Effectivity Functions and Stable Governance Structures

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Abstract

Stability properties of several governance structures are studied by means of their effectivity functions. It is shown that (strong) stability of a governance structure is mainly determined by the collegial status of the underlying assembly's decision rule.JEL c.n.:D70,D71,D72

1 Introduction

The present paper is devoted to the task of exploring 'stability' properties of governance structures in multiparty environments by means of a coalitional game-theoretic model in effectivity-function form (see Vannucci(1999) for a similar attempt in a more specialized setting) . Indeed, 'stability'–i.e existence of 'stable' outcomes on a suitably large domain of preference profiles– is widely regarded as a prominent criterion of good performance for a governance structure, and understandably so: unstable systems are prone to unpredictable behaviour and inefficiency to an extent that, arguably, stable ones are not.

Unfortunately, in a multiparty environment instability is a serious possibility, and a disturbing one in view of the following considerations:

i) as a matter of fact, several modern democracies do exhibit either well-established or incipient multipartism (i.e. existence of three or more effective parties);

ii) institutional compression of a multiparty system to a two-party format is objectionable on the grounds that it may undermine fair representation and/or party discipline;

iii) moreover, there is scarce evidence that multiparty systems can be at all effectively compressed to a two-party format by any acceptable institutional means;

iv) standard parliamentary governance structures seem to face an inescapable instability threat, but then presidential-like governance structures–as such–are no easy remedy against outcome instability. By disallowing demotion of the executive by means of parliamentary non-confidence votes presidential systems can certainly deliver a considerable amount of cabinet-stability, but need not produce stability in the outcome or policy space . (Indeed, many critics claim that standard presidential systems are a recipe for ineffective, weak government).

As a result, designing an outcome-stable governance structure in a multiparty environment is apparently a non-trivial task. In what follows, we proceed to consider several combinations of basic features of current governance structures, and emphasize the need for an explicit description of the relevant *majority formation rules*, which dictate the set of effective–i.e. 'admissible'–majority coalitions, perhaps as an outcome of the electoral process. As mentioned above, we rely on effectivity functions (EFs), a quite abstract and flexible model of coalitional power : a seat-allocation-parameterized family of EFs – or *EF-scheme (EFS)*– is attached to any governance structure as defined by a suitable combination of basic features. Then, the stability of the latter on a large domain of preference profiles is analysed using the core–in a "refined" version which is related to its Von Neumann-Morgenstern-stability properties–as a solution concept.

The resulting model provides a list of remarkably sharp results on stability that may be summarized as follows:

• if the *majority formation rule* is *collegial* i.e. (essentially) the outcome of the electoral process includes a *fixed* minimal majority coalition then–*except for standard presidential systems with perfect separation of powers– outcome-stability ensues even in multiparty environments*, whatever the respective powers of the assembly and the president (if any) concerning legislation and office termination;

•if the *majority formation rule is not collegial* (i.e. the outcome of the electoral process allows formation of several alternative minimal majority coalitions) then in multiparty environments *outcome-instability cannot be ruled out* by other institutional means, to the extent that the assembly has some policy-relevant legislation power;

• by contrast, *two-party environments* are conducive to outcome-stability *even under presidential governance structures* (albeit in a somewhat weaker sense).

To put it simply, the distinguishing features of parliamentary, presidential and mixed governance structures (namely, separation of powers, and its extent, and the allocation of termination power among assemblies and presidents) appear to be largely irrelevant when it comes to the issue of outcome-stability on a 'large' domain of preference profiles. The collegial (i.e. fixed vs.variable) nature of the leading majority within a legislature is the key factor for outcome-stability across (very) different regimes (in basic order-theoretic terms, collegiality of the leading majority amounts to requiring the set of 'winning' or all powerful coalitions to be a *principal–hence latticial–order filter*). The single prominent exception is the classical presidential system with fixed terms of office, which embodies a trivial allocation of termination powers (i.e. new early elections can only be triggered by voluntary resignations or procedurally equivalent events). Moreover, the well-known stability-supporting properties of two-party systems can be replicated under multiparty systems– to a large extent. The exception is, again, the classical presidential system, whose outcome-stability apparently requires–according to our model's predictions– a suitable two-party environment.

The paper is organized as follows. Section 2 is devoted to a short presentation of some relevant background and motivation. In Section 3, the model and results are presented. In Section 4, a short discussion on the significance of the results of the previous section (including comments on recent institutional reforms in Israel and Italy) is provided. Section 5 consists of a few concluding remarks. Proofs are confined to an Appendix.

2 Parliamentary, presidential and mixed governance structures

As previously mentioned, we take the widely held view that the distinctive feature of parliamentary systems is the requirement of a continued support from the assembly for the survival of the cabinet (including the president, if she has any role in it). This is ensured by the right of the former to issue a (non-)confidence vote under suitable conditions that can be unilaterally elicited both by the assembly and the cabinet. Critics claim that in a multiparty environment this arrangement is bound to result in unstable–hence ineffective–government. On the other hand, the most characteristic feature of presidential systems is usually taken to be a suitable separation of powers between the presidential cabinet and the assembly. That is typically meant to entail both a reasonably clear separation of the issues under their respective control, and independence of their appointment and substitution procedures. The main underlying rationale for that is apparently the aim of establishing a set of "checks and balances" to the effect of enhancing accountability of the system. Critics' rebuke is that such an arrangement is likely to be conducive to a divided–hence weak and indecisive–type of government.(Another criticism that is sometimes raised against presidential systems invokes their claimed proneness to authoritarian rule by the president. While this criticism may well be appropriate whenever poor institutional design and/or unfortunate political circumstances produce a grossly unbalanced division of powers in favour of the president, it seems to be hardly relevant against presidential systems in general, at least under the previously proposed definition that requires in fact a suitable separation of powers). Mixed governance structures amount to systems that allow non-confidence votes against the cabinet (but not against the president) and/or exhibit a less-than-perfect separation of powers. Mixed structures are often portrayed by their proponents as a sensible way out of the apparent drawbacks of parliamentary and presidential systems, and by critics as unfortunate compromises that can only compound their respective defects (see Shugart,Carey(1992), Sartori(1994), or Lijphart(1999) for comprehensive surveys, and alternative views on the classification of current governance structures).

The main aim of the present paper is to provide a broad coalitional game-theoretic framework which should enable a clear discussion and assessment of such sharply conflicting views concerning more or less vaguely defined "stability" and/or "decisiveness" properties of different governance structures. In order to accomplish this task in the simplest and most effective way, I shall rely on a highly stylized model focussing on a president and an assembly whose seats are controlled by at least three parties. Thus, there will be no explicit modeling of structures featuring both a president and a premier as distinct players, or bicameralism, just to name a couple of major examples. Rather, the focus will be on the outcome-stability effects of different arrangements concerning i) separation of powers between the assembly and the president, and ii) the set of effective or admissible majority coalitions i.e. those majority coalitions that are entitled to control the assembly, for any fixed allocation of seats. Therefore, the outcome or policy space X is decomposed into three component subspaces Y,Z,D representing the assembly's jurisdiction, the presidential jurisdiction, and the joint jurisdiction, respectively. Each jurisdiction is supplemented with a distinguished outcome representing "demotion–or resignation–of president", "assembly termination", and "termination of joint activities", respectively. Then, separation of powers can be modeled along two dimensions, namely: a) the scope of president's and assembly's jurisdictions, and b)the allocation of "termination" power, i.e. the ability (or lack of it) of the assembly to remove and substitute the president in office, and vice versa. A *majority formation rule* models the outcome of the electoral process. Indeed, we implicitly distinguish two types of electoral rules according to their output, which may consist either i) of a seat allocation with no constraint whatsoever on the coalition formation process, or ii) of a seat allocation *and* a proper subset of *admissible (or effective) majority coalitions* (possibly a single elected minimal majority coalition). Under case i) the set of effective majority coalitions is simply the set of majority coalitions induced by the given seat allocation, whereas under case ii) the set of effective majority coalitions may reduce to a collegial set of selected majority coalitions.

As mentioned in the introduction, a suitably defined EF-scheme is attached to each governance structure. Of course, one prominent advantage of an EF-model is that commitment to any specific strategic game form is avoided. However, the involved EFSs are meant to represent the EFS of *some* underlying strategic game forms: in order to ensure consistency with such an interpretation monotonicity is required. Moreover, such an interpretation calls for a further option between maximin-like and minimax-like 'forcing' power of coalitions (i.e. α-EFSs and β-EFSs, which require regularity and maximality, respectively: see e.g. Abdou,Keiding(1991), Otten,Borm,Storcken,Tijs(1995)). As mentioned above, the outcome-stability issue is addressed using core-stability and strong stability (a refinement of core-stability that prevents a certain sort of coalitional manipulation of preference revelation) as solution concepts.

3 Model and results

Let *N,X* be two non-empty sets (denoting the player set and the outcome set, respectively), and *P(N),P(X)* their power sets. An *effectivity function (EF)* on *(N,X)* is a function *E:P(N)* $\rightarrow P(P(X))$ such that :

EF 1) $E(N) = P(X) \setminus \{\emptyset\}$; *EF 2)* $E(\emptyset) = \emptyset$; *EF 3)* $X \in E(S)$ for any $S \in P(N) \setminus \{\emptyset\}$; *EF 4)* $\emptyset \notin E(S)$ for any $S \in P(N) \setminus \{\emptyset\}$.

The following properties of an EF will be repeatedly taken into consideration:

N-Monotonicity: An EF E on (N, X) is $N-$ *monotonic* if for any $A \subseteq X$ and $S \subseteq T \subseteq N$, $A \in E(S)$ entails $A \in E(T)$.

X-Monotonicity: An EF E on (N, X) is $X-$*monotonic* if for any $A \subseteq B \subseteq X$, and $S \subseteq N$, $A \in E(S)$ entails $B \in E(S)$.

Monotonicity: An EF E on (N, X) is *monotonic* if it is both N-monotonic and X-monotonic.

Regularity: An EF E on (N, X) is *regular* if for any $A, B \subseteq X$, and $S \subseteq N$, $A \in E(S)$ and $B \in E(N \setminus S)$ jointly entail $A \cap B \neq \emptyset$.

Maximality: An EF E on (N, X) is *maximal* if for any $A \subseteq X$, and $S \subseteq N$, $A \notin E(S)$ implies that a $B \subseteq X$ exists such that $A \cap B = \emptyset$ and $B \in E(N \setminus S)$.

