality.

Proof (ii):

With the case of the equality, we have

$$yRx \ \Rightarrow \ py \geq px,$$

which is H''. Conversely, from this we directly get

$$\rho(p, x) \geq px.$$

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[7] A function $f(x)$ is *conical* if its graph $\{(x, y) : y = f(x)\}$ is a cone or, what is the same, $f(xt) = f(x)t$ $(t > 0)$.

But in any case we also have the inequality (i), and hence the equality.

Corollary If H'' holds, then ρ is attained.

The Theorem shows it is attained, with the value px. Consequently we have

$$\rho(p, x) \;=\; \min\{py : yRx\} = px.$$

Theorem 4.3 For all p,

$$xRy \;\Rightarrow\; \rho(p, x) \geq \rho(p, y),$$

for all x, y.

By transitivity of R,

$$\begin{aligned} xRy \;&\Rightarrow\; (zRx \;\Rightarrow\; zRy) \\ &\Leftrightarrow\; Rx \subset Ry \\ &\Rightarrow\; \inf\{pz : \; z \in Rx\} \geq \inf\{pz : \; z \in Ry\} \\ &\Leftrightarrow\; \rho(p, \; x) \geq \rho(p, \; y), \end{aligned}$$

and hence the conclusion.

5 Cost differentiability

The approach to investigating differentiability of $\rho(p, x)$ as a function of $p > o$ will be made with reference to a given normal demand element (u, x) $(ux = 1)$, with $u > o$, strictly compatible with the utility.

Take $S = Rx$, so $x \in S$, and $\sigma(p) = \rho(p, x)$, so

$$\sigma(p) \;=\; \inf\{py : y \in S\}$$

is the *support function* of the set S. The strict compatibility provides that

$$\sigma(u) = ux = 1,$$

and

$$y \in S \wedge y \neq x \;\Rightarrow\; \sigma(u) < uy.$$

We investigate the differentiability of σ at the point $p = u$, with gradient to be given by

$$\sigma_{u'} = x.^{8}$$

Directly from the form of its definition, σ is a concave conical function, as remarked already in reference to ρ.

Since it is conical, it has any y as a *bounding gradient* if

$$\sigma(p) \leq py \text{ for all } p > o,$$

and a *support gradient* at the point $p = u$ if moreover

$$\sigma(u) = uy.^{9}$$

The bounding gradients of σ are a closed orthoconvex[10] set, identical with the closed orthoconvex closure of the original set S, or with S itself if this is already closed orthoconvex (as, for instance, when R is representable by a continuous increasing quasiconcave utility function). The support gradients describe the boundary of this set.

Since σ is concave, differentiability at $p = u$ depends on the support gradient there being unique. In that case the differential gradient exists, and coincides with the unique support gradient.[11]

From its definition as support function of S, σ is the largest function that admits S as a set of bounding gradients. Then the compatibility provides x as a support gradient at the point u, and since this is strict compatibility, it is the unique one which is also an element of S. For differentiability, we need to exclude the possibility that there might still be some others, not elements of S.

The elements of the closed orthoconvex closure $\tilde{S}$ of S are all bounding gradients of σ. This is the largest set which has σ as support function. Its boundary, the *support locus*, describes all possible support gradients, including x which is a support gradient

[8] This corresponds to the proposition frequently offered as an unrestricted mechanical rule and referred to as "Shephard's Lemma", or the "Shephard-Uzawa-McFadden Lemma".

[9] A function $f(x)$ has $l(x)$ as a *linear bound* if this is linear and $f(x) \leq l(x)$ for all x in the domain, and it is a *support* at a point y if also $f(y) = l(y)$, in which case its gradient v is a *support gradient* of f at the point $x = y$, for which a necessary and sufficient condition is that $f(x) - f(y) \leq v(x - y)$ for all x. There is some difference in dealing with a conical function.

For a conical function $f(x)$, any v is a *bounding gradient* if $f(x) \leq vx$ for all x, and this is a *support gradient* at a point y if also $f(y) = vy$. Discussion of these definitions and related points is in my 1987 book, Pt. V, Ch. 2.

[10] A set X is *orthogenous* if $x \in X \wedge x \leq y \Rightarrow y \in X$, and *orthoconvex* if also convex.

[11] The Support Theorem for convex functions assures the existence of a support at any interior point in the open set where it is defined. But here we already have a support provided at the point u, with gradient given by x. We appeal now just to the Differentiability Theorem which asserts differentiability at exactly those points where the support is unique.

of σ at the point u.

