Patterns of Freedom and Flexibility

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Abstract

A few opportunity rankings which can be defined without relying on preferences for basic alternatives are compared with respect to a set of containment monotonicity properties, and represented in terms of flexibility w.r.t. a –possibly variable–set of preferential criteria. It is shown that a large variety of patterns obtain : in particular a few 'nice' opportunity rankings are defined which cannot be possibly represented in terms of 'unanimous' indirect preference maximization with respect to a fixed set of preference orderings on the outcome set. JEL Classification Numbers : D71, 025.

1 Introduction

Let X be a non-empty finite set of basic alternatives, and $P(X)$ the power set of X i.e. the set of its *opportunity sets.* An *opportunity ranking (OR)* is a binary relation $(P(X), \succcurlyeq)$: in particular, we shall be mainly concerned with *opportunity preorders* i.e. with *transitive and reflexive* ORs.

The literature on ORs typically relies on a few quite different intuitions which result in different interpretations of the former. Prima facie, four main approaches can be distinguished in the study of ORs, namely:

i) ranking opportunity sets in terms of freedom of choice;

ii) ranking opportunity sets in terms of range or diversity;

iii) ranking opportunity sets in terms of flexibility with respect to a set of preference orderings on X;

iv) ranking opportunity sets in terms of a rigid preference ordering on X.

The 'freedom of choice' view tends to rule out any information whatsoever concerning orderings on X as far as the definition of $\succcurlyeq$ is concerned. The 'diversity' approach may be construed as relying on some notion of *range* concerning subsets of X. The 'flexibility' approach relies on information about *sets* of possibly relevant preferences on X. Finally, the 'rigid preference' approach purports to provide a plausible extension to $P(X)$ of a *fixed* preference relation on X: thus, it may be regarded as a special limit case of the 'flexibility' approach . By contrast, the relationships among the 'freedom of choice', 'flexibility' and 'diversity' approaches are still less than clear (see e.g. Sen (1988,1991), Puppe (1995,1996), Sugden (1998)). One view emphasizes the 'intrinsic value' of 'freedom of choice' as an independent preference-free criterion underlying ORs (see e.g. Sen (1988), Pattanaik,Xu (1990), Klemisch-Ahlert (1993), Bossert,Pattanaik,Xu (1994), Gravel (1994,1998), Dutta,Sen (1996)). Another –somehow opposite– view insists that preference for 'freedom of choice' essentially reduces to preference for 'flexibility' *and* equates the latter to indirect-utility(preference)-maximization with respect to a fixed set of relevant preferences (see e.g. Arrow (1995), or Nehring,Puppe (1999) for two different versions of this approach to the interpretation of ORs : Sugden (1998) endorses a similar view but insists that the 'flexibility' sobriquet should be reserved for the special case where the relevant preference preorders are–essentially– the probable future preferences of the relevant agent).

The present paper is mainly devoted to a criticism of those two widely held views. Indeed, it might be argued that the ultimate significance of 'freedom of choice' relies on availability of different alternative *choices* hence of different *revealed preferred* basic alternatives: one might perhaps be willing to allow 'inconsistent' choice functions or nontransitive revealed preference relations, and/or refer to a *context-dependent* set of preferential criteria, but reference to *some* set(s) of choice functions or preference relations (revealed or otherwise) seems to be scarcely avoidable. Thus, the very notion of an 'intrinsic value of freedom of choice' may be apparently re-interpreted in terms of flexibility with respect to suitable subsets of a universal set of preferential criteria. At the same time, it is quite unclear why the general notion of 'flexibility' with respect to suitable sets of preferences should be reduced to one particular version, namely 'unanimous' indirect-preference-maximization with respect to a (context-independent) *fixed base* of preferences.

The line of reasoning we advocate here suggests that 'freedom of choice' and 'flexibility' (and perhaps 'diversity' as well) are best regarded as labels for notions of 'opportunity' which simply emphasize reliance on different types of available information on X. In that connection, the fixed-base unanimous indirect-preference-maximization (FBU-IPM) version of 'flexibility' should be regarded as a quite specialized –if undoubtedly interesting–case of the latter notion. Thus, a gap opens up –*in principle*– between the much insisted upon FBU-IPM notion of 'flexibility' and the *general* concept of 'flexibility' with respect to some possibly variable (i.e. *context-dependent*) set of preference preorders. But then, the following issue immediately arises: *are there 'nice' opportunity preorders which actually live in the niche provided by that gap? Or, equivalently, are there sensible opportunity preorders which cannot possibly arise from unanimous indirect-preference-maximization with respect to some fixed set of* *preference preorders?*

Those issues can only be settled by providing a set of uncontroversial minimum requirements for 'nice' ORs and working out a few concrete examples of ORs of the relevant sort. The present paper is therefore devoted to a first tentative analysis along these lines. It proceeds in the most straightforward –and elementary– way according to the two steps mentioned above, namely by:

a) establishing a few fundamental preference-free axioms and criteria that are satisfied by the uncontroversial if undecisive set-containment partial order –essentially a set of *containment-monotonicity* properties– to be used as a common reference base for a comparative assessment of the relevant ORs;

b) generating a suitable list of concrete examples of ORs which have been actually proposed–or might be considered– under the most abstract label i.e. 'freedom of choice'(with no explicit reference to the fixed-base-unanimous indirect-preference-maximization(FBU-IPM) version of 'flexibility').

In particular, we focus on two (dual) families of monotonicity conditions– i.e.*improvability(I) and essentiality(E)* properties– whose significance for our purposes is enhanced from the fact that FBU-IPM ORs tend to perform reasonably well w.r.t. essentiality properties and badly w.r.t. improvability properties.Concerning minimum requirements for OR preorders in terms of I-and E-properties, we adopt here–for the sake of argument– the most permissive stance: *we implicitly treat any non-empty subset from our list of I- and E-properties as a feasible option for minimum containment-monotonicity requirements.*(Notice that the cardinality-based total preorder is perfectly well-behaved w.r.t. both $I-$and $E-$properties, as well as amenable to a FBU-IPM flexibility interpretation, but is generally regarded as an arbitrary– hence trivial– OR).

Thus, we try and use the foregoing axioms in order to elicit patterns in the set of ORs under consideration, and check their credentials as 'nice' ORs. Our findings are largely preliminary (e.g. we do not provide any characterization result), but telling. Among the 'freedom of choice'-motivated ORs we consider, we find: α)one class of ORs (convex ORs) which are a good compromise between $I-$and $E-$properties and amenable to a FBU-IPM flexibility interpretation, β)two classes of ORs (set-filtral and convex-set-filtral ORs) which are fairly good $I - E$ compromises while not being amenable to a FBU-IPM-flexibility interpretation, γ) a class of (total) ORs (the width-based ORs) which are reasonably well-behaved w.r.t. $E-$properties, badly-behaved w.r.t. $I-$properties and not amenable to a FBU-IPM flexibility interpretation, and δ)a class of (total) ORs (total set-filtral ORs) which are both $I-$and $E-$badly behaved and not generally amenable to any FBU-IPM flexibility interpretation.

In our view, such a remarkable variety of non-trivial ORs does certainly call for further analysis. However, our preliminary results (especially those under points β and γ above) apparently suggest a tentative positive answer to our question on existence of 'nice' non FBU–IPM ORs, confirming by example that the general label of 'freedom of choice'–or, for that matter, 'flexibility'– admits a far more comprehensive interpretation than the typical emphasis on FBU-IPM ORs might suggest.

The paper is organized as follows: Section 2 presents the model, and the results. Section 3 is mainly devoted to a short discussion of some related literature on 'flexibility' and 'diversity'. Section 4 provides some concluding remarks. Proofs are confined to an Appendix.

2 Model and results

Let X be the (finite) set of alternatives/opportunities, and $P(X)$ the corresponding set of opportunity sets: we assume $\#X \geq 3$ in order to avoid trivialities and the need for tedious qualifications. We are concerned here with defining a pure opportunity ranking $\succcurlyeq$ of $P(X)$, namely a binary relation $(P(X), \succcurlyeq)$ that (weakly) extends $(P(X), \supseteq)$ —i.e. $A \supseteq B$ entails $A \succcurlyeq B$— the underlying interpretation being that $A \succcurlyeq B$ means "A embodies more opportunities(or positive freedom) than B ". We also denote as usual by $\succ$ and $\sim$ the asymmetric and symmetric components of $\succcurlyeq$, respectively. Moreover, for any set Y, $\mathbf{C}(Y)$ will denote the set of all chains on Y (a chain is a linear order i.e. an antisymmetric, total and transitive binary relation). In order to meet some basic intuitions concerning the very idea of an "opportunity ranking" we shall consider some minimal restrictions on $(P(X), \succcurlyeq)$, namely

(Preorder (PR)) $(P(X), \succcurlyeq)$ is transitive and reflexive .