Superadditivity: An EF E on (N, X) is *superadditive* if for any $S, T \subseteq N$, and $A, B \subseteq X$, $A \in E(S), B \in E(T)$ and $S \cap T = \emptyset$ entail $A \cap B \in E(S \cup T)$.

Convexity: An EF E on (N, X) is *convex* if for any $S, T \subseteq N$, and $A, B \subseteq X$, $A \in E(S)$ and $B \in E(T)$ imply that either $A \cap B \in E(S \cup T)$ or $A \cup B \in E(S \cap T)$.

The foregoing properties are not mutually independent. In particular, it is easily checked that convexity entails superadditivity that in turn entails both regularity and N-monotonicity.

The usually suggested interpretation of the statement '$A \in E(S)$' is that coalition S can "force" the outcome within subset A, according to the rules of of an underlying decision mechanism that is left unspecified. In that connection, there are at least two natural interpretations of "forcing power" that command consideration. The first and strongest version of "forcing power" is $\alpha-$effectivity (w.r.t. a strategic game form) which refers to the ability of coalition S to coordinate on a strategy profile which guarantees that the final outcome will be in A no matter what the complementary coalition $N \setminus S$ does. The second and weaker is $\beta-$effectivity (w.r.t. a strategic game form), which refers to the ability of coalition S to reply to *any* strategy profile chosen by the complementary coalition $N \setminus S$ by choosing a suitably strategy profile causing the outcome to be in A. A few basic facts about $\alpha-$ and $\beta-$effectivity functions are to be recalled here. First, it is quite clear that for any fixed strategic game form the $\alpha-$effectivity value $E_\alpha(S)$ of each coalition S must be a subset of the $\beta-$effectivity value $E_\beta(S)$. Moreover, the $\alpha-$and $-\beta$ effectivity functions of a strategic game form G are identical if and only if the $\alpha-$effectivity function of G is maximal. It is also well-known, and easily checked, that the $\alpha-$EF ($\beta-$EF) of a strategic game form is monotonic and superadditive (monotonic and maximal). Finally, it can be shown (see e.g. Otten, Borm, Storcken, Tijs(1995)) that any EF E that satisfies both $X-$monotonicity and superadditivity is the $\alpha-$effectivity function of some suitably defined strategic game form.

In the sequel, we want to cover both the "guaranteeing" and the "replying" version of "forcing power" while avoiding commitment to any specific strategic game form. Therefore, we follow Abdou, Keiding(1991) and define a *polarity operator* on EFs, which amounts to an abstract coalitional game-theoretic counterpart to the strategic $\alpha-$ and $\beta-$effectivity concepts:

Definition 1 *(*Polar EF of an EF*) Let E be an EF on (N, X). Then the* polar *E^* of E is the EF on (N, X) defined by the following rule: for any $S \subseteq N, S \neq \emptyset : E^*(S) = \{B \subseteq X : B \cap C \neq \emptyset$ for any $C \in E(N \backslash S)\}$, and $E^*(\emptyset) = \emptyset$.*

It can be immediately checked that the following facts hold true (see e.g. Abdou,Keiding(1991)) : (a) E is monotonic only if $E^*$is also monotonic , (b) if E is regular then $E(S) \subseteq E^*(S)$ for any $S \subseteq N$, (c) if E is monotonic then $A \in E^*(S)$ if and only if $X \backslash A \notin E(N \backslash S)$, (d) if E is monotonic and regular then E^* is maximal, and (e) $E = E^*$ if and only if E is both regular and maximal .

In particular, it follows from fact (e) that if an EF is both superadditive and maximal then it can be regarded indifferently as the $\alpha-$effectivity function or the $\beta-$effectivity function of an underlying strategic game form. Therefore, in that particular case the choice between the "guaranteeing" and the "replying" interpretation of "forcing power" turns out to be inconsequential. As mentioned above, coalitional stability of an EF will be assessed using the core as the basic solution concept.

Definition 2 *(*Core stability of an EF*) Let E be an EF on (N, X) and $D(N, X)$ a domain of preference profiles $\succ = (\succ_i)_{i \in N}$, where– for any $i \in N$ – $\succ_i$is a suitable binary relation on X. The* core *of E at $\succ$ –written $C(E, \succ)$ – is the set of $(E, \succ)$-undominated outcomes i.e.*

$$C(E, \succ) = \left\{ \begin{array}{c} x \in X : \textit{for no } S \subseteq N, B \subseteq X, B \in E(S) \\ \textit{and } b \succ_i x \ \ \textit{for any } b \in B, i \in S \end{array} \right\}.$$

An EF E on (N, X) is (core-)stable *on $D(N, X)$ if $C(E, \succ) \neq \emptyset$ for any $\succ \in D(N, X)$ and* unstable *otherwise.*

As it happens, core-stability of E cannot guarantee that an $(E, \succ)$-dominated outcome be dominated through some subset including some core-outcome of $(E, \succ)$ (see e.g. Demange(1987) or Abdou, Keiding(1991) on this unpleasant phenomenon that creates some opportunities for coalitional manipulation of the outcome under the most common circumstances, i.e. whenever preferences are not verifiable). This motivates a *stronger* stability requirement first introduced by Demange(1987), namely:

Definition 3 *(*Strong stability of an EF*). Let E be an EF on (N, X) and $D = D(N, X)$ a domain of preference profiles as defined above. E is said to be* strongly stable *over D if for any profile $\succ \in D(N, X)$ the following holds true: for any $x \in X \backslash C(E, \succ)$ a pair $S \subseteq N, B \subseteq X$ exists such that $B \cap C(E, \succ) \neq \emptyset$, $B \in E(S)$ and $b \succ_i x$ for any $b \in B, i \in S$.*

Remark 4 *It is easily checked by direct inspection that* strong stability *over* D *of an EF* E *on* (N,X) *can also be reformulated in terms of Von Neumann-Morgenstern (VNM) stable sets, namely* E *is strongly stable if at any preference profile* $\succ\in D$ *the core* $C(E,\succ)$ *is a* VNM stable set *of the abstract game* $(X,\rhd)$ *where the* dominance relation $\rhd=\rhd(E,\succ)$ *on* X *is defined by the following rule: for any* $x,y\in X$, $x\rhd y$ *if and only if* $A\subseteq X, S\subseteq N$ *exist such that i)*$x\in A$ *, ii)* $A\in E(S)$ *, and iii)* $x\succ_i y$ *for all* $i\in S$.

Notation 5 *In what follows, we shall be concerned with the domain* $D^*=D^*(N,X)$ *of all profiles of weak orders on* X *(i.e. asymmetric and negatively transitive binary relations) having a maximal element.*

It is well-known that any convex EF is strongly stable on D^* (see Demange(1987), Abdou,Keiding(1991)), while (core-)stability on D^* has been shown to be equivalent to a condition called acyclicity of E on D^* (see again Abdou,Keiding(1991)). A necessary condition for (core-)stability of an EF E on D^* is *lower acyclicity* or *absence of lower cycles,* which are defined as follows:

Definition 6 *(*Lower cycles of an EF) *A* lower cycle *of an EF* E *on* (N,X) *is a finite sequence* $((S_i),(B_i))$ $i=1,..,k$ *such that* $\bigcup_{i=1}^{k}B_i=X$, $B_i\in E(S_i), i=1,..,k$, $\bigcap_{i=1}^{k}S_i=\emptyset$, *and* $B_i\cap B_j=\emptyset$ *for any* $i,j=1,2,3,i\neq j$ *(see e.g. Abdou,Keiding(1991)).*

We are now ready to accomplish our task of representing governance structures by means of effectivity functions. In order to allow a unified treatment of alternative governance structures we envisage the following simplified scenario. The outcome of the electoral process includes an ‘assembly’ and a ‘president’, which are to be taken as institutionally neutral labels (see e.g. Shugart,Carey(1992) for a similar choice of terminology). We shall abstract from the details of the underlying electoral processes and–at least for the moment–from the allocation of powers among them. In order to rule out irrelevant distinctions concerning electoral procedures, however, it may be useful to think of both the ‘assembly’ and the ‘president’ as appointed by means of general elections (a further argument for envisaging directly elected ‘presidents’ even in parliamentary systems is related to ‘power sharing’ requirements as explained below, in Section 4).

The ‘*president*’ is denoted as player 0. The $(n-player, h-sized)$ ‘*assembly*’ $(\mathbf{v},h)\in Z_+^n\times Z_+\setminus\{0\}$–with weight profile $\mathbf{v}=(v_1,..,v_n)$ such that $\sum_{i=1}^{n}v_i=h$– comprises h seats which are divided among a finite number of parties, denoted as players $1,2,..,n$, with $n\geq 3$: the voting weight v_i of party i denotes the (non-negative, integer) number of seats under i's control. We posit $N=\{1,.,n\}$ and denote by $V(n,h)$ the set of all n-agent weight profiles for a h-sized assembly. Decisions are taken by the assembly using a rule that is implicitly defined by a *majority formation rule* $W^*:\Delta\subseteq V(n,h)\rightarrow P(P(N))\setminus\{\emptyset\}$ such that :

$(MF1)$: for any $\mathbf{v}\in\Delta$, $W^*(\mathbf{v})$ is an *order filter* of $(P(N),\subseteq)$ i.e. for any $S,T\subseteq N$, $S\in W^*(\mathbf{v})$ and $S\subseteq T$ entail $T\in W^*(\mathbf{v})$, and

$(MF2)$: $W^*(\mathbf{v}) \subseteq W(\mathbf{v})$, where $W(\mathbf{v})$ is the set of all simple majority coalitions at $\mathbf{v}$, namely coalition $S \in W(\mathbf{v})$ if and only if $\sum_{i \in S} v_i \geq \lfloor h/2 \rfloor + 1$.

Thus, a majority formation rule $W^*(.)$ determines – for any admissible weight profile $\mathbf{v} \in \Delta$– the (non-empty) order-filter $W^*(\mathbf{v}) \subseteq P(N)$ of *admissible majority coalitions* .