While, from strict compatibility,

$$uI \cap S = \{y : uy = 1 \wedge y \in S\} = \{x\},$$

this being the set containing the element x alone, all the support gradients to σ at the point u describe the closed convex set

$$\mathcal{X} = \{y : uy = 1 \wedge y \in \widetilde{S}\} = uI \cap \widetilde{S},$$

which includes x. The question is whether $\mathcal{X}$ is properly larger than $\{x\}$.

There can be no comment about this without entertainment of some auxilliary assumption about the utility R. If R is represented by a continuous semi-increasing quasiconcave function, then $S = Rx$ is already closed orthoconvex, so $S = \widetilde{S}$. Then immediately $\mathcal{X} = \{x\}$, and the issue disappears. But without some limitation on R this is not assured.

The assumption that $S = Rx$ is closed will suffice. For, with this, and $u > o$, so that

$$uy = 1 \wedge y \lesssim z \Rightarrow uz > 1,$$

the intersection of the closed convex closure $\widetilde{S}$ of S with its supporting hyperplane uI is unchanged when $\widetilde{S}$ is replaced by the convex closure $< S >$ of S. Then

$$\begin{aligned}
\mathcal{X} &= uI \cap \widetilde{S} \\
&= uI \cap < S > \\
&= \langle uI \cap S \rangle \\
&= < \{x\} > \\
&= \{x\}.
\end{aligned}$$

Hence:

Theorem 5.1 If (p, x) $(p > o)$ is a demand element strictly compatible with a utility R for which the set Rx is closed, and $\rho(p, x)$ is the utility-cost function based on R, then ρ is differentiable as a function of p, with gradient $\rho_{p'} = x$.

6 Demand–utility logic

With prices p, and an amount M of money to be spent on some bundle of goods x, there is the budget constraint $px = M$. Given a function $x = F(p, M)$ that determines the unique maximum of a function ϕ under any budget constraint, F is a *demand function* which has ϕ as a utility function, or is *derived* from ϕ.

With any given function $F,$ it may be asked whether or not it is *consistent* in being so associated with a utility function. The function F first must have the properties

$$pF(p,M) = M,\ F(p,M) = F(M^{-1}p,1),$$

usually attributed to consumer demand functions. Then a function ϕ is sought for which, for all p *and* M, $x = F(p,M)$ is the unique maximum of ϕ under the constraint $px \leq M$, that is,

$$py \leq M \wedge y \neq x \ \Rightarrow\ \phi(y) < \phi(x).$$

For simplification, introduce the *budget vector* $u = M^{-1}p$, so the budget constraint $px = M$ is stated $ux = 1$. The *standard* demand function F determines the *normal* demand function f, its *normalization*, given by

$$f(u) = F(u,1),$$

with the property

$$uf(u) = 1,$$

from which it is recovered as

$$F(p,M) = f(M^{-1}p).$$

It is simpler, and has other advantage, to deal with the question about F through its normalization f. For similar reasons, a utility order R can take the place of the utility function. If it is the order represented by the function, it provides all that is important about the function. But it is natural to have an arbitrary order in view, free of such representation.

A demand function f is *compatible* with a utility R if every demand element (u,x) which, being such that $x = f(u)$, so it belongs to f, is compatible with R. With $H_f(R)$ denoting this condition, H_f asserts the existence of a such a compatible R, or the *consistency* of f.

Similarly $H^*_f(R)$ can assert *strict compatibility*, and H^*_f the *strict consistency* of f.

For $H^*_f(R)$ we have that for all u, and $x = f(u)$,

$$uy \leq 1 \wedge y \neq x \ \Rightarrow\ xRy \wedge\ \sim yRx.$$

Therefore, for any cyclic sequence $u_0, u_1, \ldots, u_m, u_0, \ldots,$

$$u_0x_1 \leq 1 \wedge u_1x_2 \leq 1 \wedge \cdots \wedge u_{m-1}x_m \leq 1$$
$$\Downarrow$$

$$\begin{gathered} x_0 R x_1 \ldots R x_m \\ \Downarrow \\ x_0 R x_m. \end{gathered}$$

But also

$$u_m x_0 \leq 1 \wedge x_0 \neq x_m \;\Rightarrow\; \sim x_0 R x_m.$$

Therefore

$$\begin{gathered} u_0 x_1 \leq 1 \wedge u_1 x_2 \leq 1 \wedge \cdots \wedge u_m x_0 \leq 1 \\ \Downarrow \\ x_0 = x_m. \end{gathered}$$

