QUADERNI

Università degli Studi di Siena
DIPARTIMENTO DI ECONOMIA POLITICA

Marcello Basili

KNIGHTIAN UNCERTAINTY IN FINANCIAL MARKETS: AN ASSESSMENT

n. 281 - Febbraio 2000

INTRODUCTION

In financial market theory, it is assumed that a unique subjective additive probability distribution (Bayesian prior) represents an agent's beliefs about the likelihood of future states of the world. In a rich enough market (complete markets), the assumptions of no arbitrage and no frictions induce an asset valuation by a linear functional of its discounted future payments. An asset price functional is unique and it can be expressed as a mathematical expectation by the Riesz representation theorem. Moreover, an asset price can be regarded as the formation cost of a replicating portfolio of marketed assets. In this context, financial market failures, such as price booms and crashes, excess volatility of asset prices, violation of call and put parity, bid and ask spreads and portfolio rigidities can be explained by introducing transaction costs, restricted short-sales, asymmetric information and incompleteness into the markets.

In the last decade, on the basis of development of decision theory and empirical evidence, Knightian uncertainty (Knight 1921) has been regarded as an additional source of financial market failures. Uncertainty attitude of agents may shed new light on some financial market puzzles and provides a new explanation of them.

Roughly speaking, the Knightian distinction between risk and uncertainty relies on the concept of chance (physical probability). In other words, risk is a situation in which the relative odds of events are known. By contrast uncertainty is a condition in which probabilities of events are unknown and no unique assignment of them can be obtained. Vagueness or ambiguity is a situation characterized by Knightian uncertainty. A well known test defined by Ellsberg (1961) and recent experimental results of Camerer and Weber (1992) suggest that individuals generally prefer to act on known probabilities rather than unknown ones and they are willing to pay in terms of money and probability to avoid ambiguous actions. This behavior, known as the Ellsberg paradox, contradicts both the Savage paradigm, that assumes a unique well-defined additive probability distribution that represents beliefs of an individual, and the theory of probabilistic sophisticated agent defined by Machina and Schmeidler (1992).

Two normative and descriptive decision theories under Knightian uncertainty have been presented to explain the Ellsberg paradox, namely the multiple priors approach and the related set-up with a non-necessarily-additive probability measure or capacity (Schmeidler 1982, 1986, 1989, Gilboa 1987, Gilboa and Schmeidler 1989). The multiple priors approach and the non-necessarily-additive probability measure based model have been adopted as a starting point to explain anomalies that occur in financial markets without any apparent change in fundamentals. The existence of Knightian uncertainty, represented by capacities or multiple priors, has been assumed both in static and intertemporal financial market models, with interesting results.

The paper is organized as follows. Section 2 introduces Knightian uncertainty. Results derived in static and intertemporal models are discussed in sections 3 and 4 respectively. Section 5 illustrates some empirical applications of the financial market models under ambiguity. Conclusions are drawn in section 6.

2. DECISION MAKING WITH KNIGHTIAN UNCERTAINTY

Decision theory under uncertainty describes how an individual makes and/or should make a decision between a set of alternatives, when the consequences of each action are tied to events, about which the individual is uncertain that is he/she does not know what will occur. The decision-maker formalizes the problem setting alternatives (technically, acts), states of the world and consequences. The individual acts on the basis of a well-defined utility function, which represents his/her preferences, that involve an evaluation of consequences and their likelihood. The rational decision-maker's goal is to maximize his/her expected utility in the case in which probabilities are given in advance (von Neumann and Morgenstern 1944) or derived from preferences (Savage 1954). Both these theories and their mixed version (Anscombe and Aumann 1963) weigh consequences with a unique probability measure, respectively objective and subjective, on the set of states of the world, so as to induce the linearity of the preference functional. As a consequence, the expected utility can be represented as the mathematical expectation of a real function on the set of consequences with respect to a unique probability distribution and acts are ranked with respect to their expected utility. Given First Order Stochastic Dominance[1], linearity of probabilities is a direct consequence of two very similar axioms, the Independence Axiom in the von Neumann and Morgenstern theory, and the Sure-thing Principle in the Savage theory. The Independence Axiom states that given two alternatives (lotteries in technical language), each composed of an action and a common act, preferences between them should be independent of any common consequence with identical probability. The Sure-thing Principle assumes that the decision-maker ignores states in which actions yield the same consequences when choosing between actions.

Experimental evidence has revealed systematic violations of the Independence Axiom and the Sure-thing Principle that are inconsistent with the hypothesis of expected utility maximization. The most discussed of these violations are the Allais and the Ellsberg paradoxes. The Allais paradox (Allais 1953) is a seminal counterexample to the validity of the expected utility theory because it shows a puzzle built on elements of certainty, small probability differences, high versus low stakes, common consequences, and so forth. Nonetheless, the challenge posed to the expected utility theory by the Ellsberg paradox (Ellsberg 1961) is crucial, because it focuses attention on the belief side of the decision problem and involves considerations about ambiguity and confidence.

The following experiment is by Ellsberg (1961). An individual faces two urns M and N, each containing 100 red or black balls. Urn M contains black and red balls in known equal proportion; urn N contains black and red balls but there is no information whatsoever about the number of black and red balls in it (unknown proportion). One ball is chosen at random from one of the urns and there are four events: MB, MR, NB and NR, where MB means that a black ball is drawn from urn M and so on. A similar bet is offered on each event and an individual is asked to choose between them. The individual will be indifferent between MB and MR or between NB and NR, but he/she will show a remarkable preference for betting on urn M rather than on urn N, i.e.

[1] Given two acts X and Y with cumulative distribution functions G_X and G_Y, X first-order stochastically dominates Y if $G_X(t) \leq G_Y(t)$ for all $t \in R$. If an individual feels X at least as favorable as Y, the cumulative distribution of the preferred prospect never exceeds that of the inferior prospect.

MB~MR$\succ$NB~NR. Since "any preference for drawing from one urn over the other leads to a contradiction" (Ellsberg 1961: 653), these preferences contradict the expected utility theory and every other theory of rational behavior under uncertainty that assumes a unique additive probability measure underlying choices. Hence, "it is impossible, on the basis of such choices, to infer even qualitative probabilities for the events in question…to find probability numbers in terms of which these choices could be described - even roughly or approximately – as maximizing the mathematical expectation of utility" (Ellsberg 1961: 655). Violations of both Complete Ordering of Actions and the Sure-thing Principle, confirmed through many experiments replicated in recent years, suggest that the majority of agents like making unambiguous choices rather than ambiguous ones. Individual choices are not affected by "the relative desirability of the possible payoffs and the relative likelihood of the events affecting them, but …the nature of one's information concerning the relative likelihood of events. What is at issue might be called the ambiguity of this information, a quality depending on the amount, type, reliability and unanimity of information, and giving rise to one's degree of confidence in an estimate of relative likelihood" (Ellsberg 1961: 657).