(Weak Monotonicity (WM)) For any $A, B \subseteq X$, $A \cup B \succcurlyeq A$.

A few other monotonicity properties which are satisfied by the set-containment partial order, and are usually related to 'freedom of choice' will also be considered:

(Weak Set-Improvability(WSI)) For some $X' \subset X$ and any A with $X' \subseteq A \subset X$ there exists $B \subseteq X$ such that $A \cup B \succ A$.

(Weak Point-Improvability(WPI)) For some $X' \subset X$ and any A with $X' \subseteq A \subset X$ there exists $x \in X$ such that $A \cup \{x\} \succ A$.

(Set-Improvability (SI)) For any $A \neq X$ there exists $B \subseteq X$ such that $A \cup B \succ A$.

(Point-Improvability(PI)) For any $A \neq X$ there exists $x \in X$ such that $A \cup \{x\} \succ A$.

(Weak Set-Essentiality(WSE)) For some $X' \subset X$ and any A with $X' \subset A \subseteq X$ there exists $B \subseteq A$ such that $A \succ A \backslash B$.

(Weak Point-Essentiality(WPE)) For some $X' \subset X$ and any A, $X' \subset A \subseteq X$ there exists $x \in A$ such that $A \succ A \backslash \{x\}$.

(Set-Essentiality(SE)) For any $A \neq \emptyset$ there exists $B \subseteq A$ such that $A \succ A \backslash B$.

(Point-Essentiality (PE)) For any $A \neq \emptyset$ there exists $x \in A$ such that $A \succ A \backslash \{x\}$.

Obviously, $PI \Longrightarrow SI \Longrightarrow WSI, PI \Longrightarrow WPI \Longrightarrow WSI, PE \Longrightarrow SE \Longrightarrow WSE$, and $PE \Longrightarrow WPE \Longrightarrow WSE$, but not vice versa.

It should be noticed that the entire set of monotonicity properties mentioned above has been subjected to some criticisms on the grounds that there may exist 'ugly' alternatives which do not really contribute to improve any opportunity set. Insisting on the relevance of each of those monotonicity properties clearly entails that the possibility of such 'ugly' alternatives is implicitly ruled out.

The latter assumption can be indeed readily justified by means of the following *acceptance criterion for* X: take a reference set of preference relations on some X' and choose $X \subseteq X'$ such that each $x \in X$ is *maximal* with respect to at least one such relation.

Remark 1 *In Pattanaik,Xu(1998) two 'new' total opportunity preorders are defined –and characterized– which rely on a fixed reference set* $\mathbf{P} = \{P_1, .., P_k\}$ *of total preference preorders on* X, *namely* $(P(X), \succcurlyeq_{(\#,\mathbf{P})})$ *and* $(P(X), \succcurlyeq_{(\#,\mathbf{P},-)})$. *Such total opportunity preorders are defined as follows: for any* $A, B \subseteq X$

$A \succcurlyeq_{(\#,\mathbf{P})} B$ *iff* $[\#(\bigcup_{i=1}^{k} \max_{P_i} A) \geq \#(\bigcup_{i=1}^{k} \max_{P_i} B)]$, *and*

$A \succcurlyeq_{(\#,\mathbf{P},-)} B$ *iff* $[\#((\bigcup_{i=1}^{k} \max_{P_i} A) \cap (\bigcup_{i=1}^{k} \max_{P_i} A \cup B)^c) \geq \#((\bigcup_{i=1}^{k} \max_{P_i} B) \cap (\bigcup_{i=1}^{k} \max_{P_i} A \cup B)^c)]$.

It should be remarked here that under the acceptance criterion for X *as defined above both* $(P(X), \succcurlyeq_{(\#,\mathbf{P})})$ *and* $(P(X), \succcurlyeq_{(\#,\mathbf{P},-)})$ *reduce to the cardinality-based OR* $(P(X), \succcurlyeq_{\#})$ *to be defined below. Hence, under the acceptance criterion for* X *the discussion of* $(P(X), \succcurlyeq_{\#})$ *we provide below does in fact also cover both* $(P(X), \succcurlyeq_{(\#,\mathbf{P})})$ *and* $(P(X), \succcurlyeq_{(\#,\mathbf{P},-)})$.

In particular, we are interested here –as mentioned above– in classifying by their containment-monotonicity properties (and possible preferential content) several ORs that are usually motivated in terms of 'freedom of choice', as well as in assessing the extent to which they are amenable to a fixed-base-flexible IPM-interpretation.

To begin with, we should like to substantiate our foregoing claim to the effect that *the notion of 'flexibility' with respect to a set of preference preorders is indeed general enough to support any opportunity ranking.* Thus, we start from the following

Claim 2 *Let* $(P(X), \succcurlyeq)$ *be an OR. Then there exist a set* $\mathbf{P}$ *of –possibly indexed– total preorders (indeed, chains) on* X, *a function* $f : P(X) \times P(X) \rightarrow P(\mathbf{P})$ *and a partial evaluation function* $v : P(\mathbf{P}) \rightharpoonup \Re$ *such that for any* $A, B \subseteq X$

$A \succcurlyeq B$ *if and only if* $(v \circ f)(A, B) \geq 0$.

Thus, the foregoing Claim makes precise the idea that *any* OR –hence in particular any ranking of opportunity sets in terms of 'freedom of choice'–can be expressed in terms of 'flexibility' with respect to a set of preferences (see e.g.Dennett(1984), which may be regarded as a spirited defence of one version of that view). *Here, by 'more flexible' we simply mean 'better' with respect to a possibly variable i.e.context-dependent set of 'relevant' preferences.* It follows that alternative notions of 'flexibility' rankings obtain according to the way one specifies a) what are the sets of 'relevant' preferences, in particular whether such sets are unit or non-unit and context-dependent or not, and b) what is to be meant by 'better' whenever the set of 'relevant' preferences is non-unit, i.e. what is -or are- the aggregation rule(s) to be used.

It should be emphasized that such a broad notion of 'flexibility' is quite different and considerably *more general* than the 'flexibility' concept most workers in the field have typically in mind, namely *'unanimous' indirect-preference-maximization with respect to a fixed set of preference preorders* or, possibly, some of its extensions to a total OR (see e.g. Kreps(1979), Arrow(1995), Puppe(1996), Nehring,Puppe(1999)).

In order to make precise this more specialized interpretation of 'flexibility' we introduce now the following definition:

Definition 3 *(* Fixed base unanimous indirect-preference-maximizing OR w.r.t. a set $\mathbf{P}$ of total preference preorders on X ($\mathbf{P}$-FBU-IPM*)) . Let* $(P(X), \succcurlyeq)$ *be an OR, and* $\mathbf{P} \subseteq PR(X)$ *a set of total preorders on* X*. Then,* $(P(X), \succcurlyeq)$ *is a* $\mathbf{P}$–fixed-base-unanimous indirect-preference-maximizing OR *iff for any* $A, B \subseteq X : A \succcurlyeq B$ *if [for all* $\gtrsim \in \mathbf{P} : a^* \gtrsim b^*$ *for any* $a^* \in \max_{\gtrsim} A$, $b^* \in \max_{\gtrsim} B$]*. Moreover,* $(P(X), \succcurlyeq)$ *is said to be a* fixed base unanimous indirect-preference-maximizing (FBU-IPM) OR *iff it is a* $\mathbf{P}$-fixed-base unanimous indirect-preference-maximizing *OR w.r.t. some* fixed $\mathbf{P} \subseteq PR(X)$*.In particular, if* $\#\mathbf{P} = 1$ *then a* $\mathbf{P}-IPM$ *OR reduces to an* IPM *OR.*

Remark 4 *Since* X *is finite (by hypothesis), the foregoing notion of FBU–IPM-'flexibility' of an OR is clearly equivalent to existence of a finite state space* S *and a state-dependent utility function* $u : X \times S \to R$ *such that for any* $A, B \subseteq X : A \succcurlyeq B$ *if [for all* $s \in S$ *,* $\max_{x \in A} u(x, s) \geq \max_{x \in B} u(x, s)$].