This condition on *f*, to be denoted K^*_f, has been seen to be a consequence of the strict consistency of *f*,

$$H^*_f \;\Rightarrow\; K^*_f.$$

From the cyclic symmetry, it is equivalent to the *strict cyclical consistency* condition

$$\begin{gathered} u_0 x_1 \leq 1 \wedge u_1 x_2 \leq 1 \wedge \cdots \wedge u_m x_0 \leq 1 \\ \Downarrow \\ x_0 = x_1 = \cdots = x_m. \end{gathered}$$

Then it is also equivalent to

$$\begin{gathered} u_0 x_1 \leq 1 \wedge u_1 x_2 \leq 1 \wedge \cdots \wedge u_{m-1} x_m \leq 1 \\ \wedge \\ x_0 \neq x_1 \vee x_1 \neq x_2 \vee \cdots \vee x_{m-1} \neq x_m \\ \Downarrow \\ u_m x_0 > 1. \end{gathered}$$

and to

$$\begin{gathered} u_0 x_1 \leq 1 \wedge u_1 x_2 \leq 1 \wedge \cdots \wedge u_{m-1} x_m \leq 1 \\ \wedge \\ x_m \neq x_0 \\ \Downarrow \\ u_m x_0 > 1. \end{gathered}$$

This last form shows the condition obtained by Houthakker (1950), elaborating the `revealed preference' method of Samuelson (1948). A part of it is that

$$u_0 x_1 \leq 1 \wedge x_1 \neq x_0 \;\Rightarrow\; u_1 x_0 > 1,$$

which is Samuelson's condition.[12]

Let $R_f(u)$, the *directly revealed preference* relation of *f* associated with the budget *u*, be defined by

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[12] Samuelson dealt with the two-commodity case, for which his and Houthakker's condition are equivalent, as proved by Rose (1958), and again by Afriat (1965).

$$xR_f(u)y \equiv x = f(u) \wedge \mathrm{uy} \leq 1,$$

and let R_f, the *revealed preference* relation of *f*, be the transitive closure of the union of these,

$$R_f = \overrightarrow{\bigcup_u} R_f(u).$$[13]

This is reflexive[14] because the $R_f(u)$ are reflexive, and transitive by construction as a transitive closure, so it is an order.

Another expression for K_f^*, proceeding from the original statement, is that, for $x = f(u)$,

$$uy \ \leq 1 \wedge yR_f x \ \Rightarrow \ y = x.$$

Since $uy \leq 1 \ \Rightarrow \ xR_f y$, this is equivalent to

$$uy \leq 1 \wedge y \neq x \ \Rightarrow \ xR_f y \wedge \ \sim yR_f x,$$

that is, $\mathrm{H}_f^*(\mathrm{R}_f)$, so we have

$$H_f^* \ \Rightarrow \ K_f^* \ \Rightarrow \ H_f^*(R_f) \ \Rightarrow \ H_f^*,$$

and hence:

Theorem 6.1 $\qquad H_f^* \ \Leftrightarrow \ H_f^*(R_f) \ \Leftrightarrow \ K_f^*.$

In other words, a demand function is strictly consistent, or strictly compatible with some utility order[15], if and only if it is strictly compatible with its own revealed preference order, and this is if and only if the strict cyclical consistency condition holds.[16]

The *strict revealed preference* relation of f is the strict or antisymmetric part of R_f,

$$P_f = R_f \cap \overline{R}_f',$$

and the *revealed indifference* relation is the symmetric part

$$E_f = R_f \cap R_f'.$$

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[13] Also Hirofumi Uzawa, dealing with this subject in the 1950s, made use of the transitive closure.

[14] Rather, it is reflexive just at points in the range of the demand function. Without altering anything important but to give respect to the definition of an order, it could be made reflexive by taking its reflexive closure, or union with ‘=’.

[15] The present theorem has no requirements at all about the utility order, or about the demand function. Samuelson and Houthakker sought a continuous numerical utility, involving auxilliary assumptions about the demand function and a differential equation method. The following asks less about the demand dunction and the utility: for a demand function *f* to have a lower semicontinuous numerical utility, it is necessary and sufficient that Houthakker's condition holds and the sets $f^{-1}(x)$ be closed.

[16] From recollection I believe Kotaro Suzamura offered a similar proposition.

The *directly revealed strict preference* relation is the irreflexive part of R_f,

$$S_f = R_f \cap D,$$

so this is irreflexive by construction, though not transitive. Its transitive closure,

$$T_f = \overrightarrow{S_f},$$

is the *revealed strict preference* relation, transitive by construction, not necessarily irreflexive.