The *Nakamura number* $\nu^{NK}(W^*(\mathbf{v}))$ of the set $W^*(\mathbf{v})$ is defined by the following rule (see Nakamura(1979)):

$$\nu^{NK}(W^*(\mathbf{v})) = \left\{ \begin{array}{c} \min \{\#W : W \subseteq W^*(\mathbf{v}), \cap W = \emptyset\} \text{ if } \cap W^*(\mathbf{v}) = \emptyset \text{ and} \\ \infty \text{ otherwise} \end{array} \right\}.$$

A set $W^*(\mathbf{v})$ of admissible majorities for assembly $(\mathbf{v}, h)$ is said to be *collegial* if $W^*(\mathbf{v})$ is a *principal (or, equivalently, latticial) filter* i.e. a (majority) coalition $S \subseteq N$ exists such that $W^*(\mathbf{v}) = \{T \subseteq N : S \subseteq T\}$ (S is also referred to as the *basic* coalition of $W^*(\mathbf{v})$). Also, $W^*(\mathbf{v})$ is *minimal collegial* if it is collegial and its basic coalition is *minimal* (a majority coalition $S \in W(\mathbf{v})$ is said to be *minimal* if $T \notin W(\mathbf{v})$ for any $T \subseteq S, T \neq S$).

A *majority formation rule* $W^*(.)$ is said to be *(minimal) collegial* on Δ if $W^*(\mathbf{v})$ is (minimal) collegial at any $\mathbf{v} \in \Delta$, and *non-collegial* otherwise. For any (outcome) set Y, a majority formation rule $W^*(\mathbf{v})$ is said to be *strictly non-collegial* on (Δ, Y) if there exists some $\mathbf{v} \in \Delta$ such that $W^*(\mathbf{v})$ is not collegial and $\nu^{NK}(W^*(\mathbf{v})) < \#Y$.

Moreover, without any real loss of generality, the outcome or policy space X is taken to be decomposable into three distinct non-overlapping jurisdictions, i.e. $X = Y \times Z \times U$, where Y denotes the jurisdiction of the assembly, Z denotes the presidential jurisdiction, and U denotes the joint jurisdiction. We also assume that there exist "special" outcome-components $y^* \in Y$, $z^* \in Z, u^* \in U$ (whose exact interpretation will be made dependent on context as explained below) and that $\#A \geq 3$ for any $A \in \{Y, Z, U\}$, in order to avoid trivialities. Moreover, I shall be able to avoid some irrelevant complications by focussing on *strong assemblies* i.e.those assemblies whose two-coalitions partitions must include one majority. Strong assemblies are characterized by a *strong weight profile* as defined below.

Definition 7 *(*Strong weight profiles*) An $n-$dimensional weight profile* $\mathbf{v}$ *for a h-sized assembly is said to be* strong *if for any coalition $S \subseteq N$ either $\sum_{i \in S} v_i \geq \lfloor h/2 \rfloor + 1$ or $\sum_{i \in N \setminus S} v_i \geq \lfloor h/2 \rfloor + 1$. The set of all such $n-$dimensional $h-$sized strong weight profiles is denoted by $V^*(n, h)$.*

Notation 8 *The following notation will be used: a parameterized family $E(.)$ of EFs will also be referred to as an* EF-scheme (EFS) . *An EF-scheme $E(.)$ with parameter set V is said to enjoy property P on V if and only if $E(\mathbf{v})$ is P for any $\mathbf{v} \in V$. In particular, when referring to stability properties it will be said that $E(.)$ is (strongly) stable on (V, D) if and only if $E(\mathbf{v})$ is (strongly) stable on D for any $\mathbf{v} \in V$, and* unstable *otherwise.*

We are now ready to define the weight-parameterized effectivity functions to be attached to the parliamentary, presidential, and mixed governance structures we are going to scrutinize (see Vannucci(1999) for a specialized version of the same model as applied to parliamentary governance structures without any distinction among different jurisdictions).

To begin with, we consider a stylized version of parliamentary governance structures. Such structures are characterized here by the unilateral power of the assembly –i.e. of its effective majorities–to remove the executive and/or block its actions (see however Lijphart(1999) for a quite different classification of governance structures which emphasizes direct election of 'presidents' as a distinctive feature of presidential governance structures). Moreover, we focus on the somewhat simplified arrangement that requires appointment of a –possibly new–president whenever the 'old' assembly is replaced. Thus, the weight-parameterized EF of a parliamentary governance structure such that election of a new assembly always involves election of a new president can be defined as follows:

Definition 9 *(* The EF-scheme $E_{PA}(W^*(.))$ of a parliamentary governance structure with asymmetric termination power*) Let* $\mathbf{v}$ *be an* $n-$*dimensional weight profile for a* $h-$*sized assembly,* $W^*(\mathbf{v})$ *the non-empty set of effective majorities at* $\mathbf{v}$*,* $N^* = N \cup \{0\}, X = (Y \backslash \{y^*\} \times Z \backslash \{z^*\} \times U \backslash \{u^*\}) \cup (Y \times \{z^*\} \times \{u^*\})$*. Then,* $E_{PA}(W^*(\mathbf{v}))$ *is the EF on* (N^*, X) *as defined by the following rules: for any* $S \subseteq N^*, B \subseteq X$*,* $B \in E_{PA}(W^*(\mathbf{v}))(S)$ *iff one of the following clauses applies*

(i) $B \neq \emptyset$ *and* $S \supseteq M \cup \{0\}$ *for some* $M \in W^*(\mathbf{v})$ *;*
(ii) $(y, z^*, u^*) \in B$ *for some* $y \in Y$ *, and* $S \supseteq M$ *for some* $M \in W^*(\mathbf{v})$*;*
(iii) $(y^*, z^*, u^*) \in B$ *and* $S \cap M \neq \emptyset$ *for any* $M \in W^*(\mathbf{v})$ *;*
(iv) $B \supseteq Y \times \{z^*\} \times \{u^*\}$*, and* $0 \in S$ *;*
(v) $B = X$ *and* $S \neq \emptyset$*.*

In words, a coalition is all-powerful if and only if it comprises both an effective majority and the president. An effective majority has full control over the jurisdiction of the assembly (including of course the right to call for new elections). An effectively blocking coalition can call for new elections and possibly remove the president (calling for new presidential elections). The president retains control of her jurisdiction except for the possibility of being removed (e.g. by means of a non-confidence vote) as mentioned above.

Proposition 10 *Let* $V^*(n, h)$ *the set of strong weight profiles as defined above and* $W^*(.)$ *a* majority formation rule *on* $V^*(n, h)$*. Then,*

i) $E_{PA}(W^*(.))$ *is monotonic, superadditive and maximal on* $V^*(n, h)$*. Moreover,*

ii) if $W^*(.)$ *is a* collegial *rule, then* $E_{PA}(W^*(.))$ *is* convex *on* $V^*(n, h)$ *– hence* strongly stable *on* $(V^*(n, h), D^*)$*;*

iii) if $W^*(.)$ *is* strictly non-collegial *on* $(V^*(n, h), Y)$*, then* $E_{PA}(W^*(.))$ *is* unstable *on* $(V^*(n, h), D^*)$*.*

A similar result obtains for a parliamentary governance structure with symmetric termination power, where the elected president is also enabled to call for new general elections (e.g. both non-confidence votes and presidential resignation trigger new elections), as defined below.

Definition 11 *(*The EF-scheme $E_{PA^*}(W^*(.))$ of a parliamentary governance structure with symmetric termination power *). Let* $\mathbf{v}, W^*(\mathbf{v})),N^*$ *be the same as under the previous definition above, and* $X^{\circ} = Y\setminus\{y^*\}\times Z\setminus\{z^*\}\times U\setminus\{u^*\}\cup \{(y^*, z^*, u^*)\}$.

Then, $E_{PA^*}(W^*(\mathbf{v}))$ *is the EF on* (N^*, X°) *defined as follows : for any* $S \subseteq N^*, A \subseteq X^{\circ}$, $A \in E_{PA^*}(W^*(\mathbf{v}))(S)$ *if and only if*

(i) $A \neq \emptyset$ *and* $S \supseteq M \cup \{0\}$ *for some* $M \in W^*(\mathbf{v})$ *, or*

(ii) $(y^*, z^*, u^*) \in A$ *and* $S \cap M \neq \emptyset$ *for any* $M \in W^*(\mathbf{v})$*, or*

(iii) $(y^*, z^*, u^*) \in A$ *and* $0 \in S$ *, or*

(iv) $A = X^{\circ}$ *and* $S \neq \emptyset$.

Indeed, the following proposition is easily established.

Proposition 12 *Let* $V^*(n, h)$ *the set of strong weight profiles as defined above and* $W^*(.)$ *a* majority formation rule *on* $V^*(n, h)$*. Then,*

i) $E_{PA^*}(W^*(.))$ *is monotonic, superadditive and maximal on* $V^*(n, h)$*. Moreover,*

ii) if $W^*(.)$ *is a* collegial *rule, then* $E_{PA^*}(W^*(.))$ *is* convex *on* $V^*(n, h)$*, hence* strongly stable–*on* $(V^*(n, h), D^*)$;

iii) if $W^*(.)$ *is* strictly non-collegial *on* $(V^*(n, h), Y)$*, then* $E_{PA^*}(W^*(.))$ *is* unstable *on* $(V^*(n, h), D^*)$.

We proceed now to analyze presidential-like systems. As mentioned above, the defining feature of those systems is that the 'president' is not subjected to a non-confidence vote on the part of the assembly, and is therefore entitled to keep her office till the (essentially) prefixed date of the next presidential election. We shall distinguish two basic presidential-like systems. First, we consider a presidential system where the president is enabled to call for new elections thereby terminating the assembly while retaining her office. This is made precise by the following definition.

Definition 13 *(*The EF-scheme $EF_{PS^*}(W^*(.))$ of a presidential governance structure with assembly-termination power *) . Let* $\mathbf{v}, W^*(\mathbf{v}), N^*$ *be as previously defined , and* $X^* = (Y\setminus\{y^*\} \times Z\setminus\{z^*\} \times U\setminus\{u^*\}) \cup (\{y^*\} \times Z \times \{z^*\})$*. Then,* $E_{PS^*}(W^*(\mathbf{v}))$ *is the EF on* (N^*, X^*) *defined as follows : for any* $S \subseteq N^*, A \subseteq X^*$, $A \in E_{PS^*}(W^*(\mathbf{v}))(S)$ *if and only if*

(i) $A \neq \emptyset$ *and* $S \supseteq M \cup \{0\}$ *for some* $M \in W^*(\mathbf{v})$ *or*

(ii) $(y^*, z, u^*) \in A$ *for some* $z \in Z$ *, and* $0 \in S$ *or*

(iii) $A \supseteq \{y^*\} \times Z \times \{u^*\}$ *and* $S \cap M \neq \emptyset$ *for any* $S \in W^*(\mathbf{v})$ *or*

(iv) $A = X^*$ *and* $S \neq \emptyset$.