The following notion due to Kreps(1979) will be much helpful when studying the 'flexibile IPM content' of different ORs.

Definition 5 *(*Dominance relation induced by an OR*) Let* $(P(X), \succcurlyeq)$ *be an OR. Then, the* dominance relation $\succcurlyeq^*$*induced by* $\succcurlyeq$ *is defined as follows : for any* $A, B \subseteq X$*,* $A \succcurlyeq^* B$ *iff* $A \succcurlyeq A \cup B$*.*

As mentioned above, a prominent particular case of a fixed base $\mathbf{P}-$FBU-IPM OR $(P(X), \succcurlyeq)$ obtains whenever $\#\mathbf{P} = 1$. In this case, we say that $(P(X), \succcurlyeq)$ is a *rigid IPM* OR. Clearly enough, if $(P(X), \succcurlyeq)$ is a rigid IPM OR w.r.t. a total preference preorder R on X, then for any $x, y \in X : \{x\} \succcurlyeq \{y\}$ iff xRy. This observation suggests the following criterion of rigid preferential content, which will also be used in order to compare and contrast different ORs (including preferentially-defined and preference-free ORs).

Definition 6 (Rigid (Strict) Preferential Content of an OR) *Let* $(P(X), \succcurlyeq)$ *be an OR. Then, the* rigid preferential content *of* $(P(X), \succcurlyeq)$ *is the binary relation* $(X, \succsim_{\succcurlyeq})$ *where* $\succsim_{\succcurlyeq} = \{(x, y) \in X \times X : \{x\} \succcurlyeq \{y\}\}$*. The* basic strict preferential content of $(P(X), \succcurlyeq)$ *is* $(X, \succ_{\succcurlyeq})$*, the asymmetric component of* $(X, \succsim_{\succcurlyeq})$*.*

The theoretical setting is now ready for the ensuing analysis. As mentioned above, while *any* OR is *in principle* amenable to a 'flexibility' interpretation (see Claim 2 above), one may suspect that the prevailing notion of 'flexibility' as fixed-base-unanimous indirect-preference-maximization is too narrow to accomodate every plausible notion of 'freedom of choice' (or 'diversity'). Thus, the principal aim of this paper is to give some analytical substance to the foregoing simple intuition: this will be accomplished by working on a few concrete examples, namely by defining some relevant ORs (as motivated in terms of 'freedom of choice' or 'diversity') and showing that they are not amenable to a FBU-IPM interpretation. The second aim will consist in comparing and contrasting the ORs under consideration by means of the uncontroversial set-containment monotonicity properties introduced above: indeed, such a simple set of properties will prove to be strong enough to elicit a few significant patterns of the relevant ORs.

Let us start with the following:

Remark 7 *The behaviour of* $FBU-IPM$ *ORs and rigid* IPM *ORs w.r.t. the foregoing properties must be considered here for the sake of comparisons. Indeed, a* **P**-FBF-IPM *OR clearly satisfies* PR, WM, SE,*and its basic preferential content amounts to* $\bigcap_{\gtrsim \in \mathbf{P}} \gtrsim$, *but neither* WSI *nor* WPE *need be –in general–satisfied (recall that preferences in* **P** *may be widely diverse, and may include several maxima). An* IPM *OR satisfies* PR, WM, SE,*while their rigid preferential content obviously corresponds to the underlying (unique) total preference preorder* $\gtrsim$ *on* X*: again, neither* WSI *nor* WPE *need in general hold true, because the case of several maxima is to be taken into account.If, however, the underlying* $\gtrsim$ *is a linear order (i.e. a total antisymmetric transitive relation) then* WPI *(just take* $X' = X \setminus \{\max_{\gtrsim} X\}$*)and* PE *are also satisfied. At the other extreme, it is also worthwhile to consider the containment poset* $(P(X), \supseteq)$ *: clearly enough, it satisfies* PR, WM, PI, PE,*while its rigid preferential content reduces– trivially– to the diagonal* $\{(x, y) : x, y \in X, \text{ and } x = y\}$. *Moreover,* $\supseteq^* = \supseteq$, *hence the dominance relation* $\supseteq^*$*is transitive (we shall see below the significance of this fact with respect to 'flexibility' considerations).*

In our review of ORs motivated in terms of 'freedom of choice', we start from ORs that arise in a natural way whenever a) all the alternatives are (potentially) 'good' -or 'not bad'- as implied by the acceptance principle for X mentioned above, but b) their ultimate significance depends on a threshold effect so that either each alternative is a significant opportunity or none of them is according to the location of the menu w.r.t. the minimum standard (i.e. 'above' or 'below' the standard). Those requirements – which amount to the introduction of some sort of "freedom-poverty line"–can also be regarded as an attempt to accommodate the widespread treatment of "freedom" as a 'yes-or-no' concept while insisting on the idea of many degrees of freedom. We shall rely on the following notion:

Definition 8 *(*Order filter of a preordered set*). Let* $(Y, \gtrsim)$ *be a preordered set.An* order filter *of* $(Y, \gtrsim)$ *is a non-empty set* $Z \subseteq Y$ *such that for any* x, y , $x \in Z$ *and* $y \gtrsim x$ *entail* $y \in Z$.

Notation 9 *Let* $(Y, \gtrsim)$ *a preordered set. We denote by* $\mathcal{F}(Y, \gtrsim)$ *the set of all order filters of* $(Y, \gtrsim)$.

The following specialization of the previous definition will also be used in the sequel:

Definition 10 *(*Principal order filter of a preordered set *) Let* $(Y, \gtrsim)$ *be a preordered set. A* principal order filter *of* $(Y, \gtrsim)$ *is a set* $Z \subseteq Y$ *such that* $Z = \{x \in Y : x \gtrsim y\}$ *for some* $y \in Y$.

Remark 11 *For any order filter* F *of a* $finite$ *preordered set* $(Y, \succsim)$ *there exists a finite set* $G(F) = \{g_1, .., g_k\} \subseteq Y$ *–the set of* $generators$ *of* F*–such that* $i)F = \bigcup_{i=1}^{k} \{z \in Y : z \succsim g_i\}$ *and ii) not* $g_i \succsim g_j$ *for any* $i, j \in \{1, .., k\}$, $i \neq j$*.If* $(Y, \succsim) = (P(X), \supseteq)$ *then each generator* g_i *of an order filter* F *is a set: then, we shall denote by* $G^r(F)$ *the set of generators of* F *having cardinality* r.

The notion of an order filter enables us to formulate in a natural way a special type of opportunity ranking that embodies requirements a) and b) for opportunity rankings as mentioned above (see Suppes(1987), Gekker(1999) for two earlier proposal of similar ORs, and Vannucci(1999) for a fairly detailed analysis). This is made precise by the following definitions :

Definition 12 *(*Filtral opportunity rankings *) A* filtral opportunity ranking (FOR) *is a binary relation* $(P(X), \succcurlyeq)$ *such that for some preordered set* $(P(X), \gtrsim)$*, and some order filter* $F \in \mathcal{F}(P(X), \gtrsim)$: *for any* $A, B \subseteq X$, $A \succcurlyeq B$ *if and only if either* $A \gtrsim B$ *or* $B \notin F$

Definition 13 *(*Set-filtral opportunity ranking*) A* set-filtral opportunity ranking (SFOR) *is a FOR with order filter* $F \in \mathcal{F}(P(X), \supseteq)$.

In words, SFORs replicate the set-containment partial order, except that the empty set and those opportunity sets which do not meet the minimal standard embodied by order filter F are regarded as (pairwise) indifferent.

Notation 14 *A FOR with order filter* F *will also be denoted by* $(P(X), \succcurlyeq_F)$.