Other expressions for K_f^* are

(i) $E_f = I$

(ii) $P_f = S_f$

(iii) $P_f = T_f$

(iv) S_f is transitive

(v) T_f is irreflexive

(vi) $S_f = T_f$

With revealed preferences there can be none of the "violation of transitivity" sometimes entertained, and no inconsistencies obtained from them alone. They are transitive by construction and any contradictions come only when they are taken together with the less well-noticed revealed non-preferences. With Samuelson, for instance, these are provided by

$$py \leq px \wedge y \neq x \;\Rightarrow\; \sim yRx,$$

as part of the strict compatibility H^*, or instead there are fewer coming from

$$py < px \;\Rightarrow\; \sim yRx,$$

which is the H'' part of the weaker compatibility condition H.

7 Demand–utility calculus

Consider a normal demand function f, strictly compatible with a utility R for which the sets Rx are closed. If ρ is the utility-cost function based on R then, for any u,

$$ux = 1 \wedge \rho(u, x) = 1 \;\Leftrightarrow\; x = f(u).$$

For a fixed u and $x = f(u)$, so $ux = 1$, the function

$$\rho(v, x) = \min\{vy : yRx\}$$

is defined for all v, and such that

$$\rho(v,x) \leq vy \text{ for all } y \in Rx,$$

and

$$\rho(v,x) = vz \text{ for unique } z \in Rx.$$

Now with x already fixed, and

$$\rho = \rho(v,x),\ w = \rho^{-1}v,$$

we have

$$vz = \rho,\ wz = 1.$$

Since ρ is concave in v, uniqueness of z is equivalent to ρ being differentiable as a function of v, with gradient

$$\rho_{v'} = z.$$

Also,

$$\begin{aligned}
1 &= wz \\
&\geq \rho(w,z) && \text{Theorem 4.2} \\
&= \min\{wy : yRz\} && \text{def } \rho \\
&\geq \min\{wy : yRx\} && zRx \text{ and Theorem 4.3} \\
&= \rho(w,x) && \text{def } \rho \\
&= \rho^{-1}\rho(v,x) && \text{def } w \text{ and conical } \rho \\
&= 1 && \rho = \rho(v,x).
\end{aligned}$$

so that

$$\rho(w,z) = \rho(w,x) = 1.$$

But

$$wz = 1 \wedge \rho(w,z) = 1 \Rightarrow z = f(w).$$

Hence $z = f(w)$, which also establishes uniqueness of z as a support gradient of ρ.

By the differentiability theorem, we now have ρ differentiable, with gradient

$$\rho_{v'} = f(w), \text{ where } w = \rho^{-1}v.$$

If a concave function in an open set is differentiable everywhere, then it is continuously differentiable. From this now follows the continuity of $f(\rho^{-1}v)$ as a function of v.[17] This goes some way towards continuity of f, but further restrictions on R are required to obtain that.[18]

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[17] This is continuity of the 'compensated demand functionr'.

[18] Katzner (1970) and Afriat (1980, pp 89ff) give proofs that a demand function is continuous if it has a continuous utility function. Afriat also proves continuity on assumption of a utility order for which the sets Px are open and $xPy \Rightarrow cl\,Px \subset Py$.

If f is given to be differentiable, then $\rho_{v'} = z$ can be differentiated, to obtain

$$\rho_{v'v} = z_v = z_w w'_v,$$

where

$$w'_v = (v'\rho^{-1})_v = \rho^{-1}1 - v'\rho^{-2}\rho_v = \rho^{-1}(1 - w'z'),$$

so there is the formula

$$\rho_{v'v} = \rho^{-1}z_w(1 - w'z').$$

If now f is given to have continuous derivatives, then ρ has continuous second derivatives. It then follows, by a theorem of the differential calculus[19], that the matrix $\rho_{v'v}$ of these is symmetric. Since ρ is concave in v, it must also be negative semidefinite.

Now let $v = u$. Then $\rho = 1$, $w = u$, $z = x$, and we get

$$\rho_{u'u} = s,$$

where

$$s = x_u(1 - u'x').$$

In terms of the standard demand function $x = F(p, M)$ which derives from $x = f(u)$ with $u = M^{-1}p$, we have

$$x_p = x_u u'_p,\ \ x_M = x_u u'_M,$$

and

$$u'_p = M^{-1}1,\ \ u'_M = -p'M^{-2} = -u'M^{-1},$$

so there is the alternative expression

$$s = M(x_p + x_M x').$$

From here, but for the factor M which makes no significant difference, s is seen identical with the matrix of *Slutsky coefficients*, usually given as

$$s_{ij} = \partial x_i/\partial p_j + (\partial x_i/\partial M)x_j.$$

We now have the following:

Theorem 7.1 For a demand function with continuous derivatives to be strictly compatible with a utility R for which the sets Rx are closed, it is necessary that the Slutsky matrix be symmetric and negative semidefinite.