Even if uncertainty encompasses the intuitive concepts of ambiguity and vagueness, it is possible to provide a more precise notion of uncertainty by dealing with description of the world. Consider a decision problem in which states of the world included in the model do not exhaust the actual ones. A description of the world might be considered a misspecified model whenever omitted states are not explicitly included in the model. When an agent does not know how many states are omitted (missing markets), he/she can represent his/her beliefs by either a non-necessarily-additive measure or a set of additive probability distributions on the set of events. Summing up, a decision-maker faces Knightian uncertainty if he/she has a misspecified description of the states of the world, if he/she is unable to assign a unique probability distribution to states of the world because they are ambiguous, if he/she is ignorant about the world in which he/she has to act and attaches an interval of probabilities to each event.

Since Knightian uncertainty has clear economic importance such as in insurance and search problems or in investment and environmental decisions, two non-expected utility theories have been proposed to encompass uncertainty attitude and the expected utility maximization. Schmeidler (1989) and Gilboa (1987), in the Anscombe-Aumann and Savage approaches respectively, axiomatize a generalization of expected utility, which provides a derivation of utility and non-necessarily-additive probability by the Choquet integral (Choquet 1954). Gilboa and Schmeidler (1989) extend standard expected utility representing preferences by a utility function and a set of additive probabilities, instead of a unique probability, on the set of events.

Let $\Omega=\{w_1,...,w_n\}$ be a non-empty finite set of states of the world and let $S=2^{\Omega}$ be the set of all events. A function $\mu:S\rightarrow R_+$ is a non-necessarily-additive probability measure or a capacity if it assigns a value 0 to the impossible event $\varnothing$ and value 1 to the universal event Ω, i.e. the measure is normalized, and for all $s_1,s_2\in S$ such that $s_1\supset s_2$, $\mu(s_1)\geq\mu(s_2)$, i.e. the measure is monotone. A capacity is convex (concave) if for all $s_1,s_2\in S$, $\mu(s_1\cup s_2)+\mu(s_1\cap s_2)\geq(\leq)\mu(s_1)+\mu(s_2)$ and μ is super-additive (sub-additive) if $\mu(s_1\cup s_2)\geq(\leq)\mu(s_1)+\mu(s_2)$ for all $s_1,s_2\in S$ such that $s_1\cap s_2=\varnothing$. Since μ is a non-additive measure, the integration of a real-valued function $f:\Omega\rightarrow R$ with respect to μ is impossible in the Lebesgue sense and the proper integral for a capacity is the Choquet integral. The Choquet integral with respect to a capacity is a generalization of the Lebesgue integral and it ranks states of the world from the most to the least favorable ones, or vice versa, with respect to their consequences.[2] As a result, the Choquet integral is a generalization of mathematical expectation with respect to a capacity.

The decision-maker expresses Knightian uncertainty aversion (love) if he/she assigns larger probabilities to states when they are unfavorable (favorable), than when they are favorable (unfavorable), that is, if his/her non-additive measure is convex (concave). Hence, the convexity of

[2] The Choquet integral of f with respect to μ is $\int f d\mu = \int_0^{\infty} \mu(\{w| f(w)\geq t\})dt + \int_{-\infty}^{0} [\mu(\{w| f(w)\geq t\})-1]dt$

the capacity captures the decision-maker's Knightian uncertainty aversion and encompasses the conservative statement that the decision-maker's acts "as though the worst were somewhat more likely than his best estimates of likelihood would indicate he distorted his best estimates of likelihood, in the direction of increased emphasis on the less favorable outcomes and to a degree depending on his best estimate" (Ellsberg 1961: 667).

Knightian uncertainty may be represented by a set of possible priors instead of a unique one on the underlying state space, that is "each subject does not know enough about the problem to rule out a number of possible distributions" (Ellsberg 1961: 657). In this case the agent has multiple additive probability measures P on $\Omega=\{w_1,...,w_n\}$ and his/her preferences are compatible with either the maximin or the maximax expected utility decision rule.[3] In fact, if the agent is Knightian uncertainty-averse, he/she maximizes the minimal expected utility with respect to each probability in the prior set, thus $\int fdP = \min \int fdp : p \in P$. On the contrary, if the agent is Knightian uncertainty-loving, he/she maximizes the maximal expected utility with respect to the set P, thus $\int fdP = \max \int fdp : p \in P$.[4]

Gilboa and Schmeidler (1989) and Chateauneuf (1991) prove that when an arbitrary (closed and convex) set of possible priors P is given, and a non-additive probability measure υ (convex) or ν (concave) is defined on Ω, such that all additive probability measures in P majorize υ or minorize ν, the non-additive expected utility theory coincides with the maximin or the maximax decision rule, respectively. The non-additive expected utility with respect to a convex (concave) capacity and the maximin (maximax) expected utility give the same solution if P is considered the core of υ (ν), or a proper subset of the core of υ (ν), since by definition the core of υ (respectively ν) consists of all finitely additive probability measures that majorize υ (minorize ν) event-wise.

[3] The maximin (maximax) expected utility postulates that an agent with multiple priors looks at the least (greatest) value of expected utility for any act and chooses the act for which the minimum (maximum) value is greatest. See Wald (1950); Ellsberg (1961); Arrow and Hurwicz (1972).

3. UNCERTAINTY ATTITUDE IN FINANCIAL MARKETS: STATIC MODELS

In a competitive market with no frictions, if the number of linearly independent securities equals all the possible states of the world (there is a sufficiently rich array of securities), security markets are complete and portfolios of securities can replicate any pattern of returns across states.

Let a security $a:\Omega \to R^{\Omega}$ be defined by its vector of returns in different states of the world, such that $a_j=1$ if w=j and $a_j=0$ otherwise, and let $q \in R_{+}^{\Omega}$ be the price vector of securities. Any marketable portfolio $\Psi : \Omega \to R^{\Omega}$ can be constructed and its payoff is equal to a linear combination of marketed security payoffs[5], and with no frictions the cost of the portfolio $C(\Psi)=\sum_{j=1}^{\Omega} a_j q_j$. Such a portfolio can be considered equivalent to an asset β that exactly yields an equal amount in each state. No arbitrage principle implies that two portfolios Ψ and Φ with the same payoff have the same cost, i.e. $C(\Psi) = C(\Phi)$.

Under no arbitrage and no frictions conditions, the market value of an asset is the expectation of the discounted value of the future dividends. The main feature of this argument is the price at which an asset is traded is given by the formation costs of portfolios replicating it. As a consequence, the pricing functional of the economy is unique, positive and linear.[6] In Arrow (1953) the security prices can be normalized so that they add up to one and security prices may be interpreted as a probability distribution on the space of states. Note that the derived probability distribution is not a probability distribution (subjective or objective) of the agents on the set of states of the world, but simply "a weighting of the states made by prices which express an aggregation of agents' behaviors towards uncertainty" (Chateauneuf, Kast and Lapied 1992). From a theoretical point of view, all valuation models in finance, the most famous of which is the Black

[4] See Appendix I.
[5] An asset is called marketable when it is not traded in markets but is tradable by trading the marketed securities. With linear pricing, the market value of every marketable choice is determined by the market value of the linearly independent set of securities that span the marketed space.
[6] Note that the pricing functional of the economy is:
(i) unique, because two portfolios yielding the same revenue cannot have different prices;
(ii) positive, because an asset with positive payoffs does not have a negative value;

and Scholes (1973) one, can be considered a generalization of the complete Arrow model. Given no frictions, no arbitrage conditions and an asset price that follows a log-normal diffusion process, in the Black and Scholes model there is a unique probability distribution on the measurable space (Ω,S) such that the market value of an asset is the expectation of its payments. The unique additive probability distribution is "the analogue of Arrow's probabilities of the states defined by the (Arrow) security price…that probability is revealed by market prices and has nothing to do with agents' subjective beliefs" (Ami, Kast and Lapied 1991).