What are then the 'freedom of choice' properties, and what the 'preferential' and 'flexibility' contents of a SFOR? This question is answered by the following proposition:

Proposition 15 *Let* $(P(X), \succcurlyeq_F)$ *be a SFOR with order filter* $F \in \mathcal{F}(P(X), \supseteq)$. *Then,*

i) $(P(X), \succcurlyeq_F)$ *satisfies* PR, WM, SI, WPI, WPE *but not* PI *or* SE *unless* $F \supseteq P(X) \setminus \{\emptyset\}$ *i.e.* $(P(X), \succcurlyeq_F) = (P(X), \supseteq)$ *in which case it also clearly satisfies both* PI *and* PE;

ii) $(P(X), \succcurlyeq_F^*) \subseteq (P(X), \succcurlyeq_F)$ *and the* $\succcurlyeq_F$ *–induced dominance relation* $(P(X), \succcurlyeq_F^*)$ *is a* $FBU - IPM$ *iff* $F = P(X)$ *i.e.* $(P(X), \succcurlyeq_F) = (P(X), \supseteq)$ *;*

iii) if $G^1(F) \neq \emptyset$ *then*
$\succsim_{\succcurlyeq_F} = \left\{(y, z) \in X \times X : y = z \text{ or } y \in G^1(F) \text{ and } z \notin G^1(F)\right\}$ *and*
$\succ_{\succcurlyeq_F} = \left\{(y, z) \in X \times X : y \in G^1(F) \text{ , } z \notin G^1(F()\right\}$;
if $G^1(F) = \emptyset$ *then* $\succsim_{\succcurlyeq_F} = X \times X$ *and* $\succ_{\succcurlyeq_F} = \emptyset$.

Remark 16 *It should be noticed that for any* $F \in F(P(X), \supseteq)$, $\succcurlyeq_F^* \subseteq \succcurlyeq_F$. *Indeed, let* $A \succcurlyeq_F^* B$. *Then, either* $A \supseteq A \cup B$ *i.e.* $A \supseteq B$, *or* $A \cup B \notin F$ *(whence* $B \notin F$ *): in either case, it follows that* $A \succcurlyeq_F B$.

Corollary 17 *Let* $(P(X), \succcurlyeq_F)$ *be a SFOR with order filter F. Then* $(P(X), \succcurlyeq_F)$ *is a fixed-base unanimous* IPM OR *iff* $F = P(X)$*,i.e.*$(P(X), \succcurlyeq_F) = (P(X), \supseteq)$.

Proof. Straightforward from the foregoing Proposition and Remark , by noticing that $(P(X), \succcurlyeq_{P(X)}^*) = (P(X), \succcurlyeq_{P(X)})$. $\square$

The foregoing Proposition and Corollary establish that set-filtral ORs provide a first class of examples of plausible rankings of opportunity sets in terms of 'freedom of choice' which cannot possibly be interpreted as fixed-base unanimous IPM ORs, except for the trivial case where $F = P(X)$ i.e. when $(P(X), \succcurlyeq_F = (P(X), \supseteq)$. On the other hand, our Claim 2 above entails that SFORs are representable in terms of general 'flexibility' with respect to a context-dependent set of total preference preorders. One possible 'flexibility' interpretation w.r.t. $P(X) \times P(X)$-indexed total preorders is as follows: for any order filter F, and $A, B \subseteq X$ define

$$f_F(A,B) = \left\{ \begin{array}{c} \{(\succeq_{(x,\geq_x^i)}, (A,B)) : x \in A \cup B, \ \geq_x^i \in \mathbf{C}(X \setminus \{x\})\} \\ \text{if } A \neq B \text{ and } \{A,B\} \cap F \neq \emptyset \text{, and} \\ \emptyset \text{ if either } A = B \text{ or } \{A,B\} \cap F = \emptyset \end{array} \right\}$$

where for any $y, z \in X$, and $\geq_x^i \in \mathbf{C}(X \setminus \{x\})$, $y \succeq_{(x,\geq_x^i)} z$ iff either $y = x$ or $y \geq_x^i z$. It is easily checked that the restriction of f_F to $(P(X) \times F) \bigcup (F \times P(X)) \setminus \{(A,B) \in P(X) \times P(X) : A = B\}$ is an injection.

Then, define $[f_F(A,B)]_1 = \{\succeq \in \mathbf{C}(X) : (\succeq, (A,B)) \in f_F(A,B)\}$, and posit

$$v(f_F(A,B)) = \left\{ \begin{array}{c} 1 \text{ if } [(\max_{\succeq_{(x,\geq_x^i)}} A \succeq_{(x,\geq_x^i)} \max_{\succeq_{(x,\geq_x^i)}} B) \\ \text{for all } \succeq_{(x,\geq_x^i)} \in [f_F(A,B)]_1 \neq \emptyset \\ \text{and } not(\max_{\succeq_{(x,\geq_x^i)}} B \succeq_{(x,\geq_x^i)} \max_{\succeq_{(x,\geq_x^i)}} A) \\ \text{for some } \succeq_{(x,\geq_x^i)} \in [f_F(A,B)]_1 \neq \emptyset] \\ -1 \text{ if } [(\max_{\succeq_{(x,\geq_x^i)}} B \succeq_{(x,\geq_x^i)} \max_{\succeq_{(x,\geq_x^i)}} A) \\ \text{for all } \succeq_{(x,\geq_x^i)} \in [f_F(A,B)]_1 \neq \emptyset \\ \text{and } not(\max_{\succeq_{(x,\geq_x^i)}} A \succeq_{(x,\geq_x^i)} \max_{\succeq_{(x,\geq_x^i)}} B) \\ \text{for some } \succeq_{(x,\geq_x^i)} \in [f_F(A,B)]_1 \neq \emptyset] \\ 0 \text{ if } f_F(A,B) = \emptyset \text{ and} \\ \text{undefined otherwise} \end{array} \right\}.$$

It is then easily checked that for any $A, B \subseteq X$, $v(f_F(A,B)) \geq 0$ iff $A \succcurlyeq_F B$.

It should be emphasized that the foregoing 'flexibility' representation of SFORs retains the IPM criterion while using *variable context-dependent* (non-singleton) sets of total preference preorders. It should also be remarked that set-filtral ORs perform reasonably well with respect to improvability properties, and less so with respect to essentiality properties.

Let us turn now to a total extension of a set-filtral opportunity ranking as first introduced and discussed – in a specialized version– by Suppes(1987):

Definition 18 *(*Total set-filtral opportunity ranking*) A* total set-filtral opportunity ranking *with order filter F is a binary relation $(P(X), \succcurlyeq_F^t)$ such that for any $A, B \subseteq X$: $A \succcurlyeq_F^t B$ if and only if either $A \in F$ or $B \notin F$.*

Proposition 19 *Let $(P(X), \succcurlyeq_F^t)$ be a total SFOR with order filter F as defined above. Then,*

i) $\succcurlyeq_F$ is indeed a total preorder that satisfies WM, but neither PI nor PE. Also, $\succcurlyeq_F^t$ satisfies WSI, WPI, and WPE iff $\#G^{\#X-1}(F) < \#X$, SI iff $F = \{X\}$, WSE iff $F \neq P(X)$, WPE iff F is such that $\max\{r : G^r(F) \neq \emptyset\} \geq \#X - 1$, SE iff $F = P(X) \setminus \{\emptyset\}$;

ii) $\succcurlyeq_F^{*t} \subseteq \succcurlyeq_F^t$, and $(P(X), \succcurlyeq_F^{*t})$ is not a fixed-base unanimous IPM OR;

iii) $\succsim_{\succcurlyeq_F^t} = \{(x,y) \in X \times X : x \in G^1(F)$ or $y \notin G^1(F)\}$, hence $\succ_{\succcurlyeq_F^t} = \{(x,y) \in X \times X : x \in G^1(F)$ and $y \notin G^1(F)\}$.

Thus, total set-filtral ORs provide a second example of a class of 'freedom of choice'-motivated ORs which are generally not amenable to a fixed-base-unanimous IPM interpretation. Moreover, total set-filtral ORs perform quite poorly with respect to both improvability and essentiality properties.

Remark 20 *Again, a* variable-base unanimous IPM *representation of a total SFOR $(P(X), \succcurlyeq_F^t)$ may be provided along the following lines. For any $A, B \subseteq X$, define:*

$$f_F^t(A,B) = \left\{ \begin{array}{c} \{(\geq_{X \setminus B} \oplus \geq_B)\} \text{ for some } \geq_{X\setminus B} \in \mathbf{C}(X \setminus B), \geq_B \in \mathbf{C}(B) \\ \text{if } A \in F \text{ and } B \notin F \\ \{(\geq_{X \setminus A} \oplus \geq_A)\} \text{ for some } \geq_{X\setminus A} \in \mathbf{C}(X \setminus A), \geq_A \in \mathbf{C}(A) \\ \text{if } B \in F \text{ and } A \notin F \\ \{[X, \sim]\} \text{ if } \{A,B\} \subseteq F \\ \emptyset \text{ if } \{A,B\} \cap F = \emptyset \end{array} \right\}$$

where $[X, \sim]$ denotes the universally indifferent preorder i.e. $\sim = X \times X$, and $\oplus$ denotes the ordinal or linear sum for preorders.