We ask again the– by now–familiar question concerning stability of $E_{PS^*}(W^*(\mathbf{v}))$ and its possible dependence on the features of $W^*(\mathbf{v})$. The answer is summarized by the following proposition:

Proposition 14 *Let* $V^*(n, h)$ *the set of strong weight profiles as defined above, and* $W^*(.)$ *a* majority formation rule *on* $V^*(n, h)$*. Then,* *i)* $E_{PS^*}(W^*(.))$ *is monotonic,superadditive and maximal on* $V^*(n,h)$

ii) if $W^*(.)$ *is a* collegial *rule then* $E_{PS^*}(W^*(.))$ *is* convex *on* $V^*(n,h)$, *hence* strongly stable *on* $(V^*(n,h),D^*)$

iii) if $W^*(.)$ *is* strictly non-collegial *on* $(V^*(n,h),Y)$ *– then* $E_{PS^*}(W^*(.))$ *is* unstable *on* $(V^*(n,h),D^*)$.

Hence, the collegial structure of $W^*(\mathbf{v})$ turns out to be –again– the key feature that provides strong stability even to presidential-like governance structures with presidential assembly-termination power. It should also be noticed here that presidential assembly-termination power– a feature shared by the EF-schemes $E_{PA^*}(W^*(.))$ and $E_{PS^*}(W^*(.))$ – can be regarded as a characteristic trait of *mixed governance structures* (on these grounds we are entitled to claim that mixed systems are also covered by the present analysis).

Let us now turn to the analysis of (a stylized version of) "classical" presidential governance structures with a "perfect" separation of powers. Under those systems, the president retains full control over the executive in that neither the president nor (members of) the executive can be demoted by means of non-confidence votes of the assembly. In particular, the president is entitled to keep her office till the next presidential election (to be held at an essentially prefixed date). On the other hand, the president is *not* allowed to interfere with the assembly or legislative body : in particular, she definitely cannot call for new elections and terminate the legislature. This kind of institutional arrangement may be aptly represented by the following family of effectivity functions.

Definition 15 *(*The EF-scheme $E_{PS}(W^*(.))$ of a presidential governance structure with perfect separation of powers*). Let* $\mathbf{v}, W^*(\mathbf{v}), N^*$ *be as in Definition 9 above, and* $X' = [Y\backslash\{y^*\} \times Z\backslash\{z^*\} \times U\backslash\{u^*\}] \cup [\{y^*\} \times Z \times \{u^*\}] \cup [Y \times \{z^*\} \times \{u^*\}]$. *Then,* $E_{PS}(W^*(\mathbf{v}))$ *is the EF on* (N^*, X') *that is defined by the following rules : for any* $S \subseteq N^*, A \subseteq X^\circ$, $A \in E_{PS}(W^*(\mathbf{v}))(S)$ *if and only if*

(i) $A \neq \emptyset$ *and* $S \supseteq M \cup \{0\}$ *for some* $M \in W^*(\mathbf{v})$ *, or*

(ii) $A \supseteq \{y\} \times Z \times \{u^*\}$ *for some* $y \in Y$ *and* $S \supseteq M$ *for some* $M \in W^*(\mathbf{v})$, *or*

(iii) $(y^*, z, u^*) \in A$ *for some* $z \in Z$, $0 \in S$ *and* $S \cap M \neq \emptyset$ *for any* $M \in W^*(\mathbf{v})$, *or*

(iv) $A \supseteq \{y^*\} \times Z \times \{u^*\}$ *and* $S \cap M \neq \emptyset$ *for any* $M \in W^*(\mathbf{v})$, *or*

(v) $A \supseteq Y \times \{z\} \times \{u^*\}$ *for some* $z \in Z$ *and* $0 \in S$, *or else*

(vi) $A = X'$ *and* $S \neq \emptyset$.

Core-stability and related properties of $E_{PS}(W^*(.))$ are summarized by the following proposition:

Proposition 16 *Let* $V^*(n,h)$ *the set of strong weight profiles as defined above, and* $W^*(.)$ *a* majority formation rule *as defined on* $V^*(n,h)$. *Then,*

i) $E_{PS}(W^*(.))$ *is monotonic and superadditive, but* not maximal *on* $V^*(n,h)$

ii) if $W^*(.)$ *is* collegial *then* $E_{PS}(W^*(.))$ *is* unstable *on* $(V^*(n,h),D^*)$;

iii) if $W^*(.)$ *is* strictly non- collegial *on* $(V^*(n,h),\, Y)$*, then* $E_{PS}(W^*(.))$ *is also* unstable *on* $(V^*(n,h), D^*)$*;*

iv) the polar $E^*_{PS}(W^*(.))$ *is (monotonic, maximal and)* unstable *on* $(V^*(n,h), D^*)$ *whether* $W^*(.)$ *is* collegial *(on* $V^*(n,h)$*) or* strictly non-collegial *(on* $(V^*(n,h), Y)$*).*

Thus, we have eventually singled out a class of structures where a collegial governance of the assembly is not enough to ensure stability (let alone strong stability) of outcomes: perhaps unsurprisingly, presidential systems with a "perfect" separation of powers turn out to embody such a characteristic feature. Apparently, this is so because within those systems decision power is so effectively diluted among different actors that no costraint on coalition formation can be strong enough to realign their respective interests to the effect of producing a stable outcome whatever the prevailing preferences.

4 Discussion: power sharing and two-party environments

The gist of the results provided by our stylized model should be by now quite clear , namely : when it comes to outcome-stability the key point is the collegial status of the leading majority. If –and only if–the majority coalition is elected and essentially fixed then a parliamentary or mixed governance structure can ensure outcome core-stability –and strong stability– even in multiparty systems. This is however not the case for a standard presidential structure with perfect separation of powers.

But then, it might be plausibly observed that this is so precisely because classic presidential systems were originally devised in order to ensure a proper dispersion of decision power among different bodies as opposed to outcome-stability. While this paper is indeed focussed on outcome-stability, the foregoing observation cannot be lightly dismissed as irrelevant to our present concerns. In fact, it raises a most important issue regarding the significance of our results. Are the latter to be interpreted as a (by no means new) suggestion that the'power sharing' principle –however implemented– entails lack of general outcome-stability ? If so, our results might be construed as a late addition to a long list of 'impossibility theorems'. However, it is the view of the present author that such an interpretation should be firmly resisted. Therefore, the 'power sharing' issue is to be (briefly) addressed here. In order to do that, we first need a suitably general notion of 'power sharing'. Here we propose to rely on the following definitions :

Definition 17 *(*The winning simple game of an EF*) Let* E *be an EF on* (N, X)*. Then, the* winning simple game (WSG) *of* E *is the simple game* $(N, W[E])$ *defined as follows :* $W[E] = \{S \subseteq N : E(S) = P(X) \setminus \{\emptyset\}\}$.

Thus, the *WSG* of an EF E amounts to the list of those coalitions which are able to enforce by themselves any conceivable outcome. Relying on WSGs we are now able to introduce a quite natural minimal requirement of 'power sharing' for the characteristic EFs of a governance structure.

Definition 18 *(*Minimal 'Power Sharing'*) Let* $N^* = N \cup \{0\}$ *be the (finite) player set (with* $\#N = n$*),* X *the outcome set,* $V^*(n,h)$ *the set of strong weight* n*-profiles for an* h*-sized assembly as defined above,* $W^*(.)$ *a majority formation rule on* $V^*(n,h)$*. Then, a* $V^*(n,h)-$*parameterized EF-scheme* $E(W^*(.))$ *on* (N^*, X) *is said to satisfy the* Minimal Power Sharing *property iff –for any* $\mathbf{v} \in V^*(n,h)$*–* $W[E(W^*(\mathbf{v}))]$ *is* not dictatorial (or, equivalently, is not an ultrafilter) *i.e. for* no $i \in N^*$: $W[E(W^*(\mathbf{v}))] = \{S \subseteq N^* : i \in N^*\}$.

We are now ready to state the following

Claim 19 *The EF-schemes* $E_{PA}(W^*(.))$, $E_{PA^*}(W^*(.))$, $E_{PS}(W^*(.))$, $E_{PS^*}(W^*(.))$ *(see definitions 9,11,13,15 above) do satisfy the* Minimal Power Sharing *property.*

Proof. It follows trivially from the definitions, in that –for each of those EFSs– any 'winning'(i.e. all-powerful) coalition must invariably include a certain majority-coalition in N *and* the 'president' $0 \notin N$ (hence at least *two* players!). □

There is little doubt that stronger notions of 'power sharing' might be devised that would turn out to be inconsistent with outcome-stability. However, the previous Claim –as combined with Propositions 10,12,14 above–suffices to establish that in general outcome-stability and some version of the power-sharing principle are jointly implementable.

Finally, we proceed to consider the implications of our results in a *a two-party environment.* This is a most relevant issue since two-party systems are widely regarded as a bench mark by scholars and practitioners as well. Concerning that matter, the main question is: do two-party systems significantly affect the performance of the EFSs we have been considering in this paper, and –if so– how? As a matter of fact, the issue can be immediately settled relying on the following simple observation: on the set $V^*(2,h)$ of 2-dimensional strong weight profiles, the set of *all* feasible majority coalitions is invariably an ultrafilter, hence in particular *collegial.* As far as the parliamentary and mixed EFSs are concerned, the circumstances that would engender their instability in the general $n-$party case cannot possibly occur: (strong) stability properties of the former do not need any particular restriction on the majority formation process. On the other hand, it is immediately seen that –on $V^*(2,h)$–*parliamentary governance structures without a directly elected 'president'(premier) would reduce to dictatorial simple EFs, thereby violating the 'minimal power sharing' requirement as formulated above.* (This circumstance is indeed one of the major motivations underlying our insistence on 'neo-parliamentary' systems, i.e. parliamentary governance structures with a directly elected premier).

The impact of a two-party environment on the 'pure' presidential EFSs with perfect separation of powers is much more significant, since in that case the unqualified instability-verdict of Proposition 16 is partially reversed, and strong stability is re-established for the relevant $\alpha-$EF, while instability of the $\beta-$EF is –generally speaking–confirmed. All this is summarized and made precise by the following proposition:

Proposition 20 *Let* $V^*(2,h)$ *be the set of* 2*-dimensional strong weight profiles for a h-sized assembly, and* $W(.)$ *the* unrestricted *majority formation rule on* $V^*(2,h)$ *. Then,*

i) $E_{PS}(W(.))$ *is monotonic and* convex *–hence* strongly stable *on* $(V^*(2,h), D^*(2,X'))$ *with* X' *as defined above (see Def.14)- but* not maximal *;*

ii) the polar $E^*_{PS}(W(.))$ *is monotonic and maximal but is* unstable *on* $(V^*(2,h), D^*(2,X'))$ *for some choice of* Y *and* Z *in* X'*.*

To put it in other terms, standard two-party systems with no unanimity constraints on majority formation confer strong stability even to a 'pure' presidential governance structure with perfect separation of powers, *but not in a clear cut way.* Indeed, strong stability only results according to the stricter maxmin-like $\alpha-$effectivity notion of 'enforcing power', and not according to the broader minmax-like notion of $\beta-$effectivity. Thus, two-party environments greatly enhance stability properties of 'pure' presidential systems, but not to a full extent.