Some recent articles have shown that the valuation of an asset is not a linear pricing rule (Lebesgue integral of the asset payments) but is obtained by Choquet integral of the asset payments (non-linear pricing rule), if an agent faces Knightian uncertainty. By a non-linear pricing rule, there may be portfolio inertia that is an interval of prices in which each agent neither buys nor sells the asset short (Simonsen and Werlang 1991, Dow and Werlang 1992a) or thin markets in which pessimistic or optimistic agents wish to trade (Basili 1999). Moreover, the Choquet integral of asset payments is consistent with price puzzles such as the bid and ask spread and call and put parity (Chateauneuf, Kast and Lapied 1992, 1993).[7]

3.1. PORTOLIO INERTIA

In the classical theory of portfolio selection if a risk-neutral agent has to allocate his/her wealth between a riskless asset and a risky asset he/she will act on the basis of the discounted expected value of the risky asset, given no frictions. Indeed, if the current price of the risky asset were less than its discounted expected value, he/she would buy it; conversely he/she would sell it, if the current price of the risky asset were greater than its discounted expected value.[8] In an economy with one individual, defined by a strictly increasing utility function and his/her endowment, and two

---

(iii) linear, because an asset value is defined as its formation cost, that is, by the linear combination of the prices of the assets in the replicating portfolio.

[7] Given no arbitrage and frictionless assumptions, violation of call and put parity is inconsistent with the linear pricing rule; indeed, the value of the put is not equal to the formation cost of the replicating portfolio. Under the same conditions, bid and ask spread is inconsistent with a unique, positive, linear pricing functional.

assets, there will exist an optimal portfolio which maximizes the expected utility of his/her wealth. Any change in relative current asset prices must displace the portfolio equilibrium.

Introducing Knightian uncertainty, the individual no longer has an additive probability distribution on states of the world (future asset prices), but he/she attaches a non-necessarily-additive probability measure on Ω. Simonsen and Werlang (1991) and Dow and Werlang (1988, 1992a) postulate a pessimistic agent and derive that his/her uncertainty attitude might lead to portfolio inertia. An investor may not adjust his/her portfolio to small variations in asset prices; he/she holds positive quantities of all assets or there would be a range of asset prices at which holds no risky asset.

Dow and Werlang (1988, 1992a) determine inertia in a portfolio with money (riskless asset) and a risky asset with a current price q, assuming a non-satiable, risk-neutral and Knightian uncertainty-averse agent. The risky asset is traded without transaction costs and short sales are unrestricted. The individual has to allocate his/her wealth between money and the risky asset, taking into account that the expected risky asset value could be either higher, with a measure υ_1, or lower, with a measure υ_2, than its current price. By Knightian uncertainty aversion $\upsilon_1(.)+\upsilon_2(.)<1$, that is, the individual attaches a super-additive probability measure to states of the world. The expected values of the risky asset determine an interval in which the individual neither buys nor sells the asset and all the wealth is in money. Formally, by non-additivity of priors, the agent's indifference curve, which represents the choices of the portfolio optimal, has a kink, and a corner equilibrium occurs with no risky asset. The interval of inertia depends on Knightian uncertainty aversion: the more uncertainty-averse the individual is, the larger the interval of reservation prices.[9]

Simonsen and Werlang (1991) generalize inertia to a portfolio with risky assets, assuming that the investor is pessimistic about future asset prices. The risk-neutral and non-satiable investor has to allocate his/her initial wealth between two risky assets x and y, with a price q_x and q_y, and

---

[8] See Arrow 1970.
[9] For details see Appendix II.

maximizes his/her expected utility. Both assets are traded with no transaction costs, but short sales are restricted. Simonsen and Werlang consider two possible scenarios (states of the world), namely the expected price of *x* is greater than the expected price of *y*, and the converse, with respective measures υ_1 and υ_2, such that $\upsilon_1(.)+\upsilon_2(.)<1$. By super-additivity of the capacity, the agent's indifference curve has a kink, and there exists a corner equilibrium in the initial position. Until the asset price ratio (q_x/q_y) is in the interval defined by the asset price ratio obtained in the two scenarios, the individual optimal portfolio will not change and there will be inertia.[10] There exists an interval of asset prices in which pessimistic agents do not trade in the markets and this only depends on the agent's beliefs and Knightian uncertainty aversion. The two bounds of the reservation price interval can be considered "the non-additive analogue of the local risk neutrality result" (Dow and Werlang 1992a: 198).

3.2. BID AND ASK SPREADS

Under Knightian uncertainty, Chateauneuf, Kast and Lapied (1992, 1993) assume that asset market prices reveal a capacity instead of a probability distribution, as in the standard theory. The expected value of an asset β is the Choquet integral of its payments with respect to a non-additive measure v, i.e. $\int_\Omega \beta dv$, and "this capacity expresses risks appreciation by the market in the same way as the linear pricing in tight [complete] markets reveals a probability" (Chateauneuf, Kast and Lapied 1993: 303). Chateauneuf, Kast and Lapied consider a dealer who has both a long and a short position on the asset β, $\int_\Omega -\beta dv$ and $\int_\Omega \beta dv$ respectively. The dealer is Knightian uncertainty-loving and he/she has a sub-additive probability measure on the set of states. By the sub-additivity of the Choquet integral $\int_\Omega \beta dv + \int_\Omega -\beta dv \geq \int_\Omega (\beta - \beta) dv$, $\int_\Omega \beta dv - \int_\Omega -\beta dv \geq 0$ and $\int_\Omega \beta dv \geq . \int_\Omega -\beta dv$. In this condition the dealer makes a profit, which is a representation of the bid and ask spread on the asset

[10] See Appendix II.

β. This positive gain of the dealer could be regarded as the transaction costs that an individual has to pay if he decides to buy and sell an asset (non-comonotonic assets[11]). Chateauneuf, Kast and Lapied extend the non-linear pricing rule to price puzzles, such as the premium paid for a prime and a score on the same underlying security, and the violation of the famous call and put parity. Chateauneuf, Kast and Lapied observe that the prime, the score and the underlying asset are not comonotonic if dividends are included in the prime payments. The Knightian uncertainty-loving dealer is aware that there is a demand for score coming from different individuals and he/she makes a profit buying the underlying security and selling prime and score. Violation of call-put parity is encompassed, proving that the value of the replicating portfolio obtained by the Choquet pricing rule may differ from the value of the put. Once more, the uncertainty-loving dealer makes profits trading non-comonotonic assets and derivatives.

3.3. THIN MARKETS

Basili (1999) assumes that agents are risk-neutral and price-taker but they are split into the classes of uncertainty-loving (optimists) and uncertainty-averse (pessimists).[12] The beliefs of the optimistic and pessimistic agents determine the lower and upper bounds, respectively, of an asset price. In this closed interval there is partial inertia and a thin market; outside the interval agents always trade and there is a thick market.