Then, posit

$$v(f_F^t(A,B)) = \left\{ \begin{array}{c} 1 \text{ if } \max_{\succsim} A \succsim \max_{\succsim} B \text{ for all } \succsim \in f_F^t(A,B) \text{ and} \\ not(\max_{\succsim} B \succsim \max_{\succsim} A \text{ for all } \succsim \in f_F^t(A,B) \\ -1 \text{ if } \max_{\succsim} B \succsim \max_{\succsim} A \text{ for all } \succsim \in f_F^t(A,B) \text{ and} \\ not(\max_{\succcurlyeq} A \succcurlyeq \max_{\succcurlyeq} B \text{ for all } \succsim \in f_F^t(A,B) \\ 0 \text{ if } \max_{\succsim} A \succsim \max_{\succsim} B \text{ for all } \succsim \in f_F^t(A,B) \text{ and} \\ \max_{\succsim} B \succsim \max_{\succsim} A \text{ for all } \succsim \in f_F^t(A,B) \end{array} \right\}.$$

It follows from the foregoing definitions that for any $A, B \subseteq X$, $v(f_F^t(A,B)) \geq 0$ iff $A \succcurlyeq_F^t B$.

Next, let us consider for the sake of comparisons the cardinality-based (total) opportunity ranking as defined below:

Definition 21 *(*Cardinality-based opportunity ranking*) The* cardinality-based *OR $(P(X), \succcurlyeq_C)$ is defined by the following rule: for any $A, B \subseteq X$, $A \succcurlyeq_\# B$ iff $\#A \geq \#B$.*

Claim 22 *Let* $(P(X), \succcurlyeq_{\#})$ *the cardinality- based OR. Then,*
i) $(P(X), \succcurlyeq_{\#})$ *is a total preorder and satisfies* WM, PI, PE*;*
ii) $(P(X), \succcurlyeq_{\#}) = (P(X), \supseteq)$ *is a fixed-base unanimous* IPM *OR ;*
iii) $\succsim_{\succcurlyeq_{\#}} = X \times X$ *, and* $\succ_{\succcurlyeq_{\#}} = \emptyset$ *.*

Hence, as mentioned above, the cardinality-based OR is perfectly well-behaved with respect to our improvability and essentiality properties, is devoid of non-trivial rigid preferential content, and its dominance subrelation is amenable to a fixed-base-unanimous IPM interpretation.

We proceed now to consider a somewhat hybrid class of opportunity rankings, where the usual motivation in terms of 'freedom of choice' is further qualified by considerations concerning outcome-range (or 'diversity'). This move requires, however, some reference to the underlying structure of X . Most typically, a metric on X is invoked(see e.g. Klemisch-Ahlert(1993) who works with a Euclidean outcome space). Here, we pursue a quite different route to diversity-modelling, namely we simply require the outcome set to be a partially ordered set or *poset* $(X, \geq)$ (i.e. $\geq$ is a transitive, reflexive, antisymmetric binary relation on X). We are not going to push forward any particular interpretation of $(X, \geq)$, but – clearly enough– , such a poset is meant to represent a 'physical' as opposed to a 'preferential' structure (e.g. think of X as a set of points in a multiattribute space where attribute-values are chains, $\geq$ is the meet of those chains, and (at least) some of the relevant preferences exhibit non $\geq$-extremal bliss points in X, hence are *not* $\geq$-monotonic). It is easily checked that a poset structure is indeed rich enough to support some significant notions of outcome-diversity as made precise by the following definitions:

Definition 23 *(*Convex hulls in a poset*). Let* $(X, \geq)$ *be a poset. Moreover, for any* $x, y \in X$ *let us define*
$[x, y] = \max_{\supseteq} \{\{z \in X : x \geq y \geq z\}, \{z \in X : y \geq z \geq x\}, \{x, y\}\}$.
Then, for any $A \subseteq X$ *the* convex hull $co_{\geq}(A)$ *of* A *w.r.t.* $\geq$ *is defined by the following rule :*
$co_{\geq}(A) = \bigcap \{B \subseteq X : B \supseteq A \text{ and } B \supseteq [x, y] \text{ for any } x, y \in B\}$.

Notation 24 *We posit* $Co(X, \geq) = \{A \subseteq X : A = co_{\geq}(B) \text{ for some } B \subseteq X\}$. *Moreover, for any* $A \subseteq X$ *we denote the set of* $\geq -$*extreme points of* A *by* $E_{\geq}(A) = E^{+}_{\geq}(A) \cup E^{-}_{\geq}(A)$ *where*
$E^{+}_{\geq}(A) = \{x \in A : y > x \text{ for no } y \in A\}$
$E^{-}_{\geq}(A) = \{x \in A : x > y \text{ for no } y \in A\}$.

Definition 25 *(*Convex-hull-based opportunity ranking*). Let* $(X, \geq)$ *be a poset. Then, the* convex-hull-based *OR –or* $\geq -$*convex OR–*$(P(X), \succcurlyeq_{co_{\geq}})$ *is defined by the following rule : for any* $A, B \subseteq X$ *,*
$A \succcurlyeq_{co_{\geq}} B$ *iff* $co_{\geq}(A) \supseteq co_{\geq}(B)$.

A set-filtral version of convex-hull-based opportunity rankings may also be defined, namely:

Definition 26 *(*Convex-hull-based set-filtral opportunity ranking*) Let* $(X, \geq)$ *a poset, and* F *an order filter of* $(Co(X, \geq), \supseteq)$*. Then, the* convex-hull-based set-filtral *OR* $(P(X), \succcurlyeq_{(co_{\geq}, F)})$*with order filter* F *(or* $set-filtral \geq -convex$ *OR– is defined as follows: for any* $A, B \subseteq X$*,* $A \succcurlyeq_{(co_{\geq}, F)} B$ *iff either* $co_{\geq}(A) \supseteq co_{\geq}(B)$ *or* $co_{\geq}B \notin F$*.*

The relevant monotonicity and flexibility properties of $(P(X), \succcurlyeq_{co_{\geq}})$ and $(P(X), \succcurlyeq_{(co_{\geq}, F)})$ are summarized by the following propositions:

Proposition 27 *Let* $(X, \geq)$ *be a (finite) poset , and* $(P(X), \succcurlyeq_{co_{\geq}})$ *the* $\geq -convex$ *OR. Then,*

i) $(P(X), \succcurlyeq_{co_{\geq}})$ *satisfies* PR, WM, WPI, PE *but not* SI *whenever* $(X, \geq)$ *includes a chain of size* $s \geq 3$*;*

ii) $\succcurlyeq^{*}_{co_{\geq}} = \succcurlyeq_{co_{\geq}}$*, and* $(P(X), \succcurlyeq^{*}_{co_{\geq}})$ *is a* $fixed-base-unanimous$ IPM *OR ;*

iii)$\succsim_{\succcurlyeq co_{\geq}} = \{(x, y) \in X \times X : x = y\}$ *, hence* $\succ_{\succcurlyeq co_{\geq}} = \emptyset$ *.*

Thus, a $\geq$-convex OR shares a considerable amount of 'nice' monotonicity properties with the set-containment poset, is devoid of any non-trivial rigid preferential content and is amenable to a fixed-base-flexible IPM interpretation. Hence, $\geq$-convex ORs are in that respect a remarkable exception among the 'freedom of choice'-ORs we consider in this paper. Those somehow 'distinctive' properties are however immediately lost if we move to a 'filtral' version of $\geq$-convex ORs.