5 Concluding remarks

"Stability requires a two-party system", "stability only requires a bi-polar system", "stability is inconsistent with multiparty parliamentary systems", "stability is inconsistent with presidential systems", "electoral rules are crucial to stability", "electoral rules are largely irrelevant to stability": those statements are arguably a fairly good summary of some of the views widely held and vocally expressed by analysts and politicians in current debates on institutional reforms, especially wherever the latter are –or have been recently– on the public agenda (e.g. Italy, Israel). Prima facie, the foregoing list of statements suggest a hopeless clash of deeply and irreducibly opposed opinions. A quite remarkable point made by the EF model proposed in the present paper is that most of those views may be indeed jointly accomodated, if properly qualified. In particular, the most significant predictions of our EF model may be summarized as follows:

• Majority formation rules that effectively dictate–possibly through the operation of electoral rules– a collegial set of admissible majorities confer strong stability to parliamentary and mixed governance structures with a directly elected 'president' even in a multiparty environment (while respecting a minimal power sharing requirement).

It follows that

• the most obvious virtue of two-party systems –namely, stability– is amenable to replication in a multiparty environment via collegial majority formation rules. In that respect, the widespread obsession with two-party systems may well turn out to be beyond the point. At least in resilient multiparty environments, refocusing on the task of designing effective procedures for actual enforcement of collegial majority formation rules might well prove to be a most rewarding move.

By contrast,

• standard presidential governance structures with perfect separation of powers are apparently most impervious to the requirements of outcome-stability. In this case, two-party environments can support stability w.r.t. the relevant α-EFs (but not w.r.t. the β-EFs), while multiparty environments definitely cannot.

It should also be emphasized that such results are detailed enough to assess -and provide some guidance to- actual reforms aimed at enhancing stability in multiparty environments. Recent institutional reforms prescribing a direct election of the premier in Israel and of presidents of regional councils in Italy are two cases in point. Those reforms share one basic rule: *the directly elected 'president' can be demoted by the 'assembly' through a non-confidence majority vote, which in turn automatically triggers new general elections.* Our EF-model predicts that such an institutional arrangement cannot ensure stability (see also Vannucci(1999)), and suggests a way out that consists in linking 'survival' of the 'president' to support *from a single minimal majority-the collegial majority.* (Notice that–if we ignore impeachment procedures–the current arrangement links the 'president' to support from an *arbitrary majority* coalition, while a full-fledged presidential reform would amount to a delinking rule i.e. equivalently to linking the president's 'survival' in office to support from an *arbitrary–possibly empty!–* coalition: in that respect the new stability-supporting reform suggested by our EF-model is just *the opposite* of a move towards a classic presidential system).

To be sure, the EF-model of governance structures presented in this paper embodies some crudely simplifying assumptions including i) lack of structure of the outcome space , ii) a relatively undiscriminating notion of stability, which allows 'deadlock' and 'office(or legislature) termination' as possibly stable outcomes, and iii) existence of entirely non-overlapping jurisdictions. Thus, it would certainly be most unwise to take at face value all of the predictions mentioned above when considering actual governance structures. However, it seems to me that such an EF-model shows promise when it comes to the broader aim of providing an analytical framework for some of the stability-related issues mentioned in the Introduction. Indeed, the EF-model we propose and similar approaches might perhaps bring us closer to the point where, in public debates on constitutional reforms, the clash between largely idiosyncratic views and statements alluded to above could be eventually supported –or even replaced–by a contest between neatly articulated arguments on institutional design.

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6 Appendix: proofs

Proof of Proposition 10. i) *Monotonicity* of $E_{PA}(W^*(.))$ is obvious from the definition. *Superadditivity* is readily checked: let $A \in E_{PA}(W^*(\mathbf{v}))(S), B \in E_{PA}(W^*(\mathbf{v}))(T)$, and $S \cap T = \emptyset$. Then, there are four sort of cases that may occur: a) $(y^*, z^*, u^*) \in A \cap B$ and $S \cap M \neq \emptyset \neq T \cap M$ for any $M \in W^*(\mathbf{v})$. In this case, $A \cap B \in E_{PA}(W^*(\mathbf{v}))(S \cup T)$ follows immediately from clause iii); b) $X \in \{A, B\}$: then $A \cap B \in \{A, B\}$, hence $A \cap B \in E_{PA}(W^*(\mathbf{v}))(S \cup T)$ (by hypothesis, and monotonicity); c) $(y, z^*, u^*) \in A'$ for some $y \in Y$, $B' \supseteq Y \times \{z^*\} \times \{u^*\}$, $S' \supseteq M$ for some $M \in W^*(\mathbf{v})$, $0 \in T'$, where $\{A', B'\} = \{A, B\}$ and $\{S', T'\} = \{S, T\}$. Here, $A \cap B \in E_{PA}(W^*(\mathbf{v}))(S \cup T)$ follows from clause i); d) $(y^*, z^*, u^*) \in A', B' \supseteq Y \times \{z^*\} \times \{u^*\}$, $S' \cap M \neq \emptyset$ for any $M \in W^*(\mathbf{v})$, and $0 \in T'$, where $\{A', B'\} = \{A, B\}$ and $\{S', T'\} = \{S, T\}$. Here $(y^*, z^*, u^*) \in A \cap B$ hence $A \cap B \in E_{PA}(W^*(\mathbf{v}))(S \cup T)$ follows from monotonicity and clause iii) .

In order to check *maximality*, take $\mathbf{v} \in V^*(n, h), \emptyset \neq S \subset N^*, \emptyset \neq A \subset X$ such that $A \notin E_{PA}(W^*(\mathbf{v}))(S)$. Two cases are to be distinguished: a) $(y^*, z^*, u^*) \in A$, and b) $(y^*, z^*, u^*) \notin A$. If a) obtains, then–by definition of $E_{PA}(W^*(\mathbf{v}))$, clause iii)– $S \cap M = \emptyset$ i.e. $N^* \backslash S \supseteq M$ for some $M \in W^*(\mathbf{v})$. Two subcases are then to be considered, namely a')$A \supseteq Y \times \{z^*\} \times \{u^*\}$, a") $A \not\supseteq Y \times \{z^*\} \times \{u^*\}$. If a') is the case then–according to clause iv)–$0 \notin S$ i.e. $N^* \backslash S \supseteq M \cup \{0\}$. It follows that $X \backslash A \in E_{PA}(W^*(\mathbf{v}))(N^* \backslash S)$ (since $A \neq X$). If a") is the case then $(y, z^*, u^*) \notin A$ for some $y \in Y$. Thus, by clause ii), $(y, z^*, u^*) \in X \backslash A \in E_{PA}(W^*(\mathbf{v}))(M) \subseteq E_{PA}(W^*(\mathbf{v}))(N^* \backslash S)$. If b) obtains then–by clause i)–either $0 \in N^* \backslash S$ or $(N^* \backslash S) \cap M \neq \emptyset$ for any $M \in W^*(\mathbf{v})$. In either case–from clauses iii) and iv), respectively– it follows that $X \backslash A \supseteq \{(y^*, z^*, u^*)\} \in E_{PA}(W^*(\mathbf{v}))(N^* \backslash S)$.

ii) To check *convexity* of $E_{PA}(W^*(\mathbf{v}))$ whenever $W^*(.)$ is collegial, i.e. –for any $\mathbf{v} \in V^*(n, h)$–$W^*(\mathbf{v})$ is a (principal) latticial filter, with M as its unique generating (majority) coalition, a simple if tedious direct proof by enumeration of cases will do. Indeed, take $A, B \subseteq X, S, T \subseteq N, \mathbf{v} \in V^*(n, h)$ such that $(\alpha) A \in E_{PA}(W^*(\mathbf{v}))(S)$, and $(\beta) B \in E_{PA}(W^*(\mathbf{v}))(T)$. Several different cases are to be distinguished according to which of the defining clauses of $E_{PA}(W^*(\mathbf{v}))$ underlie (α) and (β).

If both (α) and (β) rely on clause i -written $(\alpha : i, \beta : i)$ namely $S \supseteq M \cup \{0\}$ and $T \supseteq M \cup \{0\}$ for some $M \in W^*(\mathbf{v})$,then $S \cap T \supseteq M^*$ hence clearly $A \cup B \in E_{PA}(W^*(\mathbf{v}))(S \cap T)$ (by clause i). Similarly, if $(\alpha : ii, \beta : ii)$, i.e. $S \supseteq M$, $T \supseteq M, A \supseteq \{(y, z^*, u^*)\}$, $B \supseteq \{(y', z^*, u^*)\}$,for some $y, y' \in Y$, then $A \cup B \in E_{PA}(W^*(\mathbf{v}))(S \cap T)$ (by clause ii). The case $(\alpha : iv, \beta : iv)$, with $0 \in S \cap T, A \cap B \supseteq Y \times \{z^*\} \times \{u^*\}$ entails $A \cup B \in E_{PA}(W^*(\mathbf{v}))(S \cap T)$ (as well as $A \cap B \in E_{PA}(W^*(\mathbf{v}))(S \cup T)$). If $(\alpha : iii, \beta : iii)$ i.e. $S \cap M \neq \emptyset \neq T \cap M$, and $(y^*, z^*, u^*) \in A \cap B$ then $A \cap B \in E_{PA}(W^*(\mathbf{v}))(S \cup T)$ (by clause iii and monotonicity). Moreover, whenever $(\alpha : v)$ or $(\beta : v)$ is the case – hence $X \in \{A, B\}$– it follows that $A \cap B \in E_{PA}(W^*(\mathbf{v}))(S \cup T)$ (by monotonicity again). Next consider the cases $(\gamma : i, \delta : ii)$, $\{\alpha, \beta\} = \{\gamma, \delta\}$ where w.l.o.g. $A \neq \emptyset, (y, z^*, u^*) \in B$ for some $y \in Y, S \supseteq M \cup \{0\}, T \supseteq M$: here, $A \cup B \in$