For instance the demand function

[19] For instance, if f_{xy} and f_{yx} both exist and either one is continuous, then they are equal.

$$x = a\,(M/pa),$$

for any $a \in C$, with normalization

$$x = a\,(1/ua),$$

is strictly compatible with the utility function

$$\phi(x) = \max\{t : at \leq x\},$$

and the Slutsky coefficients all vanish, so the matrix is both symmetric and negative semidefinite.

This example of "The Case of the Vanishing Slutsky Matrix"[20] should have mystery for any follower of Slutsky, who required a condition properly intermediate between s being negative semidefinite and negative definite, impossible if s vanishes.

Slutsky, and others, consider the problem

$$\max\ \phi(x) :\ px{=}M,$$

where ϕ is assumed differentiable, whose solution is to determine unique $x = F(p, M)$. From first order Lagrange conditions

$$\phi_{x'} = \lambda p,$$

the symmetry of s is obtained. Then from second order conditions, the symmetric matix s is required to be something more than negative semidefinite, going towards its being negative definite, though it cannot possibly be that since $sp' = o$ is an identity. This *Slutsky negativity* condition is that

$$q \nparallel p \ \Rightarrow\ qsq' < 0,$$

where $q \parallel p$ means $q = tp$ for some $t \neq 0$, and $q \nparallel p$ is the denial.

Another excess requirement, related to this one, also comes from the approach. For, from the Lagrange conditions with the budget constraint, we have

$$\phi_{x'}x = \lambda px = \lambda M,$$

and so, eliminating the Lagrange multiplier,

$$M^{-1}p = (\phi_{x'}x)^{-1}\phi_{x'}\,,$$

which determines the budget vector $u = M^{-1}p$ as a function of x. The demand function, which at first just determines x as a function of u, therefore has an inverse.

We have followed a path initiated by McKenzie (1957) for obtaining necessary conditions, requiring s to be symmetric and

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[20] *Journal of Economic Theory* 5 (1972), 208-223.

negative semidefinite. Though lesser requirements than Slutsky's, these still had promise as sufficient conditions.

The proof of sufficiency[21] has several parts.

1. With the elements x_i of x and $x_{ij} = \partial x_i/\partial u_j$ of x_u, we may form

   $$x_{ijk} = x_i\,(x_{jk} - x_{kj}) + x_j\,(x_{ki} - x_{ik}) + x_k\,(x_{ij} - x_{ji}),$$

   Such coefficients are important for Frobenius's theorem on the integrability of linear differential forms. The identity[22]

   $$\textstyle\sum_k x_{ijk} u_k = s_{ij} - s_{ji}$$

   assists the discovery that

   $$x_{ijk} = 0 \;\Leftrightarrow\; s_{ij} = s_{ji},$$

   by which the *symmetry of s* is identified with *classical integrability conditions* for the linear differential form with coefficients $x = f(u)$.

2. From symmetry of s there is obtained the existence in the neighbourhood of any point of functions μ and ψ, the *integrating factor* and *integral*, such that $x\mu = \psi_{u'}$.

3. From this local form of the condition there is passage to a global form, with a single integral ψ defined everywhere in the budget space. This can be either an increasing or a decreasing function, and it can be chosen decreasing, with $\mu < 0$, if necessary by replacing μ by $-\mu$.

4. Then s being negative semidefinite assures ψ is quasiconvex.

5. Since ψ is decreasing quasiconvex, ϕ given by

   $$\phi(x) = \min\{\psi(v) : vx \leq 1\}$$

   in any case increasing quasiconcave, is such that

   $$\psi(u) = \max\{\phi(y) : uy \leq 1\}.$$

6. Then

   $$ux = 1 \wedge \psi(u) = \phi(x) \;\Leftrightarrow\; x = f(u),$$

   and this shows f is strictly compatible with ϕ.

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[21] In Afriat (1980); it is a main objective of this entire volume.
[22] Afriat (1954), reproduced in Afriat (1980), pp 214ff.

With the expression $s = x_u\,(1 - u'x')$, the Slutsky matrix is expressed as a product of a Jacobian, and a factor which, because $ux = 1$, is idempotent and so a projector, of rank $n-1$. This exposes otherwise obscured features about the Slutsky matrix, for instance the identity $su' = o$, which excludes the possibility of s being negative definite, or that the rank of s is $n-1$ for the invertible case and otherwise less. It also provides a way of viewing Hicks's distinction of *income* and *substitution effects,* using resolution of a budget differential du, and hence the corresponding differential $dx = x_u\,du'$, into components by means of the projector and its complement:

$$du' \;=\; (u'x')du' + (1 - u'x')du'.$$

The part

$$(u'x')du' = u'(x'du')$$

leaves the budget direction unchanged, or corresponds to an income change while prices are fixed, making the “income effect". The complementary part, when there is a utility, keeps this constant, and is the “substitution effect". Here we see any change resolved by means of projections into a sum of Hicksian “effects".