Proposition 28 *Let* $(X, \geq)$ *be a (finite) poset, and* $(P(X), \succcurlyeq_{(co_{\geq}, F)})$ *the* $set-filtral \geq -convex$ *OR with order filter* F*. Then,*

i) $(P(X), \succcurlyeq_{(co_{\geq}, F)})$ *satisfies* PR, WM, WPI, WPE *;* SI *holds iff* $(X, \geq)$ *does not include a chain of size* $s \geq 3$*;* PI *holds iff (* $(X, \geq)$ *does not include a chain of size* $s \geq 3$ *and there exists* $x \in X$ *such that* $\{x\} \in F$*);* SE *holds iff* $F \supseteq \{\{x\} : x \in F\}$*;* PE *holds iff* $F \supseteq \{\{x\} : x \in X\}$ *and* $(X, \geq)$ *does not include a chain of size* $s \geq 3$*;*

ii) $\succcurlyeq^{*}_{(co_{\geq}, F)} \subseteq \succcurlyeq_{(co_{\geq}, F)}$*, and* $(P(X), \succcurlyeq_{(co_{\geq}, F)})$ *is not a* $fixed-base$ $unanimous$ IPM *OR if there exist* $A, B, C \subseteq X$ *such that* $co_{\geq}(A) \nsupseteq co_{\geq}(C), co_{\geq}(A \cup B) \notin F, co_{\geq}(B \cup C) \notin F$ *and* $co_{\geq}(A \cup C) \in F$*;*

iii) $\succsim_{\succcurlyeq (co_{\geq}, F)} = \{(x, y) \in X \times X : x = y$ *or* $\{y\} \notin F\}$ *hence* $\succ_{\succcurlyeq (co_{\geq}, F)} = \{(x, y) \in X \times X : \{x\} \in F$ *and* $\{y\} \notin F\}$*.*

Remark 29 *For any order filter* $F \in \mathcal{F}(P(X), \supseteq)$ *a* (context-dependent) variable-base unanimous IPM *representation of* $(P(X), \succcurlyeq_{(co \geq, F)})$ *may be defined along the same lines as for* $(P(X), \succcurlyeq_{F})$*, provided that* $A \cup B$ *is consistently replaced with* $E_{\succcurlyeq}(A \cup B)$ *in the relevant definitions.*

A total OR based on another order-theoretic notion of outcome-diversity can be defined as follows:

Definition 30 *(*Width-based opportunity ranking*). Let* $(X, \geq)$ *be a poset. Then, the* $\geq$ $-$width-based *OR* $(P(X), \succcurlyeq_{w(\geq)})$ *is defined as follows: for any* $A, B \subseteq X$, $A \succcurlyeq_{w(\geq)} B$ *iff* $w(\geq)(A) \geqslant w(\geq)(B)$ *(where for any* $C \subseteq X$,
$w(\geq)(C) = \max_{D \subseteq C} \{\#\{D : x \ngeq y \text{ and } y \ngeq x \text{ for any } x, y \in D \text{ such that } x \neq y\}\}$*).*

Remark 31 *It should be emphasized that if* $(X, \geq)$ *is an antichain or equivalently* $\geq = \{(x, y) \in X : x = y\}$ *, then for any* $A \subseteq X$ *,* $w(\geq)(A) = \#A$ *hence* $(P(X), \succcurlyeq_{w(\geq)}) = (P(X), \succcurlyeq_{\#})$*.*

Proposition 32 *Let* $(X, \geq)$ *be a poset, and* $(P(X), \succcurlyeq_{w(\geq)})$ *the* $\geq$ $-$width-based *OR. Then,*

i) $(P(X), \succcurlyeq_{w(\geq)})$ *is a* total preorder *that satisfies* WM *and* SE; WSI, WPI *and* WPE *hold iff* $(X, \geq)$ *has a unique antichain* A^* *of maximum size ;* SI, PI *and* PE *hold iff* $(X, \geq)$ *is an antichain (i.e. when* $(P(X), \succcurlyeq_{w(\geq)}) = (P(X), \succcurlyeq_{\#})$*);*

ii) $\succcurlyeq^*_{w(\geq)} \subseteq \succcurlyeq_{w(\geq)}$ *and* $(P(X), \succcurlyeq^*_{w(\geq)})$ *is a* $fixed-base-unanimous\ IPM$ *OR if* $(X, \geq)$ *is a chain or an antichain, but need not be in the general case;*

iii) $\succsim_{\succcurlyeq_{w(\geq)}} = X \times X$ *hence* $\succ_{\succcurlyeq_{w(\geq)}} = \emptyset$ *.*

Remark 33 *A* variable-base non-unanimous-IPM *representation of* $(P(X), \succcurlyeq_{w(\geq)})$ *w.r.t. to* $P(X)$*-indexed chains may be defined as follows: for any* $Y \subseteq X$, *let* $\left\{\geq_{Y_1}, .., \geq_{Y_{k(Y)}}\right\} \in \mathbf{C}(Y)$ *be a minimum-size chain union decomposition of* $(Y, \geq_Y)$ *i.e. a minimum-cardinality-set of chains on some sets* $Y_i \subseteq Y$ *such that* $\geq_Y = \bigcup_{i=1}^{k(Y)} \geq_{Y_i}$ *(where* $\geq_Y = \geq \cap (Y \times Y)$*) , and posit –for any* $A, B \subseteq X$:

$$f_{(w(\geq))}(A, B) = \left\{ \begin{array}{c} (\geq_{A_1} \oplus \geq_{(X \setminus A_1)}, A), .., (\geq_{A_{k(A)}} \oplus \geq_{(X \setminus A_{k(A)})}, A), \\ (\geq_{B_1} \oplus \geq_{(X \setminus B_1)}, B), .., (\geq_{B_{k(B)}} \oplus \geq_{(X \setminus B_{k(B)})}, B) \end{array} \right\}, \text{ where}$$

$\geq_{(X \setminus A_i)} \in \mathbf{C}(X \setminus A_i)$, $\geq_{(X \setminus B_i)} \in \mathbf{C}(X \setminus B_i)$ *and* $\oplus$ *denotes the ordinal or linear sum operation for preorders. Then, for any* $A, B \subseteq X$ *denote*

$c(A : (A, B)) = \#\left\{(\succeq, C) \in f_{(w(\geq))}(A, B) : C = A\right\} = k(A)$,
$c(B : (A, B)) = \#\left\{(\succeq, C) \in f_{(w(\geq))}(A, B) : C = B\right\} = k(B)$, *and define*

$$v(f_{w(\geq)}(A, B)) = \left\{ \begin{array}{c} 1 \text{ if } c(A : (A, B)) > c(B : (A, B) \\ -1 \text{ if } c(B : (A, B)) > c(A : (A, B)) \\ 0 \text{ if } c(A : (A, B)) = c(B : (A, B)) \end{array} \right\}.$$

Now, consider the well-known Dilworth decomposition theorem for posets which establises that the width of a finite poset equals the cardinality of its minimum-size chain (union) decomposition (see e.g. Anderson(1987)). It is then immediately checked that for any $A, B \subseteq X$, $v(f_{w(\geq)}(A, B)) \geq 0$ *iff* $A \succcurlyeq_{w(\geq)} B$*.*

Thus, it is confirmed that the width-based OR –while satisfying the set-essentiality (SE) property– is also not amenable in general to a fixed-base-unanimous IPM representation (and embodies a trivial rigid preferential content). The exceptions only occur in trivial cases, i.e. when $(X, \geq)$ is a chain and the width-based total preorder collapses to universal indifference, or when $(X, \geq)$ is an antichain and the width-based OR reduces to the cardinality total preorder.

Moreover, it should be emphasized that the variable-base flexible representation of $(P(X), \succsim_{w(\geq)})$ which has been presented above is *not of the 'unanimous' IPM type* (as opposed to the other variable-base flexible representations previously considered). This circumstance apparently suggests that the width-based OR is indeed quite different from the other ORs we have discussed in the present paper.

3 Related literature

Since the focus of this paper is on alternative rankings of opportunity sets according to several notions of 'freedom of choice', 'flexibility' and 'diversity'– and on the relationships among them– the amount of related literature is simply enormous. We can only attempt to provide a very succinct review of such work in order to substantiate our earlier remarks concerning what is –and what is not– usually offered in the previous literature under those general labels.

As for 'freedom of choice', the relevant literature on opportunity rankings essentially originates with Sen (1985,1988) and Suppes (1987). Indeed, while Sen's seminal contributions have inspired the well-known characterization of the cardinality-based preorder due to Pattanaik,Xu(1990) and the subsequent contributions on opportunity rankings as previously discussed in the Introduction, Suppes's work is the original source of a distinct strand of literature which is mainly focused on numerical (ratio scale) representations of opportunity rankings (see e.g. Gravel,Laslier,Trannoy (1998)).