$E_{PA}(W^*(\mathbf{v}))(S\cap T)$, by clause ii and monotonicity. If $(\gamma : i, \delta : iii), \{\gamma,\delta\} = \{\alpha,\beta\}$ applies, i.e. w.l.o.g. $A \neq \emptyset, (y^*, z^*, u^*) \in B, S \supseteq M \cup \{0\}$,and either $T\cap M \neq \emptyset$ or $0 \in T$, again $A\cup B \in E_{PA}(W^*(\mathbf{v}))(S\cap T)$. Now consider the two remaining relevant classes of cases with $(\gamma : iii), \gamma \in \{\alpha,\beta\}$, namely –w.l.o.g.– $(y, z^*, u^*) \in A$ for some $y \in Y$, and $S \supseteq M$. If $(\delta : iv), \delta \in \{\alpha,\beta\}$ i.e. –w.l.o.g.– $T\cap M \neq \emptyset$, and $(y^*, z^*, u^*) \in B$, then –by clause iii–$A\cup B \in E_{PA}(W(\mathbf{v}))(S\cap T)$. Otherwise $(\delta : v), \delta \in \{\alpha,\beta\}$, i.e. –w.l.o.g.– $0 \in T$ and $B \supseteq Y \times \{z^*\} \times \{u^*\}$, which imply $(y^*, z^*, u^*) \in A\cap B$, whence $A\cap B \in E_{PA}(W(\mathbf{v}))(S\cup T)$. Finally, we have the last two relevant cases $(\gamma : iv, \delta : v), \{\gamma,\delta\} = \{\alpha,\beta\}$, namely–w.l.o.g.– $(y^*, z^*, u^*) \in A \cap B$, $S \cap M \neq \emptyset$, $0 \in T$, which jointly imply –by clause iii or iv– $A \cap B \in E_{PA}(W^*(\mathbf{v}))(S \cup T)$.

iii) Let $W^*(.)$ be *strictly non-collegial* at Y, and $\mathbf{v} \in V^*(n,h)$ such that $v^{NK}(W^*(\mathbf{v}\ ) < \#Y$. Then, a lower cycle of $E_{PA}(W^*(\mathbf{v}))$ can be defined as follows: take $y_1.., y_{v^{NK}(W^*(\mathbf{v}))} \in Y\backslash\{y^*\}$, $B_i = \{y_i\} \times Z\backslash\{z^*\} \times U\backslash\{u^*\}$, $i = 1,.., v^{NK}(W^*(\mathbf{v}))$, $B_{v^{NK}(W^*(\mathbf{v}))+1} = [Y\backslash\left\{y_1,..,y_{v^{NK}(W^*(\mathbf{v}))}\right\} \times Z\backslash\{z^*\} \times U\backslash\{u^*\}]\cup[Y\times\{z^*\}\times\{u^*\}]$, and $S_i \subseteq N^*, S_i\cap N \in W^*(\mathbf{v}), i = 1,.., v^{NK}(W^*(\mathbf{v}))+ 1$ such that $0 \in S_i, i = 1,.., \nu^{NK}(W^*(\mathbf{v}))$, $0 \notin S_{\nu^{NK}(W^*(\mathbf{v})+1}$, and $\bigcap_{i=1}^{v^{NK}(W^*(\mathbf{v}))}(S_i\cap N) = \emptyset$ (such a set of coalitions must exist, by definition of $\nu^{NK}(W^*(\mathbf{v}))$) .

Thus, $B_i \in E_{PA}(W^*(\mathbf{v}))(S_i), i = 1,.., \nu^{NK}(W^*(\mathbf{v}))$ (by clause (i)), $B_{\nu^{NK}(W^*(\mathbf{v}))+1} \in E_{PA^*(W^*(\mathbf{v}))}(S_{\nu^{NK}(W^*(\mathbf{v}))+1})$ (by clause (ii)), $\bigcup_{i=1}^{\nu^{NK}(W^*\mathbf{v}))+1} B_i = X$, $B_i \cap B_j = \emptyset$, $i, j = 1,.., \nu^{NK}(W^*(\mathbf{v})) + 1, i \neq j$, and $\bigcap_{i=1}^{\nu^{NK}(W^*(\mathbf{v}))+1} S_i = \emptyset$ (by definition).□

Proof of Proposition 12. i) *Monotonicity* is obvious from the definition. *Superadditivity* is checked as follows. Let $A \in E_{PA^*}(W^*(\mathbf{v}))(S), B \in E_{PA^*}(W^*(\mathbf{v}))(T)$ such that $S \cap T = \emptyset$ (for some $\mathbf{v} \in V^*(n,h)$). Then, there are three sort of cases to be distinguished: a) $(y^*, z^*, u^*) \in A \cap B$ and $S \cap M \neq \emptyset \neq T \cap M$ for any $M \in W^*(\mathbf{v})$; in this case, $A \cap B \in E_{PA^*}(W^*(\mathbf{v}))(S \cup T)$ by monotonicity and clause (ii) ; b) $X^\circ \in \{A,B\}$,hence $A \cap B \in \{A,B\}$ and $A \cap B \in E_{PA^*}(W^*(\mathbf{v}))(S \cup T)$ follows immediately from monotonicity; c) $(y^*, z^*, u^*) \in A \cap B$, $S' \cap M \neq \emptyset$ for any $M \in W^*(\mathbf{v})$, $0 \in T'$, with $\{S', T'\} = \{S,T\}$: in this case $A \cap B \in E_{PA^*}(W^*(\mathbf{v}))(S \cup T)$ follows immediately from monotonicity.

In order to check *maximality*, take $\mathbf{v} \in V^*(n,h), \emptyset \neq A \subset X, \emptyset \neq S \subset N^*$ such that $A \notin E_{PA^*}(W^*(\mathbf{v}))(S)$. If $(y^*, z^*, u^*) \in A$ then it must be the case that $0 \notin S$ and $S \cap M = \emptyset$ for some $M \in W^*(\mathbf{v})$, hence $\emptyset \neq X\backslash A \in E_{PA^*}(W^*(\mathbf{v}))(M \cup \{0\}) \subseteq E_{PA^*}(W^*(\mathbf{v}))(N^*\backslash S)$ (by clause (i)and monotonicity). If $(y^*, z^*, u^*) \notin A$ then either $0 \notin S$ (whence $0 \in N^*\backslash S$) or $(N^*\backslash S)\cap M \neq \emptyset$ for any $M \in W^*(\mathbf{v})$. In either case–by monotonicity plus clauses iii) and ii), respectively– $X\backslash A \in E_{PA^*}(W^*(\mathbf{v})(N^*\backslash S)$.

ii) *Convexity* of $E_{PA^*}(W^*(\mathbf{v}))$ whenever $W^*(\mathbf{v})$ is collegial (with majority M^* as its collegium or generating coalition) is proved again by enumeration as follows. Take $A, B \subseteq X$, $S,T \subseteq N^*, \mathbf{v} \in V^*(n,h)$ such that (α) $A \in E_{PA^*}(W^*(\mathbf{v}))(S)$, and (β) $B \in E_{PA^*}(W^*(\mathbf{v}))(T)$. Using the same notation as under the proof of the previous proposition, it is easily checked that $(\alpha : i, \beta : i)$, and $(\gamma : iv)$ for some $\gamma \in \{\alpha, \beta\}$ amount to cases already covered in the proof of the previous proposition. The remaining cases are $(\alpha : ii, \beta : ii), (\alpha : iii, \beta : iii)$,and $(\gamma : i, \delta : ii), (\gamma : i, \delta : iii), (\gamma : ii, \delta : iii)$ with $\{\gamma, \delta\} = \{\alpha, \beta\}$. If $(\alpha : ii, \beta : ii)$ obtains, i.e. $(y^*, z^*, u^*) \in A \cap B$, and $S \cap M \neq \emptyset$,$T \cap M \neq \emptyset$ for any $M \in W^*(\mathbf{v})$: hence clearly –by clause ii) and monotonicity– $A \cap B \in E_{PA^*}(W^*(\mathbf{v}))(S \cup T)$. If $(\alpha : iii, \beta : iii)$, namely $(y^*, z^*, u^*) \in A \cap B$ and $0 \in S \cap T$,then $A \cup B \in E_{PA^*}(W^*(\mathbf{v}))(S \cap T)$ (as well as $A \cap B \in E_{PA^*}(W^*(\mathbf{v}))(S \cup T)$).

Next, let us assume –w.l.o.g.– that $S \supseteq M \cup \{0\}$ for some $M \in W^*(\mathbf{v})$, $(y^*, z^*, u^*) \in B$, and $T \cap M \neq \emptyset$ for any $M \in W^*(\mathbf{v})$. Then, $\emptyset \neq (S \cap T) \cap M$ for any $M \in W^*(\mathbf{v})$, whence $A \cup B \in E_{PA^*}(W^*(\mathbf{v}))(S \cap T)$ (by clause (ii) and monotonicity).

Now, consider–w.l.o.g.– the case with $S \supseteq M \cup \{0\}$ for some $M \in W^*(\mathbf{v})$, $(y^*, z^*, u^*) \in B$, $0 \in T$. In this case, $(y^*, z^*, u^*) \in A \cup B \in E_{PA^*}(W^*(\mathbf{v}))(S \cap T)$ (by clause (iii) and monotonicity).