Basili assumes that all agents face Knightian uncertainty about future events (prices). Agents have common expectations, that is "they associate the same future prices to the same event [but] this does not necessarily imply that they agree on the joint distribution of future prices, since different traders may assign different subjective probabilities to the same event" (Radner 1972: 289). Roughly speaking, agents assume that an asset is completely defined by its flows of payments and take as given the structure of the replicating portfolio, that is, the probability distribution that is

[11] Two assets are comonotonic (common monotonic) if they vary in the same way, that is, if their covariance is positive for any probability measure on the state space. For details, see Appendix III.

usually applied in the market to recover the asset price. However, since the probability distribution induced by the replicating portfolio does not represent the probability distribution on future events, they may have quite different probability distributions, that depend on their beliefs about these events.

By common expectations, the value of an asset β is revealed by the replicating portfolio and equals $\int_{\Omega} \beta dp$. Because of Knightian uncertainty-aversion, the pessimistic agents have a unique super-additive measure υ such that $\int_{\Omega} \beta d\upsilon = \int_{\Omega} \beta dp$. Pessimistic agents consider that the Choquet expected value of the asset β with respect to υ equals to their worst expectation and it is the highest price at which they will buy the asset β. Pessimistic agents' short position in asset β may be represented by the lowest price from which pessimists will wish to sell β. By the asymmetry of the Choquet integral, there exists a unique capacity υ^*, called conjugate of υ, such that $\int_{\Omega} \beta d\upsilon^* = \int_{\Omega} \beta dp^*$. The Choquet integral of β with respect to υ^* reveals the best expectation of the pessimists or the maximum expected value of the asset β with respect to all measures consistent with their beliefs. This threshold price is the upper price (upper bound) from which the pessimistic agents will sell β. At prices between the lower and the upper price pessimists do not hold the asset β.[13]

Consider the class of Knightian uncertainty-loving agents. By common expectations there exists a unique sub-additive measure ν such that $\int_{\Omega} \beta d\nu = \int_{\Omega} \beta dp$. This threshold price can be considered the lowest price from which optimistic agents will wish to sell β. Optimistic agents' long position may be represented by the maximum price up to which they will wish to buy the asset β. By the asymmetry of the Choquet integral $\int_{\Omega} -\beta d\nu = -\int_{\Omega} \beta d\nu^{\circ}$, the Choquet expected value of the

[12] Without loss of generality it can be assumed that optimists are professional stock or option traders and pessimistic agents are all the others, see Fox, Rogers and Tversky (1996).

asset β with respect to $\nu°$, conjugate of ν, reveals the worst expectation of the optimistic agents or the highest price (lower bound) up to which they will wish to buy the asset β. Between the upper and the lower prices there is inertia.

Summing up, at prices between $[\int_\Omega \beta d\nu°; \int_\Omega \beta d\upsilon *]$ there is a thin market for the asset β because of partial inertia. In the sub-interval $[\int_\Omega \beta d\nu°; \int_\Omega \beta dp]$ the optimists do not hold the asset and only the pessimists would be wishing to buy; vice versa when the price of β is in the sub-interval $[\int_\Omega \beta dp; \int_\Omega \beta d\upsilon *]$, the pessimists do not hold the asset and only the optimists would be wishing to sell.[14] This conclusion could have applications; for example it could indicate the minimum price at which a hostile takeover can be bidden and the maximum price at which a company can float on the stock market or an initial public offering can be launched.

4. KNIGHTIAN UNCERTAINTY IN FINANCIAL MARKETS: INTERTEMPORAL MODELS

In order to resolve some empirical regularities that are anomalous in financial markets, such as excess volatility of asset prices, market breakdowns or booms and crashes in asset prices, intertemporal models, in which risk-neutral agents faced with Knightian uncertainty, have been developed. Instead of considering a unique additive probability measure, these formal models of asset price determination assume that an agent's beliefs and attitude towards uncertainty are expressed by a capacity (Dow and Werlang 1992b) or a set of priors (Hu 1994; Epstein and Wang 1994, 1995). In intertemporal pricing models, under Knightian uncertainty, agents update their beliefs according to Bayes' rule, when a set of additive probabilities is involved, or by pseudo-

[13] See Appendix IV.
[14] A method of evaluating intervals of partial inertia with three states of the world is in Basili 1999.

Bayes rules due to Dempster (1968) and Shafer (1976) and axiomatized by Gilboa and Schmeidler (1993), when agents' beliefs are expressed by non-additive probability measures.[15]

4.1. EXCESS VOLATILITY OF ASSET PRICES

The variance, Var(.), bounds restrictions, due to Le Roy and Porter (1981) and Shiller (1981), are based on the assumption that agents are risk-neutral and that an asset price is the expectation, $E(.)$, of the discounted value of its future payments. For the sake of simplicity, consider an asset with a current price q, which pays a single dividend with a present value D. The current asset price depends on the agent's information set I. As a consequence, $q=E(D|I)$ and the current asset price is the expected value of asset dividends conditional to the agent's information set. By the iterative law of expectations[16] $E(D)=E[E(D|I)]$, the variance bounds inequality states that $Var(D)\geq Var(q)$, with $Var(D)=Var[E(D|I)]+E[Var(D|I)]$.

Dow and Werlang (1992b) introduce Knightian uncertainty to explain the systematic violation of the variance bounds inequality and high stock price volatility in financial markets. In an intertemporal model, Dow and Werlang assume Knightian uncertainty-averse agents who have to allocate their wealth between a stock (risky asset) and money (riskless asset). There are finite future states of the world (possible dividends) and the stock pays a dividend D at the end. Knightian uncertainty is resolved at the end, but risk-neutral agents receive new information I (finer partition of the states space) about the value of the asset after a period. Since agents are Knightian uncertainty-averse, their beliefs are represented by a convex capacity υ. As a result, there exists a set of additive probability measures, which is called core of υ, that majorizes the convex capacity[17]

---

[15] Let $s_i,s_j\in S$ and $p(s_i)>0$. Bayes' rule states that $p(s_i|s_j)=p(s_i\cap s_j)/p(s_i)$. The Dempster-Shafer rule, which is a pessimistic interpretation of Bayes' rule, states that $\upsilon(s_i|s_j)=\{\upsilon[(s_i\cap s_j)\cup s_i^C]-\upsilon(s_i^C)\}/[1-\upsilon(s_i^C)]$, such that (s_i^C) is the complement of s_i and $[1-\upsilon(s_i^C)]\neq 0$. The Gilboa rule, which is an optimistic interpretation of Bayes' rule, states that $\nu(s_i|s_j)=\nu(s_i\cap s_j)/\nu(s_i)$.

[16] In a probability space (Ω,S,p), the iterative law of expectations states that if I is a sub-algebra, such that $I\subseteq S$, then for any random variable D, $E[E(D|I)]=E(D)$.