8 Demand correspondences

A *demand correspondence* is any collection of demand elements, so it is any $D \subset B \times C$ for which $pDx \;\Rightarrow\; px > 0.$ The *domains* $\mathcal{B} \subset B, \mathcal{C} \subset C$ are given by

$$\mathcal{B} = \{p : pD \neq O\}, \mathcal{C} = \{x : Dx \neq O\},$$

so also $D \subset \mathcal{B} \times \mathcal{C}$.

The relation E defined by

$$uEx \equiv pDx \wedge u = (px)^{-1}p$$

is such that $uEx \;\Rightarrow\; ux = 1$, so it is a *normal* demand correspondence, in this case the *normalization* of D.

A normal demand function f provides the correspondence E for which

$$uEx \;\Leftrightarrow\; x = f(u).$$

Distinguishing the case of an E that represents a function, we have that, for all u, possibly in some restricted domain,

$$uE \neq O,\; uEx, y \;\Rightarrow\; x = y.$$

With a standard demand function F, the correspondence D given by

$$pDx \equiv x = F(p, px)$$

determines the demand elements that belong to it.

Questions that up to now have concerned a single demand element or a demand function can be applied equally well to an arbitrary demand correspondence.

It can be noted that strict consistency, or strict compatibility with some utility, of a correspondence, or possibly many-valued demand function, implies it is single valued and so an ordinary demand function. For if commodity bundles x_0, x_1 are associated with the same budget u, so $ux_0 = 1, ux_1 = 1$, from Samuelson's condition we have

$$ux_0 \leq 1 \wedge ux_1 \leq 1 \Rightarrow x_0 = x_1,$$

and hence $x_0 = x_1$.

In dealing with correspondences which are not functions, instead of strict consistency it is appropriate to entertain their consistency, in any case the more basic requirement from economic principles.

Another point is that the usual treatment of demand functions, following Slutsky, and then Hicks and Allen, deals with a differentiable utility function, required by the Lagrangian method employed. An unobserved consequence is that the demand function has to be invertible. In “The Case of the Vanishing Slutsky Matrix" the demand function is not invertible and the utility function not differentiable.

We deal now with arbitrary demand correspondences, and later finite ones for which constructive methods become possible.

A demand correspondence D will be dealt with through its normalization E, so any condition on E becomes one on D. The elements are taken to be indexed in an arbitrary set I, so

$$E = \{(u_r, x_r)\}_{r \in I}.$$

The case of a demand function requires

$$u_r = u_s \Rightarrow x_r = x_s,$$

and for a 1–1 correspondence,

$$u_r = u_s \Leftrightarrow x_r = x_s.$$

With

$$r \neq s \Rightarrow u_r \neq u_s \vee x_r \neq x_s,$$

to exclude duplicate elements, as will be assumed, these conditions are equivalent to

$$r \neq s \;\Rightarrow\; u_r \neq u_s,$$

and

$$r \neq s \;\Rightarrow\; u_r \neq u_s \wedge x_r \neq x_s.$$

The *cross-coefficients*

$$D_{rs} = u_r x_s - 1$$

are determined from ordered couples of demand elements belonging to D. Then there are *chain-vectors*

$$D_{rij\ldots ks} = (D_{ri}, D_{ij}, \ldots, D_{ks})$$

formed from these.

Utility has so far been attributed to the commodity space, from which derives an indirect utility, for the budget space. Instead out of regard for a basic symmetry, we can deal with relations defined directly between demand elements. These can then induce relations in the commodity space, as usual, and equally well and in just the same manner, also in the budget space.

A relation $W_D \subset I \times I$ between demand elements is defined by

$$rW_D s \equiv (\vee\, i \ldots) D_{ri\ldots s} \leq 0.$$

In terms of the $W \subset B \times C$ which makes

$$uWx \;\Leftrightarrow\; ux \leq 1,$$

a relation $\mathbb{W}_D \subset I \times I$ between demand elements is given immediately by

$$r\mathbb{W}_D s \equiv u_r W x_s,$$

so

$$r\mathbb{W}_D s \;\Leftrightarrow\; D_{rs} \leq 0.$$

Then W_D is identical with the transitive closure,

$$W_D = \overrightarrow{\mathbb{W}}_D.$$

It is both reflexive, since

$$D_{rr} = u_r x_r - 1 = 0,$$

and transitive, from this expression. Hence it is an order, of the demand elements that make the given correspondence D.