Finally, consider–w.l.o.g.– the case with $(y^*, z^*, u^*) \in A \cap B$, $S \cap M \neq \emptyset$ for any $M \in W^*(\mathbf{v})$,and $0 \in T$. Here we have $A \cap B \in E_{PA^*}(W^*(\mathbf{v}))(S \cup T)$ – by either clause (iii) or clause (iv), plus monotonicity.

iii) Let $W^*(.)$ be a majority formation rule which is *strictly non-collegial* at Y, and $\mathbf{v} \in V^*(n, h)$ such that $\nu^{NK}(W^*(\mathbf{v})) < \#Y$. Then, a lower cycle of $E_{PA^*}(W^*(\mathbf{v}))$ may be defined as follows: take $y_1, .., y_{\nu^{NK}(W^*(\mathbf{v}))} \in Y \backslash \{y^*\}$, $B_i = \{y_i\} \times Z \backslash \{z^*\} \times U \backslash \{u^*\}$, $i = 1, .., \nu^{NK}(W^*(\mathbf{v}))$, $B_{\nu^{NK}(W^*(\mathbf{v}))+1} = [Y \backslash \{y_1, .., y_{\nu^{NK}(W^*(\mathbf{v}))}\} \times Z \backslash \{z^*\} \times U \backslash \{u^*\}] \cup [\{y^*\} \times \{z^*\} \times \{u^*\}]$, and consider a family of coalitions $S_i \subseteq N^*, i = 1, .., \nu^{NK}(W^*(\mathbf{v}))+1$, such that $S_i \cap N \in W^*(\mathbf{v}), i = 1, .., \nu^{NK}(W^*(\mathbf{v}))$, $0 \in S_i, i = 1, .., \nu^{NK}(W^*(\mathbf{v}))$, $\bigcap_{i=1}^{\nu^{NK}(W^*(\mathbf{v}))}(S_i \cap N) = \emptyset$ (such a family of coalitions exists by definition of $\nu^{NK}(W^*(\mathbf{v}))$) , and $S_{\nu^{NK}(W^*(\mathbf{v}))+1} = N$. Again, it is easily checked that –by definition– $\bigcup_{i=1}^{\nu^{NK}(W^*(\mathbf{v}))+1} B_i = X^\circ, B_i \cap B_j = \emptyset,\ i, j = 1, .., \nu^{NK}(W^*(\mathbf{v}))+1, i \neq j$, $B_i \in E_{PA^*}(W^*(\mathbf{v}))(S_i), i = 1, .., \nu^{NK}(W^*(\mathbf{v}))+1$, and $\bigcap_{i=1}^{\nu^{NK}(W^*(\mathbf{v}))+1} S_i = \emptyset$.□

Proof of Proposition 14 . i) Again, *monotonicity* is obvious. *Superadditivity* is also easily checked as follows: let $A \in E_{PS^*}(W^*(\mathbf{v}))(S)$, $B \in E_{PS^*}(W^*(\mathbf{v}))(T)$, and $S \cap T = \emptyset$ (for some $\mathbf{v} \in V^*(n, h)$).Then, it must be the case that either $X^* \in \{A, B\}$–whence $A \cap B \in E_{PS^*}(W^*(\mathbf{v}))(S \cup T)$ by monotonicity– or $A \cap B \supseteq \{y^*\} \times Z \times \{u^*\}$ and $S \cap M \neq \emptyset \neq T \cap M$ for any $M \in W^*(\mathbf{v})$ –which also entails, by monotonicity, $A \cap B \in E_{PS^*}(W^*(\mathbf{v}))(S \cup T)$ – or else $(y^*, z, u^*) \in A'$ for some $z \in Z$, $B' \supseteq \{y^*\} \times Z \times \{u^*\}$, $0 \in S'$, $T' \cap M \neq \emptyset$ for any $M \in W^*(\mathbf{v})$, with $\{A', B'\} = \{A, B\}$, $\{S', T'\} = \{S, T\}$, which again jointly entail $A \cap B \in E_{PS^*}(W^*(\mathbf{v}))(S \cup T)$, by clause (ii). To check *maximality*, take A, S such that $\emptyset \neq A \neq X^*$, $\emptyset \neq S \neq N^*$, and $A \notin E_{PS^*}(W^*(\mathbf{v}))(S)$. We may distinguish the following cases: a) $A \supseteq \{y^*\} \times Z \times \{u^*\}$, $0 \notin S$, and $S \cap M = \emptyset$ for some $M \in W^*(\mathbf{v})$; here, we may conclude that $X^* \backslash A \in E_{PS^*}(W^*(\mathbf{v}))(N^* \backslash S)$ by clause (i), since $N^* \backslash S \supseteq M \cup \{0\}$; b) $A \not\supseteq \{y^*\} \times Z \times \{u^*\}$, $(y^*, z, u^*) \in A$ for some $z \in Z$, and $0 \notin S$: in this case, $(y^*, z, u^*) \in X^* \backslash A$ for some $z \in Z$, and $0 \in N^* \backslash S$, whence $X^* \backslash A \in E_{PS^*}(W^*(\mathbf{v}))(N^* \backslash S)$ (by clause (ii)) ; c) $A \not\supseteq \{y^*\} \times Z \times \{u^*\}$, $(y^*, z, u^*) \notin A$ for any $z \in Z$, and $S \not\supseteq M \cup \{0\}$ for any

$M \in W^*(\mathbf{v})$: here, $(y^*, z, u^*) \in X^* \backslash A$ for any $z \in Z$ and either $0 \in N^* \backslash S$ or $(N^* \backslash S) \cap M \neq \emptyset$ for any $M \in W^*(\mathbf{v})$. Hence $X^* \backslash A \in E_{PS^*}(W^*(\mathbf{v}))(N^* \backslash S)$ by clause (ii) or (iii), respectively.

ii) As usual, *convexity* when $W^*(\mathbf{v})$ is collegial –with collegium M^*–is proved by considering the list of relevant cases. Let (α) $A \in E_{PS^*}(W^*(\mathbf{v}))(S)$, and (β) $B \in E_{PS^*}(W^*(\mathbf{v}))(T)$ for some $\mathbf{v} \in V^*(n, h)$. Using the same notation for clause-combinations as in the proofs of Proposition 10 and 12 above, it is immediately checked that cases $(\alpha : i, \beta : i)$, $(\gamma : iv)$ (with $\gamma \in \{\alpha, \beta\}$) are covered by cases already considered in the above mentioned proofs. Moreover, if $(\alpha : ii, \beta : ii)$ occurs (namely $(y^*, z, u^*) \in A \cap B$, $0 \in S \cap T$) then obviously $A \cup B \in E_{PS^*}(W^*(\mathbf{v}))(S \cap T)$ – as well as $A \cap B \in E_{PS^*}(W^*(\mathbf{v}))(S \cup T)$– by clause (ii). If $(\alpha : iii, \beta : iii)$, i.e. $A \cap B \supseteq \{y^*\} \times Z \times \{u^*\}$, $S \cap M \neq \emptyset \neq T \cap M$ for any $M \in W^*(\mathbf{v})$ then $A \cap B \in E_{PS^*}(W^*(\mathbf{v}))(S \cup T)$ (by clause (iii) itself). If $(\gamma : i, \delta : ii)$ with $\{\gamma, \delta\} = \{\alpha, \beta\}$ holds true, i.e. $S' \supseteq M \cup \{0\}$ for some $M \in W^*(\mathbf{v}), 0 \in T'$, $(y^*, z, u^*) \in B'$ for some $z \in Z, A' \in E_{PS^*}(W^*(\mathbf{v}))(S')$, $B' \in E_{PS^*}(W^*(\mathbf{v}))(T')$ (with $\{S', T'\} = \{S, T\}$ and $\{A', B'\} = \{A, B\}$) then $A \cup B \in E_{PS^*}(W^*(\mathbf{v}))(S \cap T)$ by clause (ii). If $(\gamma : i, \delta : ii)$ with $\{\gamma, \delta\} = \{\alpha, \beta\}$ holds true, i.e. w.l.o.g. $S \supseteq M \cup \{0\}$ for some $M \in W^*(\mathbf{v})$, $T \cap M \neq \emptyset$ for any $M \in W^*(\mathbf{v})$, and $B \supseteq \{y^*\} \times Z \times \{u^*\}$ entail $A \cup B \in E_{PS^*}(W^*(\mathbf{v}))(S \cap T)$ –by clause (iii)– since clearly $S \supseteq M^*$ and $T \cap M^* \neq \emptyset$, whence $(S \cap T) \cap M \neq \emptyset$ for any $M \in W^*(\mathbf{v}) = \{S \subseteq N : M^* \subseteq S\}$. It remains to be checked the class of cases $(\gamma : ii, \delta : iii)$ with $\{\gamma, \delta\} = \{\alpha, \beta\}$, namely w.l.o.g. $(y^*, z, u^*) \in A$ for some $z \in Z$, $B \supseteq \{y^*\} \times Z \times \{u^*\}$, $0 \in S$, and $T \cap M \neq \emptyset$ for any $M \in W^*(\mathbf{v})$: here, it follows that $A \cap B \in E_{PS^*}(W^*(\mathbf{v}))(S \cup T)$ by clause (ii).

iii) Same as the proof of Proposition 12(iii), by defining a lower cycle $(S_i, B_i), i = 1, .., \nu^{NK}(W^*(\mathbf{v})) + 1$ with the same S_i, $i = 1, .., \nu^{NK}(W^*(\mathbf{v})) + 1$, $B_i = 1, .., \nu^{NK}(W^*(\mathbf{v}))$, while positing $B_{\nu^{NK}(W^*(\mathbf{v}))+1} = [Y \backslash \left\{y_1, .., y_{\nu^{NK}(W^*(\mathbf{v}))}\right\} \times Z \backslash \{z^*\} \times U \backslash \{u^*\}] \cup [\{y^*\} \times Z \times \{u^*\}$. $\square$

Proof of Proposition 16. i) Again, *monotonicity* is obvious from the definition. If –for some $A, B \subseteq X', S, T \subseteq N^*$ $\mathbf{v} \in V^*(n, h)$– $A \in E_{PS}(W^*(\mathbf{v}))(S), B \in E_{PS}(W^*(\mathbf{v}))(T)$, and $S \cap T = \emptyset$, then (recalling that $\mathbf{v} \in V^*(n, h)$ being a *strong* weight profile entails that any two majorities in $W^*(\mathbf{v})$ must overlap) it is immediately checked by direct inspection of clauses that either $X' \in \{A, B\}$ (whence $A \cap B \in E_{PS}(W^*(S \cup T))$ by clause (v)), or w.l.o.g. $S \supseteq M$ for some $M \in W^*(\mathbf{v})$, $0 \in T, A \supseteq \{y\} \times Z \times \{u^*\}$ for some $y \in Y$, and $B \supseteq Y \times \{z\} \times \{u^*\}$ for some $z \in Z$ (under which case(s) $A \cap B \neq \emptyset$, hence $A \cap B \in E_{PS}(W^*(\mathbf{v}))(S \cup T)$ by clause (i)), or else w.l.o.g. $S \cap M \neq \emptyset$ for any $M \in W^*(\mathbf{v})$, $0 \in T, A \supseteq \{y^*\} \times Z \times \{u^*\}$, and $B \supseteq Y \times \{z\} \times \{u^*\}$ for some $z \in Z$ (under which case(s) $A \cap B \in E_{PS}(W^*(\mathbf{v}))(S \cup T)$ by clause (iii)). Thus, *superadditivity* is also established.