[17] Given a capacity υ, the core of υ is a non-empty set (convex hull) of additive probabilities such that: $core(\upsilon)=\{p\,|\,p(s_i)\geq\upsilon(s_i),\forall s_i\in S;\,p(\Omega)=\upsilon(\Omega)\}$.

and one of them is assumed to be the frequency distribution observed in the market. Dow and Werlang calculate $q_1=E(D|I)$, the expected asset price conditional to I, and point out that $Var(q_1)\geq Var(D)$ can occur for probabilty measures in the core of υ. Moreover, they show that "both endpoints of the interval of possible variances are larger for the price than for the value" (Dow and Werlang 1992: 636).

Knightian uncertainty-averse agents' beliefs are represented by a convex capacity, and violation of the variance bounds inequality may occur if the current distribution of prices (observed data) is expressed by an additive probability in the core of the convex capacity.[18]

4.2. MARKET BREAKDOWNS, BOOMS AND CRASHES IN ASSET PRICES

Hu (1994) considers market breakdowns in the form of trading suspensions and price crashes.[19] These phenomena frequently occur in financial markets, and are unrelated to fundamental value changes or release of relevant information. Standard financial literature explains these financial market failures on the basis of asymmetric information and liquidity shortage.[20] Hu provides an alternative explanation by introducing Knightian uncertainty, proving that ambiguity of information, represented by multiple priors, induces a set of asset prices that could determine financial market failures.

In a generalized version of the model of Grossman and Stiglitz (1980), Hu considers a two-period economy, t=0,1, with an informed and an uninformed agent. He assumes a riskless asset with zero return and a risky asset, with a price q at time t=0, and a terminal value V. The terminal value is the sum of two normally and independently distributed components, θ and ε, such that $\theta \sim N(\theta_0,1)$ and $\varepsilon\sim(0,1)$. The supply of the risky asset at period 0 is X, which has a normal distribution $X\sim(X_0,1)$, and is independent of θ and ε. The demand of the informed agent is a function of signal θ and price

[18] See Appendix V.

[19] It is reported that on average four stocks in the NYSE may be suspended from trading every day. For details see Bhattacharya and Spiegel (1993).

q; the uniformed agent only observes the price and his/her demand is a function of q. Due to ambiguous information, the agents, who know $\varepsilon\sim(0,1)$, assign the same multiple priors to $\theta\sim N(\theta_0,1)$, with $\theta_0 \in [\theta_1,\theta_2]$, and $X\sim(X_0,1)$, with $X_0 \in [X_1,X_2]$. Neither agent is sure of the exact distributions of θ and X. Even if the informed agent still observes the realization of θ, he/she does not know to exact distribution. Under Knightian uncertainty, both agents maximize expected utility of wealth, given their budget constraints, and apply the maximin decision rule. Multiple priors produce two direct effects: equilibrium in the economy does not exist in a particular interval of θ and X, and the equilibrium price function is not continuous in the interval.[21] As a result, market breakdowns and price crashes may occur, not because of any fundamental value changes or release of information, but simply because of ambiguity that scares the agents off. Indeed, the caution of agents against ambiguity of information could transform a small change in the asset demand into a crash. Analogue results are obtained with a modified version of Kyle's model (1985). Hu concludes that ambiguity of information is a sufficient condition for asset pricing puzzles.

Epstein and Wang (1994, 1995) apply the Gilboa and Schmeidler (1989) multiple priors approach to an intertemporal multiple-asset framework. Consider a Lucas-style pure exchange economy with infinitely many periods and a representative agent.[22] In this economy there is one perishable good (c), the total supply of which is described by the endowment process, which is a time-homogeneous function of the current state of the world (time-homogeneous Markov process[23]), and a finite set of securities (N), that provides dividends (time-homogeneous Markov process). In each period the securities are treated at prices denominated in units of consumption and the representative consumer formulates plans for the present and all future periods. There will exist

[20] Information asymmetry is considered in Glosten 1989, Bhattacharya and Spiegel 1993. Liquidity shortage is pointed outin Grossman and Miller 1988.

[21] It is worth noting that equilibrium does not exist if $\theta_1+X_1<\theta-X<\theta_2+X_2$ and equilibrium price function $q(\theta,X)$ is not continuous in (θ,X). Therefore 'price crash can happen when $\theta-X$ move from θ_2+X_2 to θ_1+X_1' (Hu 1994).

[22] See Lucas 1978.

[23] Given a Markov process, the past and future are statistically independent when the present is known. A Markov process is said to be time-homogeneous if its transition probability is stationary, that is, it only depends on the length of the time interval, regardless of its actual position. See Duffie (1996).

a Pareto optimal equilibrium if it is possible to define an asset price process (q_t) and a trading strategy that maximize the consumer's utility function.

Rather than assuming an additive utility function, which is the dominant utility specification in intertemporal stochastic modeling, the representative consumer maximizes a more general recursive utility function. The additive expected utility form $U(c)=E\left[\sum_0^T \rho^t u(c_t)\right]$, where $\rho\in(0,1)$ is discount rate, has some disadvantages. It does not enable representation of the consumer's preference with respect to the timing of the resolution of uncertainty, and it fixes the dual roles of the function u as a determinant of the consumer's risk aversion and of the rate of intertemporal substitution. The standard utility function fixes the elasticity of intertemporal substitution and relative risk aversion in terms of one another, hence "the utility function can only model the pairs: high risk aversion, low substitution and low risk aversion, high substitution" (Epstein, 1995). The recursive utility form $U(c)=u(c_1)+\rho \mathrm{E}U(c_2,c_3,..)$ enables the effect of varying the degree of risk aversion to be examined, without affecting the intertemporal substitution elasticity of the utility function.

Epstein and Wang (1994, 1995) introduce vagueness of the beliefs component of the utility function into this Lucas-style economy, by replacing the unique probability measure with a convex set of probability measures, the size of which indicates the degree of uncertainty. The presence of uncertainty aversion introduces an important technical difference into their Lucas-style economy, namely a lack of Gateaux differentiability of the consumer utility function.[24] Non-differentiability is derived directly from application of the maximin criterion to the recursive utility function of the Knightian uncertainty-averse consumer and leads to the non-uniqueness or indeterminacy of the equilibrium asset price (non-unique supporting prices). In fact, "the utility is not Gateaux differentiable in general, because utility is defined via a point-wise minimum corresponding to the Gilboa and Schmeidler (1989) way of modeling uncertainty aversion, and a positive minimum of functions is not differentiable in general” (Epstein and Wang 1994: 295). As a result, price indeterminacy may “leave room for sunspots or Keynesian animal spirits to determine a particular equilibrium process” (Epstein and Wang 1995), leading to more volatility in asset prices than is predicted in the standard model. Moreover, price discontinuity may explain booms and crashes, since “given such discontinuity, small changes in market fundamentals can be associated with abrupt changes in the security price” (Epstein 1995).

V. APPLICATIONS

Interesting applications to financial markets have recently been proposed. Melino and Epstein (1995a, 1995b) develop an empirical model that allows the excess volatility of asset prices to be investigated by formal statistical methods. Cherubini (1997) proposes a parametric representation of Knightian uncertainty, by means of a class of fuzzy measures called g_λ-measures, in which the parameter λ is an indicator of the degree of uncertainty, to determine the price of a corporate debt contract and a version with Knightian uncertainty of the Black and Scholes model.