The usual revealed preference relation $R_D \subset C \times C$, in the commodity space, is now given by the formula

$$xR_D y \equiv (\vee\, rW_D s)\, x = x_r \wedge u_s y \leq 1.$$

But just as well, in a dual fashion, a relation $S_D \subset B \times B$ instead in the budget space can be defined by

$$uS_D v \equiv (\vee\, rW_D s)\, ux_r \leq 1 \wedge v = u_s.$$

Any order $R \subset C \times C$ has a dual $R^{\triangleright} \subset B \times B,$ where

$$uR^{\triangleright} v \equiv (\vee\, uWx)(\wedge\, vWy)\, xRy,$$

and any $S \subset B \times B$ has a dual $S^{\triangleleft} \subset C \times C,$ where

$$uS^{\triangleleft} v \equiv (\vee\, vWy)(\wedge\, uWx)\, xSy.$$

Then we find

$$S_D = R_D^{\triangleright},\ R_D = S_D^{\triangleleft}.$$

These are not complete orders. But in a comparable fashion, when we have direct and indirect utility functions ϕ, ψ with the usual properties, semi-increasing and semi-decreasing, quasiconcave and quasiconvex, so they are connected by

$$\psi = \phi^{\triangleright},\ \phi = \psi^{\triangleleft},$$

where

$$\phi^{\triangleright}(u) = \max\{\phi(x) : uWx\},\ \psi^{\triangleleft}(x) = \min\{\psi(u) : uWx\},$$

if $R,\ S$ are the complete orders they represent, then these are connected by

$$S = R^{\triangleright},\ R = S^{\triangleleft}.$$

The difference between $\triangleright$ and $\triangleleft$, for going from C to B and C to B, arises just because we want the utility equality $\phi(x) = \psi(u)$ for a compatible demand element $(u, x),$ and not $\phi(x) = -\psi(u),$ or want R and S to be matched similarly. With S replaced by the converse S', or ψ by $-\psi,$ the difference disappears.[23]

9 Canonical order

A *canonical order* of D is any complete order $\mathcal{W}$ such that

$$s\mathcal{W}r \ \Rightarrow\ D_{rs} \geq 0,\ s\mathcal{V}r \ \Rightarrow\ D_{rs} > 0,$$

$\mathcal{V}$ being the antisymmetric part of $\mathcal{W}$. This is given by $\mathcal{V} = \overline{\mathcal{W}}'$ since $\mathcal{W}$ is complete. Hence the conditions, taken in opposite order, are equivalent to

$$D_{rs} \leq 0 \ \Rightarrow\ r\mathcal{W}s,\ D_{rs} < 0 \ \Rightarrow\ r\mathcal{V}s.$$

[23] This is quite like the way in LP the dual of a standard max problem is given as a standard min problem instead of another standard max problem, though there it is just for simplicity.

From the first condition here, since $\mathcal{W}$ is transitive, it follows that $W_D \subset \mathcal{W}$. From this, with the second condition as originally stated, we have

$$rW_D s \;\Rightarrow\; r\mathcal{W}s \;\Rightarrow\; D_{sr} \geq 0,$$

and hence

$$rW_D s \;\Rightarrow\; D_{sr} \geq 0.$$

This last condition is restated by the condition K_D given by

$$K_D \equiv D_{r\ldots s} \leq 0 \;\Rightarrow\; D_{sr} \geq 0,$$

which therefore has appeared necessary for the existence of a canonical order.

Also it is sufficient. For K_D, now taken in contrapositive form, is equivalent to

$$D_{rs} < 0 \Rightarrow \;\sim sW_D r,$$

and from the definition of W_D,

$$D_{rs} \leq 0 \Rightarrow rW_D s.$$

These combine to give

$$D_{rs} < 0 \Rightarrow rV_D s.$$

By Theorem 1.2, Corollary, there exists a complete order $\mathcal{W}$ such that

$$W_D \subset \mathcal{W},\; V_D \subset \mathcal{V}.$$

Immediately, this has the properties of a canonical order, hence the following.

Theorem 9.1 For any demand correspondence D, the condition K_D is necessary and sufficient for the existence of a canonical order.