The literature on opportunity rankings based upon notions of 'diversity' is even more sparse. There is indeed a wide literature on measures of 'diversity' in physics, biology, statistics. One early approach only concerns communities with a fixed number of species, proceeds to normalize the numerosity of the latter in terms of biomass units, and provides a 'diversity' ranking by relying on the Lorenz -or majorization- partial order (see e.g. Solomon (1979)). More recently other classes of 'diversity' measures –which typically rely on a suitable metric as defined on the underlying outcome space- have been introduced in environmental economics. In particular, Weitzman (1992) and Solow, Polasky, Broadus (1993) have recently proposed two closely related metric-based classes of biodiversity measures which rank ecological communities in terms of number of species and numerosity of each of them. In a more abstract vein, Van Hees (1999) considers several requirements for metric-based 'diversity' rankings of opportunity sets and provides a few 'impossibility' results.

By contrast, the literature on 'flexibility' has a long and quite complex tradition whose roots are mostly in economic theory. A special emphasis on the role of preference for flexibility as revealed by option values –under the label 'speculative liquidity preference'– may be dated back at least to Keynes. The notion of preference for flexibility was subsequently taken up and developed within several different models by Marschak (1938), Hart (1942), Marschak,Nelson (1962), Koopmans (1964), Kreps (1979, 1992), Jones,Ostroy (1984) among others. Moreover, the notion of 'flexibility' is arguably closely related to existence of ambiguity and irreversibility. Indeed, intertemporal flexibility encompasses the notion that a good current decision is sometimes a choice which permits good later responses to later observations (see e.g. Arrow,Fisher (1974), Henry (1974) for early developments of those ideas in the context of environmental economics, and Pindyck (1991) for further refinements and applications within standard models of investment decisions).

The first full-fledged axiomatization of a 'flexibility'-motivated (total) opportunity preorder is however due to Kreps (1979), who builds upon Koopmans (1964). The basic idea is that if an agent is uncertain about her future preferences then she exhibits a preference for flexibility, which can be represented by a state-dependent utility function. Thus, each state or relevant contingency 'activates'(indeed, corresponds to) *one* relevant total preference preorder : an opportunity set A is ranked above another one –say, B– if and only if at any state (i.e. according to each relevant preference preorder) its local maxima are not worse than B's, and there exists a state s^* such that the local maxima of A are strictly better than B's. Clearly enough, all this amounts to a dominance– or unanimity– principle with respect to the state space-indexed set of relevant total preference preorders: Kreps's total opportunity preorder is just an extension of such a dominance partial preorder. Moreover, Kreps (1992) advances the idea that this model of 'preference for flexibility' is best suited to the case where the state-space corresponds to the set of contingencies the agent is able to explicitly foresee, while being at the same time perfectly aware that there are more –*unforeseen*– relevant contingencies. Nehring (1999) further develops Kreps's approach within a full-fledged Savage-style framework, while Arlegi (1999) provides an alternative characterization of Kreps's 'flexibility'-motivated total opportunity preorder, relying on the notion of consistency with respect to a single asymmetric preference relation on X. Arrow (1995) – while downplaying the significance of an axiomatic approach to 'preference for flexibility'– does indeed equate 'freedom of choice' to 'flexibility' and endorses an amended version of Kreps's state-dependent indirect-preference-maximization model: by imposing a probability space structure on the state space, he defines an expected indirect utility index which extends Kreps's dominance partial preorder as described above.

Pursuing another line of reasoning which denies exclusive prominence to *future* contingent utilities, many authors have reached a considerable degree of consensus on the notion that 'freedom of choice'-motivated opportunity rankings can–and should– be represented in terms of indirect preference maximization with reference to a set of possible or plausible preference preorders on X (see e.g. Jones,Sugden(1982), Foster (1993), Pattanaik,Xu(1998), Sugden (1998), Puppe (1998)). In particular, Foster (1993) defines an opportunity ranking in terms of what he calls 'effective freedom' by applying the (unanimity) dominance principle to a suitable class of possible preference preorders on X. By introducing in a suitably general way their multi-preference approach to evaluating opportunities, Nehring,Puppe (1999) effectively show a tight connection between those 'unanimity'-based rankings and 'flexibility' rankings of the sort proposed and axiomatized by Kreps, in that the latter reduces to a total extension of the former . In fact, those two types of rankings do essentially coincide when the domain of opportunity sets to be considered reduces – as indeed proposed by Nehring,Puppe (1999)– to the set of containment-comparable subsets of X. As repeatedly mentioned above in the previous sections, such opportunity rankings share a heavy reliance on the criterion of *unanimity of indirect preference maximization with respect to a fixed base of preference preorders on* X.

A final remark concerning existence of multi-preference representations of any OR as stated by our previous Claim 2 is in order here. Due to the basic and elementary nature of such a result, we are inclined to think that some version of it might well be already known, perhaps in disguised form. As a matter of fact, we have not been able to trace any such proposition in the extant literature. To be sure, our partial evaluation functions as defined in Claim 2 amount to a partial version of 'comparison functions' as introduced in Dutta, Laslier(1999) (where they are put to a quite different use). Moreover, Claim 2 bears some resemblance to the well-known McGarvey's theorem on tournaments which establishes that for any tournament (i.e. a quasi-total asymmetric binary relation) there exist a population of voters and a profile of linear preference orders such that the given tournament is the resulting majority voting tournament (see McGarvey(1953)). However, it should be emphasized that McGarvey's theorem refers to preorders and tournaments which are defined on the same set, while our Claim 2 establishes a connection between preorders on a set and binary relations on its *power* set.

4 Concluding remarks

While tentative and preliminary, the comparative analysis of 'freedom of choice'-motivated ORs offered in the present paper enables us to establish two main points, namely:

• Unanimous indirect-preference-maximization with respect to a *fixed base* set of preferences is too narrow a specialization of the general notion of flexibility w.r.t. a set of preferences to accommodate every plausible opportunity ranking. In fact, if the foregoing link between the 'flexibility' interpretation of opportunity rankings and unanimous indirect-preference-maximization w.r.t. a fixed set of preferences is removed, then several new interesting ORs emerge.

• Similarities and differences among those 'new' and unconventional ORs are effectively–if partially– elicited by the set of uncontroversial containment monotonicity properties introduced and discussed above as well as by their basic preferential content. In particular, we have shown by example that some non FBU-IPM ORs are indeed consistent with the IPM-unanimity principle, but require a *variable-and typically non-unit-* set of preferences, while others (e.g. the width-based ORs) *might* be inconsistent with *the IPM-unanimity principle* *as such.*

It remains to be seen if and how our elementary base of monotonicity properties could be possibly augmented in order to obtain some simple characterizations of the ORs discussed in the present paper. This is however best left as a topic for further research.

5 Appendix: Proofs

Proof of Claim 2. First, posit $\#X = k$, denote by $\mathbf{R}(X)$ ($\mathbf{C}(X)$) the sets of all total preorders (chains) on X, and notice that $\#\mathbf{R}(X) > \#\mathbf{C}(X)$ and $\#(P(X) \times P(X)) = 2^{2k}$, while $\#(P(\mathbf{C}(X))) = 2^{k!}$. Since clearly $k! \geq 2k$ for $k \geq 3$ (indeed, $k! > 2k$ for $k \geq 4$), an injection $f : P(X) \times P(X) \rightarrow P(\mathbf{C}(X))$ can be defined.