Melino and Epstein point out an empirically tractable intertemporal model of asset returns that incorporates a role for uncertainty as opposed to risk. They describe an intertemporal multiple priors model of asset prices with recursive utility and non-Bayesian rational expectations, which is tractable and refutable with respect to an observed data set. They assume that the true probability law governing the asset price state process is described by a probability measure, which is contained in the set of the agent’s beliefs. The agent’s beliefs (set of probability distributions) embody non-Bayesian rational expectations, indeed expectations are non-Bayesian since they are not representable by a single prior, but they are rational because they agree with the truth. That is, “non-Bayesian rational expectations capture that idea that there is no uncertainty about what the agent has been doing, though there may very well be uncertainty associated with deviations, say through

[24] Given a function and a point in a convex set of a vector space, the derivative of the function at that point in a specific direction in the set of feasible directions is called the directional or Gateaux derivative (Duffie 1996).

diversification into certainty securities" (Epstein 1995). Melino and Epstein show that Generalized Moments Method econometric techniques are fully applicable to the model equations, which happen to be empirically falsifiable.[25] Indeed, they "first show that standard preferences that ignore the distinction between risk and uncertainty are overwhelmingly rejected by the data …extending the preferences to allow for aversion to uncertainty generates mixed results. The Euler equations are consistent with the data, but only if [they] allow for an aversion to uncertainty that appears high when compared to the level obtained from introspection" (Melino and Epstein 1995b).

Cherubini represents Knightian uncertainty attitude by g_λ-measures. A continuous fuzzy measure η is a g_λ-measure if for all $s_1,s_2\in S$ such that $s_1\cap s_2=\emptyset$, $\eta(s_1\cup s_2)=\eta(s_1)+\eta(s_2)+\lambda\eta(s_1)\eta(s_2)$. The parameter $\lambda\in(0,\infty)$ expresses sub-additivity and super-additivity if $\lambda\in(-1,0)$. The duality property holds for g_λ-measures.[26] Cherubini obtains two expected utility functions, which can be considered the upper and lower bounds (Choquet integrals) of the expected utility with respect to a set of probability distributions, and applies this approach to the problem of pricing debt subject to default risk and levered option replication. In the case of corporate debt, Cherubini highlights that under risk $(\lambda=0)$ an increase in nominal debt induces a decrease in its unit value, due to an increase of default risk, whereas under uncertainty aversion $(\lambda>0)$ "the increase in nominal debt also causes an enlargement of the set of probability measures as distance between the upper and lower bounds" (Cherubini 1997:144). As a result, "if we assume the corporate debt to be traded on an organized market, this would predict that there will be a greater difference between the price at which investors will be ready to short the corporate bond and that at which they will be ready to buy it" (Cherubini 1997: 144). In the case of levered option replication, Cherubini assumes the value of the option at time t is equal to the value of the underlying asset, which is traded in organized markets, plus a debt contract, which is issued in unofficial markets. Uncertainty is faced in unofficial markets

[25] Melino and Epstein look at Euler equations for equity and Treasury Bill returns, using monthly aggregate U.S. data.
[26] Details are in Cherubini 1997.

and applying “the fuzzy valuation technique to the debt position, we immediately compute upper and lower bounds, i.e. the bid and ask quotes, for the option” (Cherubini 1997:145). Cherubini tests his fuzzy version of the Black and Scholes model using data from quotes of Standard & Poor’s 500-stock index options reported by Bloomberg and points out that the bid and ask spread, predicted by the model, increases for higher levels of the strike price. The fuzzy Black and Scholes model shows that “a higher strike price implies a higher leverage, i.e. a higher debt position in the replicating portfolio, and this increases not only its value, leading to a decrease in the value of option, but also the uncertainty: consistently with the replication strategies, this shows up in the bid-ask spreads in option prices” (Cherubini 1997:146).

VI. CONCLUDING REMARKS

Attitude towards Knightian uncertainty sheds new light on some common puzzling behaviors of the financial market and allows some of them to be explained, simply by relaxing Bayesian rationality. Bayesian rationality states that each agent forms a prior of future states of the world, identifies each decision with an act over that state space and maximizes a standard form of expected utility. The crucial assumption underlying this paradigm is that the agent has a sort of *divine knowledge*, namely he/she knows all the future states of the world perfectly. Relaxing this strong assumption is a sufficient condition to permit Knightian uncertainty. Once the existence of a unique probability measure has been rejected, the prior of an agent can be represented by a set of probability measures or a non-additive probability measure, and Knightian uncertainty occurs. Approaches with both a set of probability measures and a non-additive probability measure have been used in the financial market models. Results proved by these models offer an alternative explanation of price booms and crashes, excess volatility of asset prices, violation of call and put parity, bid-ask spreads and portfolio rigidities.

Some empirical financial models under Knightian uncertainty have been discussed and they happen to confirm the validity of the non-expected utility approach from a twofold point of view, permitting a test of the market behavior assumptions (falsifiability) and reproducing observed puzzling phenomena (bid and ask spreads, corporate claims evaluation). However, implementation of empirical models with capacities or multiple additive probabilities is just beginning and some mathematical problems still have to be solved, particularly in models with some kind of inertia and intertemporal asset pricing set-up. In any case, applications to the market-based economic questions are not the only criterion for testing the validity of non-expected utility theories, particularly if they are tractable and make a difference. Quoting Epstein (1995), non-expected utility theories have made a difference, indeed “they either add analytical power and therefore new theoretical insights or new testable implications regarding market behaviors”.

APPENDIX I

By a well known theorem (Schmeidler 1989), the Choquet integral of a function *f* with respect to a convex capacity υ is equal to the minimum of a family of Lebesgue integrals with respect to a set *P* of additive probability distributions consistent with the agent's beliefs and $\int_{\Omega} f d\upsilon = \min_{p \geq \upsilon} \int_{\Omega} f dp$.

Similarly, Chateuneuf (1991) proves that the Choquet integral of a function *f* with respect to a concave capacity ν is equal to $\int_{\Omega} f d\nu = \max_{p \leq \nu} \int_{\Omega} f dp$. The Choquet integral of *f* with respect to ν equals the maximum of a family of Lebesgue integrals with respect to a family of probability distributions *P*.

APPENDIX II

Consider an individual with wealth $W=x+qy$, where x is money (riskless asset) and y is a stock (risky asset) with a price q. The risk-neutral agent can sell or buy assets without transaction costs. The agent maximizes an expected utility function $F(y)=EU[W+y(q^*-q)]$, subject to the budget constraint. In this simple model the future stock price is q^* and the expected utility is evaluated by the Choquet integral. For the sake of simplicity only two possible states of the world are assumed: $q_1^*>q$ and $q_2^*<q$, respectively with measures υ_1 and υ_2, such that $\upsilon_1(.)+\upsilon_2(.)<1$.