The strict cyclical consistency condition, of Houthakker, formerly applied to a demand function, now in application to an arbitrary demand correspondence D produces a condition

$$K_D^* \equiv D_{r\ldots s} \leq 0 \wedge x_r \neq x_s \;\Rightarrow\; D_{sr} > 0.$$

A restatement of this condition is that

$$rW_D s \wedge x_r \neq x_s \;\Rightarrow\; D_{sr} > 0,$$

and equivalently,

$$D_{sr} \leq 0 \wedge x_r \neq x_s \Rightarrow \;\sim rW_D s.$$

But we already have

$$D_{rs} \leq 0 \Rightarrow rW_D s,$$

so this is equivalent to

$$D_{sr} \leq 0 \wedge x_r \neq x_s \Rightarrow sV_D r.$$

Here it can be noted that another statement of the condition K_D^* is that it requires the relation $I_D = W_D \cap W_D'$ of equivalence in W_D to be such that

$$rI_D s \Leftrightarrow x_r = x_s,$$

and with this it follows that

$$rV_D s \Leftrightarrow rW_D s \wedge x_r \neq x_s.$$

By Theorem 1.3, Corollary, there exists a complete order $\mathcal{W}$ such that

$$W_D \subset \mathcal{W},\ V_D \subset \mathcal{V},$$

and for this we now have

$$D_{rs} \leq 0 \Rightarrow r\mathcal{W}s,$$

and

$$D_{sr} \leq 0 \wedge x_r \neq x_s \Rightarrow s\mathcal{V}r.$$

Since $\mathcal{W}$ is complete so that $\mathcal{V} = \overline{\mathcal{W}}'$, these conditions in reverse order are equivalent to

$$r\mathcal{W}s \wedge x_r \neq x_s \Rightarrow D_{sr} > 0,\ \ r\mathcal{V}s \Rightarrow D_{sr} > 0.$$

These are the conditions required for a complete order $\mathcal{W}$ to be a *strict canonical order.*

It has appeared that K_D^* is a sufficient condition for the existence of a strict canonical order.

Also it is necessary. For from

$$D_{rs} \leq 0 \Rightarrow r\mathcal{W}s,$$

equivalent to the second requirement since $\mathcal{W}$ is complete, taken with the transitivity of $\mathcal{W}$, it follows that $W_D \subset \mathcal{W}$. Then from this with the first,

$$rW_D s \wedge x_r \neq x_s \Rightarrow D_{sr} > 0,$$

which is another statement of K_D^*.

We now have the following.

Theorem 9.2 For any demand correspondence D, the condition K_D^* is necessary and sufficient for the existence of a strict canonical order.

Beside the strict cyclical consistency condition K_D^*, of

Houthakker, stated

$$D_{rs\ldots r} \leq 0 \Rightarrow x_r = x_s,$$

which assures uniqueness of a bundle chosen under a budget, there is the condition

$$D_{rs\ldots r} \leq 0 \Rightarrow u_r = u_s,$$

which assures uniqueness of a budget under which a bundle is chosen, and also

$$D_{rs\ldots r} \leq 0 \Rightarrow r = s,$$

which, with exclusion of duplicates, assures both, and provides a 1—1 correspondence between bundles and budgets having the relation *D.*

Instead, to abandon both of these inessential uniqueness requirements, there is the *cyclical consistency* condition K_D stated

$$D_{rs\ldots r} \leq 0 \Rightarrow D_{rs} = 0,$$

which is implied by all the foregoing, and is most appropriate for dealing with a general demand correspondence. An alternative statement is that

$$D_{r\ldots r} \leq 0 \Rightarrow D_{r\ldots r} = 0,$$

or, what is the same,

$$\sim D_{r\ldots r} \lesssim 0.$$

Theorem 9.3 A demand correspondence is (strictly) consistent, or (strictly) compatible with some utility order, if and only if it is (strictly) compatible with its own revealed preference order, and this is if and only if the (strict) cyclical consistency condition holds.

The proof is similar in each case to that of Theorem 6.1.

In application to a finite demand correspondence, cyclical consistency becomes a finitely testable condition. The last theorem represents it as a test for consistency of the correspondence, or the existence of a compatible utility. But it is also a test for the solubility of a certain finite system of homogeneous linear inequalities.[24] The algorithm for finding a solution depends on first taking the demand elements in a canonical order. Any solution is associated with compatible utility functions with the classical

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[24] The 1960 paper contains the earliest account, followed by 1964, 1970 (contains a synopsis), 1973, 1974, 1981 (deals with utility subject to the conical restriction important for price indices), and 1987; also Varian (1992).

properties, concave and semi-increasing, finitely constructible in either polyhedral or polytope form. Alternatively, utility functions are found in the budget space, convex and semidecreasing, from which compatible quasiconcave semi-increasing functions in the commodity space are derived by linear programming formulae.

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