In order to check *lack of maximality* just consider the following class of counterexamples : if $\#Y \leq \#Z$ take a surjective function $f : Z \to Y$ and define $A = \{(y, z, u) \in X' : (y, z, u) = (f(z), z, u^*)$ for some $z \in Z\}$. Next, consider a $S \subseteq N^*$ such that $0 \in S$, and $S \cap M = \emptyset$ for some $M \in W^*(\mathbf{v})$. It transpires that $A \notin E_{PS}(W^*(\mathbf{v}))(S)$ (since $\#Y \geq 3$ implies that for any $z \in Z$ a $y \in Y$ exists such that $(y,z,u^*)\notin A)$. Moreover, surjectivity of f entails that for any $y\in Y$ a $z\in Z$ exists such that $(y,z,u^*)\notin X'\backslash A$, whence $X'\backslash A\notin E_{PS}(W^*(\mathbf{v}))(N^*\backslash S)$. Similarly, if $\#Z\leq \#Y$ take a surjective function $g:Y\rightarrow Z$ and posit $B=\{(y,z,u)\in X':(y,z,u)=(y,g(y),u^*)$ for some $y\in Y\}$, and $T\subseteq N^*$ such that $0\notin T$ and $T\supseteq M$ for some $M\in W^*(\mathbf{v})$. Then, $B\notin E_{PS}(W^*(\mathbf{v}))(T)$ (since $\#Z\geq 3$ entails that for any $y\in Y$ a $z\in Z$ exists such that $(y,z,u^*)\notin B)$, and $X'\backslash B\notin E_{PS}(W^*(\mathbf{v}))(N^*\backslash T)$ (since –by surjectivity of g – for any $z\in Z$ a $y\in Y$ exists such that $(y,z,u^*)\in B$).

ii) Let $W^*(.)$ be a *collegial majority formation rule*, and $\mathbf{v}\in V^*(n,h)$.(Thus, $W^*(\mathbf{v})=\{S\subseteq N:M^*\subseteq S\}$ where $M^*=M^*(\mathbf{v})$ is such that $\sum_{i\in M^*}v_i\geq \lfloor h/2\rfloor+1$). To check *instability* of $E_{PS}(W^*(\mathbf{v}))$, take $\mathbf{v}\in V^*(n,h)$ such that $v_i\leq \lfloor h/2\rfloor+1$ for any $i\in N$, so that $\#M^*(\mathbf{v})\geq 2$.Then, consider the following lower cycle : $A_1=\{y\}\times Z\times\{u^*\}$ with $y\in Y\backslash\{y^*\}$, $A_2=\{(y^*,z,u^*)\}$ for some $z\in Z$, $A_3=\{(y^*,z',u^*)\}$ with $z'\in Z\backslash\{z\}$, $A_4=X'\backslash[A_1\cup A_2\cup A_3]$, $S_1=M^*(\mathbf{v})$, $S_2=\{0,h^*\}$ for some $h^*\in M^*(\mathbf{v})$, $S_3=\{0,k^*\}$ with $k^*\in M^*(\mathbf{v})\backslash\{h^*\}$, $S_4=M^*(\mathbf{v})\cup\{0\}$. Indeed, it is immediately checked that $A_i\in E_{PS}(W^*(\mathbf{v}))(S_i)$, $i=1,2,3,4$ (by clauses (ii),(iii),(iii),(i) respectively). Moreover, $\bigcup_{i=1}^4 A_i=X'$ and $\bigcap_{i=1}^4 S_i=\emptyset$ (indeed, $\bigcap_{i=1}^3 S_i=\emptyset$) by definition, while –as it is easily checked by direct inspection– $A_i\cap A_j=\emptyset$ for any $i,j=1,2,3,4$, $i\neq j$.

iii) Let $W^*(.)$ be *strictly non-collegial* at Y, and $\mathbf{v}\in V^*(n,h)$ a strong weight profile such that $\nu^{NK}(W^*(\mathbf{v}))<\#Y$. Then, a lower cycle of $E_{PS}(W^*(\mathbf{v}))$ can be defined as follows: take a family of coalitions $S_i\subseteq N^*$, $S_i\cap N\in W^*(\mathbf{v})$, $i=1,..,\nu^{NK}(W^*(\mathbf{v}))$ such that $\bigcap_{i=1}^{\nu^{NK}(W^*(\mathbf{v}))} S_i=\emptyset$. Next, posit $A_i=\{y_i\}\times Z\times\{u^*\}$, $i=1,..,\nu^{NK}(W^*(\mathbf{v}))$ with $y_i\neq y_j$ for any i,j such that $i\neq j$. Finally, define $A_{\nu^{NK}(W^*(\mathbf{v}))+1}=X'\backslash\bigcup_{i=1}^{\nu^{NK}(W^*(\mathbf{v}))} A_i$, and $S_{\nu^{NK}(W^*(\mathbf{v}))+1}=M\cup\{0\}$ for some $M\in W^*(\mathbf{v})$. Clearly enough, $\bigcup_{i=1}^{\nu^{NK}(W^*(\mathbf{v}))+1} A_i=X'$, $\bigcap_{i=1}^{\nu^{NK}(W^*(\mathbf{v}))+1} S_i=\emptyset$, and $A_i\in E_{PS}(W^*(\mathbf{v}))(S_i)$, $A_i\cap A_j=\emptyset$, $i\neq j$, $i,j=1,..,\nu^{NK}(W^*(\mathbf{v}))+1$.

iv) Instability is a straightforward corollary of points i)-ii)-iii) above, since a) by definition, for any pair E,E' of EFs on (N,X) : $C(E,\succ)\subseteq C(E',\succ)$ at any $\succ\in D^*$ whenever $E'(S)\subseteq E(S)$ for every $S\subseteq N$, b) $E\subseteq E^*$, for any regular EF E , and c) $E_{PS}(W^*(\mathbf{v}))$ is superadditive (by point i) above), hence regular for any $\mathbf{v}\in V^*(n,h)$. Monotonicity and maximality of $E^*_{PS}(W^*(.))$ also follow immediately from monotonicity and regularity of $E_{PS}(W^*(.))$ (respectively), according to the elementary properties of polar EFs as stated above in the text.□

Proof of Proposition 20. i) Again, *monotonicity* follows trivially from the definition (see Definition 15). *Non-maximality* of $E_{PS}(W(.))$ on $V^*(2,h)$ follows at once– as a corollary– from the proof of Proposition 16 regarding the general case. In order to check *convexity*, observe that whenever $W(\mathbf{v})$ is an ultrafilter clauses (iii) and (iv) of Definition 15 collapse to special cases of clauses (i) and (ii), respectively. Now, for any $\mathbf{v}\in V^*(2,h)$, we have $\emptyset\neq W(\mathbf{v})=\{S\subseteq\{1,2\}:i^*\in S\}$ for some $i^*\in\{1,2\}$. Therefore, for any

$A \subseteq X', S \subseteq N^* = \{0,1,2\}$, $A \in E_{PS}(W(\mathbf{v}))(S)$ if and only if (i) $A \neq \emptyset$, $S \supseteq \{0, i^*\}$, or (ii) $A \supseteq \{y\} \times Z \times \{u^*\}$ for some $y \in Y$, and $i^* \in S$, or (iii) $A \supseteq Y \times \{z\} \times \{u^*\}$ and $0 \in S$, or (iv)$A = X'$ and $S \neq \emptyset$. Then, consider $A, B \subseteq X'$ and $S, T \subseteq N^*$ such that (α) $A \in E_{PS}(W(\mathbf{v}))(S)$ and $(\beta) B \in E_{PS}(W(\mathbf{v}))$. Using again the notation previously introduced in the proof of Proposition 10 above, it is easily checked by direct inspection that $A \cup B \in E_{PS}(W(\mathbf{v}))(S \cap T)$ obtains under cases $(\alpha : i, \beta : i), (\alpha : ii, \beta : ii)$, $(\alpha : iii, \beta : iii), (\alpha : i, \beta : ii), (\alpha : ii, \beta : i), (\alpha : i, \beta : iii), (\alpha : iii, \beta : i)$. Moreover, it is immediately seen–again– that if either (α) or (β) satisfies clause (iv) then $A \cap B \in \{A, B\}$, whence –by monotonicity– $A \cap B \in E_{PS}(W(\mathbf{v}))(S \cup T)$. It remains to be checked that cases $(\alpha : ii, \beta : iii)$ and $(\alpha : iii, \beta : ii)$ are also consistent with convexity. Let us consider w.l.o.g. case $(\alpha : ii, \beta : iii)$, i.e. $A \supseteq \{y\} \times Z \times \{u^*\}$ for some $y \in Y$, $i^* \in S$, $B \supseteq Y \times \{z\} \times \{u^*\}$ for some $z \in Z$, and $0 \in T$. Here, $A \cap B \neq \emptyset$ by definition. Hence $A \cap B \in E_{PS}(W(\mathbf{v}))(S \cup T)$ by clause (i). Convexity of $E_{PS}(W(.))$ on $V^*(2, h)$ is therefore established.

ii) It is easily checked by a tedious combinatorial argument that the polar EF $E^*_{PS}(W(.))$ is defined as follows: for any $\mathbf{v} \in V^*(2, h), A \subseteq X'$, $S \subseteq N^*$, $A \in E^*_{PS}(W(\mathbf{v}))(S)$ if and only if (i) $A \neq X'$ and $S \supseteq \{0, i^*\}$, or (ii) for each $z \in Z$ a $y \in Y$ exists such that $(y, z, u^*) \in A$, and $i^* \in S$, or (iii) for each $y \in Y$ a $z \in Z$ exists such that $(y, z, u^*) \in A$, and $0 \in S$, or (iv) $A = X'$ and $S \neq \emptyset$. Then, take Y and Z such that $\#Y = \#Z$, and define two bijections $f : Y \to Z$, $g : Z \to Y$ such that $g \circ f$ is a permutation of Y without fixed points. Next, choose $A = \{(y, z, u^*) : z = f(y') \text{for some } y' \in Y\}$, $B = \{(y, z, u^*) : y = g(z') \text{for some } z' \in Z\}$, and consider the sequence $\mathbf{C} =((S_1 = \{0\}, A_1 = A), (S_2 = \{i^*\}, A_2 = B), (S_3 = \{0, i^*\}, A_3 = X' \backslash [A \cup B]))$. It is easily checked that $A_i \cap A_j = \emptyset$, $i, j = 1, 2, 3$, $i \neq j$, and $\bigcap_{i=1}^{3} S_i = \emptyset$, i.e.$\mathbf{C}$ is indeed a *lower cycle* of $E^*_{PS}(W(\mathbf{v}))$. □

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