The indifference curve has a kink at point $y=0$, where "the individual will neither buy nor sell short the risky asset, concentrating all his wealth on money" (Simonsen and Werlang 1991: 10). At $y=0$ the right derivative is $F'_+(0)=U'(W)[E(q^*)-q]$ and the left derivative is $F'_-(0)=-U'(W)[q+E(-q^*)]$. As a consequence, expected utility will be maximum for $y=0$ if $F'_+(0)\leq 0 \leq F'_-(0)$, i.e. $E(q^*)\leq q \leq -E(-q^*)$. There will be portfolio inertia until the asset price lies in this interval. In a similar framework, Simonsen and Werlang prove a close result with two risky assets and restricted short sales.

APPENDIX III

Two assets β and γ are comonotonic if there is no pair w_1, $w_2 \in \Omega$ such that $\beta(w_1) < \beta(w_2)$ and $\gamma(w_1) > \gamma(w_2)$. The comonotonic additivity of the Choquet integral with respect to a monotone capacity μ assures that $\int_\Omega (\beta + \gamma) d\mu = \int_\Omega \beta d\mu + \int_\Omega \gamma d\mu$ if and only if the assets β and γ are comonotonic.

Note that sub-additivity of a capacity is sufficient for sub-additivity of the Choquet integral (Denneberg 1994). Let ν be a continuous sub-additive capacity; then for functions $\beta, \gamma : \Omega \to R$ there exists a sub-additive integral such that $\int_\Omega (\beta + \gamma) d\nu \leq \int_\Omega \beta d\nu + \int_\Omega \gamma d\nu$.

APPENDIX IV

Let μ: $S \to [0,1]$ be a normalized and monotone capacity on the measurable space (Ω, S).

(A 1) A capacity μ is said to be compatible with p if $\forall\, s_i, s_j \in S$, $p(s_i) \leq p(s_j)$ implies $\mu((s_i) \leq \mu(s_j)$.

(A 2) A capacity μ is said to be monotonically sequentially continuous if $\forall s \in S$ $s_n \uparrow s$ implies $\mu\,(s_n) \uparrow \mu\,(s)$ and $s_n \downarrow s$ implies $\mu\,(s_n) \downarrow \mu\,(s)$.

(A 3) Two assets β, γ are comonotonic if for any w_1, $w_2 \in \Omega$, $[\beta(w_2) - \beta(w_1)][\gamma(w_2) - \gamma(w_1)] \geq 0$

Consider the class of Knightian uncertainty-averse agents, who have a convex capacity υ on (Ω, S). Assuming (A1), (A2), (A3), no frictions and common expectations, there exists a unique convex capacity υ on (Ω, S) such that the Choquet expectation of β with respect to υ is $\int_\Omega \beta d\upsilon = \int_\Omega \beta dp$. By a well known theorem (Chateauneuf 1991), for every convex capacity υ on (Ω, S) and every function $\beta : \Omega \to R$ there exists a set P of additive probabilities on (Ω, S), such that for all events $p(.) \geq \upsilon\,(.)$ and $\int_\Omega \beta d\upsilon = \min\left\{ \int_\Omega \beta dp \middle| p \in P \right\}$. The price from which the pessimists will wish to sell the asset β may be defined by the Choquet integral of β with respect to υ^*, which is the conjugate or dual capacity for υ. The dual capacity υ^* is the extent to which an agent believes the negation of s_i, that is $\upsilon^*(s_i)=\upsilon(\Omega)-\upsilon(s_i^C)$. By the asymmetry of the Choquet integral (Denneberg 1994), $\int_\Omega -\beta d\upsilon = -\int_\Omega \beta d\upsilon^*$, there exists a unique conjugate capacity υ^* on (Ω,S) such that the Choquet expectation of β with respect to υ^* is $\int_\Omega \beta d\upsilon^* = \max\left\{\int_\Omega \beta dp^* \middle| p^* \in P\right\}$. As a consequence, the upper Choquet integral with respect to a convex measure on (Ω,S) may be computed as the Choquet integral with respect to its dual (Gilboa 1989).

Consider the class of Knightian uncertainty-loving agents, who have a concave capacity ν on (Ω,S). Since the optimists are uncertainty-loving, they consider the value of the asset β as the maximum expected value consistent with their beliefs. If usual assumptions hold, the optimists have a unique concave capacity ν on (Ω,S), such that $\int_\Omega \beta d\nu = \max\left\{\int_\Omega \beta dp \middle| p \in P^\circ\right\}$. By a theorem proved by Chateauneuf (1991), for every concave capacity ν on (Ω,S) and every function β: $S \rightarrow R$, there exists a set P° of additive probabilities on (Ω,S), such that for all events $p(.) \leq \nu(.)$ and $\int_\Omega \beta d\nu = \max\left\{\int_\Omega \beta dp \middle| p \in P^\circ\right\}$. The optimists would buy at the lowest price and they define the maximum price up to which they will wish to buy a given asset. That minimum price is defined by the Choquet integral of β with respect to ν°, namely the conjugate or dual capacity of ν, which is the minimum expected value of the asset β consistent with their beliefs. If given assumptions hold, the optimists have a unique dual capacity ν° on (Ω,S), such that $\int_\Omega \beta d\nu^\circ = \min\left\{\int_\Omega \beta dp^\circ \middle| p^\circ \in P^\circ\right\}$. By a well-known theorem (Denneberg 1994), the lowest Choquet integral of the asset β with respect to a concave ν on (Ω,S) may be computed as the Choquet integral with respect to its dual.

APPENDIX V

Assume a three-period model (t=0,1,2) and three states of the world, such that $\Omega=(w_1,w_2,w_3)$ and three possible values of D (1,1/2,0) respectively. At time t=1, agents receive additional information about the possible value of D, such that $I=\{\{w_1\},\{w_2,w_3\}\}$. Risk-neutral agents are Knightian uncertainty-averse and their beliefs are represented by the following convex capacity υ, such that $\upsilon(w_1)=\upsilon(w_2)=\upsilon(w_3)=1/4$ and $\upsilon(w_1\cup w_2)=\upsilon(w_1\cup w_3)=\upsilon(w_2\cup w_3)=1/2$. There exists a closed and convex set of additive probability measures that majorizes υ, i.e. the following three probability distributions: $p(w_1)=1/2$, $p(w_2)=p(w_3)=1/4$; $p(w_1)=p(w_2)=1/4$, $p(w_3)=1/2$; $p(w_2)=1/2$,$(w_1)=p(w_3)=1/4$.

Dow and Werlang calculate the variance of q_1 and D with respect to each probability distribution and obtain: $Var(q_1)=p(w_1)[1-p(w_1)]+p(w_2\cup w_3)[1-p(w_2\cup w_3)]/36-p(w_1)[p(w_2\cup w_3)]/3$ and $Var(D)= p(w_1)[1-p(w_1)]+ p(w_2)[1-p(w_2)]/4-p(w_1)p(w_2)$.

Results: the variance of q_1 lies in the interval [0,1302; 0,1736], whereas the variance of D lies in the interval [0,1250; 0,1719]. Hence, the variance bounds restrictions are violated and the variance of q_1 could be larger than the variance of D.

REFERENCES

Allais, M. (1953), Le comportemenet de l'homme rationel devant le risque: critique des postulates et axioms de l'ecole americaine, *Econometrica*, 21: 503-546.

Ami, D., Kast R., and Lapied A. (1991), Generalizing Arrow pricing to understand financial markets, *Document de Travail* G.R.E.Q.E. 91A04, Universites d'Aix-Marseille II et III.

Anscombe, F.G., and Aumann R.J. (1963), A definition of subjective probability, *The Annals of Mathematics and Statistics*, 34: 199-205.

Arrow, K.J. (1953), Le role des valeurs boursieres pour la repartition la meillure des risques, *Econometrie*, 40: 41-47; English translation (1964), *Review of Economic Studies*, 31: 91-96.

Arrow, K.J. (1970), *Essays in the Theory of Risk Bearing*, London: North Holland.

Arrow, K.J., and Hurwicz L. (1972), *An optimal criterion for decision making under ignorance*, in C.F. Carter and J.L. Ford, *Uncertainty and Expectations in Economics*, Basil Blackwell, Oxford.

Basili, M. (1999), Knightian uncertainty causes price intervals in financial markets, *Quaderni del Dipartimento di Economia* n. 268, University of Siena, Siena.

Bhattacharya, U., and Spiegel M. (1993), Cross-sectional patterns in NYSE trading suspensions: theory and evidence, mimeo.

Black, F., and Scholes M. (1973), The pricing of options and corporate liabilities, *Journal of Political Economy*, 81: 637-654.

Camerer, C.F., and Weber M. (1992), Recent developments in modeling preferences: uncertainty and ambiguity, *Journal of Risk and Uncertainty*, 5: 325-370.

Chateauneuf, A. (1991), On the use of capacities in modeling uncertainty aversion and risk aversion, *Journal of Mathematical Economics*, 20: 343-369.

Chateauneuf, A., Kast R., and Lapied A. (1992), Pricing in slack markets, *Document de Travail* G.R.E.Q.E. 92A05, Universites d'Aix-Marseille II et III.

Chateauneuf, A., Kast R., and Lapied A. (1993), *Market preferences revealed by prices: non-linear pricing in slack markets*, in M. Machina and B. Munier, *Models and Experiment on Risk and Rationality*, Kluwer Academic Publishers, Dordrecht.

Cherubini, U. (1997), Fuzzy measures and asset prices: accounting for information ambiguity, *Applied Mathematical Finance*, 4: 135-149.

Choquet, G. (1953): Theorie des Capacites, *Annales de l'Institut Fourier*, 5: 131-233.

Dempster, A.P. (1968), A generalization of Bayesian inference, *Journal of the Royal Statistical Society*, Series B 30: 205-237.

Denneberg, D. (1994), *Non-additive Measure and Integral*. Dordrecht: Kluwer Academic Publishers.

Dow, J., and Werlang S. (1988), Uncertainty aversion and the optimal choice of portfolio, *Ensaios Economicos da EPGE* n. 115, Fundação Getulio Vargas.

Dow, J., and Werlang S. (1992a), Uncertainty aversion, risk aversion, and the optimal choice of portfolio, *Econometrica*, 37: 197-204.

Dow, J., and Werlang S. (1992b), Excess volatility of stock prices and Knightian uncertainty, *European Economic Review*, 36: 631-638.

Duffie, D.(1996), *Dynamic Asset Pricing Theory*, second edition, Princeton University Press, Princeton, New Jersey.

Ellsberg, D. (1961), Risk, ambiguity and the Savage axioms, *Quarterly Journal of Economics*, 75: 643-669.

Epstein, L.G. (1995), Consumption, savings and asset returns with non-expected utility, International School of Economics Research, University of Siena, Siena.

Epstein, L.G., and Wang T. (1994), Intertemporal asset pricing under Knightian uncertainty, *Econometrica*, 62: 283-322.

Epstein, L.G., and Wang T. (1995), Uncertainty, risk neutral measures and security price booms and crashes, *Journal of Economic Theory*, 67: 40-82.

Fox, C., Rogers, B. and Tversky A. (1996), Option traders exhibit subadditive decision weights, *Journal of Risk and Uncertainty*, 3: 5-17.

Gilboa, I. (1987), Expected utility theory with purely subjective non-additive probabilities, *Journal of Mathematical Economics,* 16: 65-88.

Gilboa, I. (1989), Duality in non-additive expected utility theory, *Annals of Operations Research*, 19: 405-414.

Gilboa, I., and Schmeidler D. (1989), Maximin expected utility with non-unique prior, *Journal of Mathematical Economics*, 18: 141-153.

Gilboa, I., and Schmeidler D. (1993), Updating ambiguous beliefs, *Journal of Economic Theory*, 59: 33-49.

Glosten, L.R. (1989), Insider trading, liquidity, and the role of the monopolist specialist, *Journal of Business,* 62: 211-235.

Grossman, S., and Miller M.H. (1988), Liquidity and markets structure, *Journal of Finance*, 43: 617-637.

Grossman, S., and Stiglitz J. (1980), On the impossibility of informationally efficient markets, *American Economic Review*, 70: 393-407

Hu, J. (1994), Market breakdowns and price crashes explained by information ambiguity, *working paper series* 94-16, Federal Reserve of Atlanta.

Kyle, A. (1985), Continuous auctions and insider trading, *Econometrica* 53: 1315-1335.

Knight, F.H. (1921), *Risk, Uncertainty and Profit*, Houghton Mifflin, Boston. Reprinted 1985 University of Chicago Press, Chicago.

Le Roy, S.F, and Porter R. (1981), The present value relation. tests based on implied variance bounds, *Econometrica*, 49: 555-574.

Lucas, R.E. (1978), Asset prices in an exchange economy, *Econometrica*, 46: 1429-1445.

Machina, M., and Schmeidler D. (1992), A more robust definition of subjective probability, *Econometrica*, 60: 745-780.

Melino A., and Epstein, L.G. (1995a), An empirical analysis of asset returns under 'non-Bayesian rational expectation', mimeo.

Melino A., and Epstein, L.G. (1995b), An empirical analysis of asset returns under 'non-Bayesian rational expectation', *Journal of Finance*, 50: 969.

Radner, R. (1972), Existence of equilibrium of plans, prices, and price expectations in a sequence of markets, *Econometrica*, 40: 289-303.

Savage, L. J. (1972): *The Foundation of Statistics*. New York: Wiley; 2nd ed. New York: Dover.

Schmeidler, D. (1982), Subjective probability without additivity, *working paper*, Foerder Institute for Economic Research, Tel Aviv University.

Schmeidler, D. (1986), Integral representation without additivity, *Proceedings of the American Mathematical Society*, 97: 571-587.

Schmeidler, D. (1989), Subjective probability and expected utility without additivity, *Econometrica*, 57: 571-587.

Shafer, G. (1976), *A Mathematical Theory of Evidence*, Princeton University Press, Princeton, New Jersey.

Shiller, R. (1981), Do stock prices move too much to be justified by subsequent changes in dividends?, *American Economic Review*, 71: 421-436.

Simonsen, M. and Werlang S. (1991), Subadditive probabilities and portfolio inertia, *Revista de Econometrica*, 11: 1-19.

Von Neumann, J., and Morgenstern O. (1944), *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, New Jersey.

Wald, A. (1950), *Statistical Theory Decisions*, Wiley, New York.