Then posit
$\mathbf{P} = \{R \in \mathbf{C}(X) : R \in f(A, B) \text{ for some } (A, B) \in P(X) \times P(X)\}$,
and for any $\mathbf{L} \in P(\mathbf{P})$, define

$$v(\mathbf{L}) = \left\{ \begin{array}{c} 1 \text{ if there exists } (A,B) \in P(X) \times P(X) \text{ such that} \\ \mathbf{L} = f(A,B) \text{ and } A \succ B \\ 0 \text{ if there exists } (A,B) \in P(X) \times P(X) \text{ such that} \\ \mathbf{L} = f(A,B) \text{ and } A \sim B \\ -1 \text{ if there exists } (A,B) \in P(X) \times P(X) \text{ such that} \\ \mathbf{L} = f(A,B) \text{ and } B \succ A \\ \text{and undefined otherwise} \end{array} \right\}$$

Hence, for any $A, B \subseteq X$, $(v \circ f)(A, B) \geq 0$ iff $A \succcurlyeq B$. The case of indexed total preorders or chains follows as an immediate corollary. □

Proof of Proposition 15. i) Weak monotonicity is obvious: indeed, for any $A, B \in P(X)$, $A \cup B \supseteq A$ whence $A \cup B \succcurlyeq_F A$. But then, reflexivity of $\succcurlyeq_F$is also obvious. To check transitivity, consider $A, B, C \subseteq X$ such that $A \succcurlyeq_F B$ and $B \succcurlyeq_F C$. Four cases are to be distinguished: a) $A \supseteq B, B \supseteq C$:here, $A \succcurlyeq_F C$ is a trivial consequence of transitivity of $\supseteq$; b) $A \supseteq B, C \notin F$, c)$B \notin F, C \notin F$: under cases b) and c) $A \succcurlyeq_F C$ follows immediately from the definition; d) $B \notin F, B \supseteq C$: here, $C \notin F$ by definition of order filter, whence $A \succcurlyeq_F C$ (by definition of $\succcurlyeq_F$). To check SI, take any $A \subset X$, and $B = X \backslash A$. Clearly, $X \in F$, and $X = A \cup B \supset A$. Therefore, $A \cup B \succcurlyeq_F A$ and not $A \succcurlyeq_F A \cup B$. To check WPI, take $X' \in \max_{\supseteq} \{B \subseteq X : B \notin F\}$ i.e. for any $B \subseteq X'$,$C \supset X'$, $B \notin F$ and $C \in F$. Then, for any A such that $X' \subseteq A \subset X$, and any $x \in X \backslash A$, $A \cup \{x\} \in F$ whence $A \cup \{x\} \succ_F A$. To check WSE, take again any $X' \in \max_{\supseteq} \{B \subseteq X : B \notin F\}$, $A \supset X'$and $x \in A \backslash X'$. By definition, $A \in F$: hence $A \supset A \backslash \{x\}$ implies $A \succ_F A \backslash \{x\}$(whether $A \backslash \{x\} \in F$ or not). Concerning PI, take any order filter F of $(P(X), \supseteq)$, $F \subset P(X) \backslash \{\emptyset\}$, and any $A \subset X$ such that $\emptyset \neq A \notin F$. Then $A \backslash \{x\} \sim_F A$ for any $x \in A$. Similarly, if $F \subset P(X) \backslash \{\emptyset\}$ and $\emptyset \neq A \notin F$, then in particular $B \succcurlyeq_F A$ for all $B \subseteq A$, hence SE is not satisfied. Conversely, if $F \supseteq P(X)\backslash\{\emptyset\}$ then $(P(X), \succcurlyeq_F) = (P(X), \supseteq)$ hence both PI and PE are trivially satisfied .

ii) For any $A, B \subseteq X$, if $A \succcurlyeq_F^* B$ then by definition either $A \supseteq A \cup B$ or $A \cup B \notin F$. If $A \supseteq A \cup B$ then obviously $A \supseteq B$ hence by monotonicity $A \succcurlyeq_F B$. If $A \cup B \notin F$ then of course $B \notin F$ whence again $A \succcurlyeq_F B$. Moreover, a well-known result due to Puppe (see Puppe(1996), lemma 5) establishes that for any $(P(X), \succcurlyeq)$,the $\succcurlyeq$ −induced dominance relation $(P(X), \succcurlyeq^*)$ is a fixed-base flexible IPM OR if and only if $\succcurlyeq^*$is transitive. Thus, our proof reduces to showing that for any $F \in \digamma(P(X), \supseteq)$, $(P(X), \succcurlyeq_F^*)$ is transitive iff $F = P(X)$. Indeed, consider the following counterexample. Let $F = \bigcup_{i=1}^k \{A \subseteq X : A \supseteq B_i\}$ where $B_i \subseteq X, i = 1,..,k$. Then, take $B = \emptyset$, $\emptyset \neq A = B_1 \neq X$, and C such that $C\backslash A \neq \emptyset, (X\backslash C) \cap B_i \neq \emptyset$ for any $i = 1,..,k$. Notice that the only implied restriction on F is that $F \notin \{\{X\}, P(X)\}$ .Hence, clearly enough, $A \supseteq A \cup B, B \cup C \notin F, A \cup C \in F$, and $A \not\supseteq C$, i.e. $A \succcurlyeq_F^* B, B \succcurlyeq_F^* C$ and *not* $A \succcurlyeq_F^* C$. Next, consider the case $F = \{X\}$, and posit $A = X\backslash\{x\}$ for some $x \in X$ (recall that $\#X \geq 2$ by assumption),$B = \emptyset$, and $C = \{x\}$. Here, again, $A \succcurlyeq_F^* B$ (because $A \supseteq B$), $B \succcurlyeq_F^* C$ (because $B \cup C = C \notin F$), and *not* $(A \succcurlyeq_F^* C)$ (because $A \not\supseteq C$ and $A \cup C = X \in F$). Conversely, if $F = P(X)$ then $A \succcurlyeq_F^* B, B \succcurlyeq_F^* C$ necessarily imply $A \supseteq B$ and $B \supseteq C$, whence $A \supseteq C$ and therefore $A \succcurlyeq_F^* C$.

iii) Let F be such that $G^1(F) = \{x_1, .., x_k\} \neq \emptyset$. Then, for any $x \in \{x_1, .., x_k\}$ and any $y \in X\backslash\{x_1, ., x_k\}$, $\{x\} \in F$ and $\{y\} \notin F$, whence –by definition of $\succcurlyeq_F$– $\{x\} \succcurlyeq_F \{x\}, \{x\} \succcurlyeq_F \{y\}$ and *not* $\{y\} \succcurlyeq_F \{x\}$. By contrast, if $G^1(F) = \emptyset$ then $\{x\} \notin F$ for any $x \in X$, hence $\{y\} \succcurlyeq_F \{z\}$ for any $y, z \in X$.□

Proof of Proposition 19. i) Totality is easily checked (indeed, for any $A, B \in P(X)$ if $A \in F$ then $A \succcurlyeq_F^t B$,and if $A \notin F$ then $B \succcurlyeq_F^t A$). In order to check transitivity, take A, B, C such that $A \succcurlyeq_F^t B, B \succcurlyeq_F^t C$. Three cases are to be considered: $A \in F$ and $B \in F, A \in F$ and $C \notin F, B \notin F$ and $C \notin F$. In any case, $A \succcurlyeq_F^t C$ follows (by definition). *Weak Monotonicity* is also easily checked: for any $A, B \subseteq X$, if $A \cup B \notin F$ then $B \notin F$ as well, hence in any case $A \cup B \succcurlyeq_F^t A$. To see that PI cannot possibly hold, consider $x_1, x_2 \in X, x_1 \neq x_2$. Then, take $A^0 = X\backslash\{x_1, x_2\}, A^1 = A^0 \cup \{x_1\}, A^2 = A^0 \cup \{x_2\}$, and assume PI. Hence $A^j \succ_F^t A^0$ for some $j \in \{1, 2\}$i.e. $A^j \in F$ and $A^0 \notin F$. It follows that $A^j \succcurlyeq_F^t B$ for any $B \subseteq X$, which contradicts PI. Similarly, take $C \subseteq X$ such that $\#C \geq 3$ and assume PE. Then $C \succ_F^t B$ (i.e $C \in F$ and $B \notin F$), for some $B \subseteq X$ such that $\#B = 2$, and $B \succ_F^t A$ (i.e. $B \in F$ and $A \notin F$) for some $A \subseteq X$ such that $\#A = 1$, a contradiction.Concerning the other relevant monotonicity properties, WPI is satisfied provided that $\#G^{\#X-1}(F) < \#X$: indeed, take $x \in X$ such that $X\backslash\{x\} \notin F$ (such a x must exist, by hypothesis). Then, choose $X' = X\backslash\{x\}$ and posit $B = \{x\}$: hence $X = X' \cup \{x\} \succ_F^t X'$. Conversely, if $\#G^{\#X-1}(F) = \#X$ then $X\backslash\{x\} \in F$ for each $x \in X$:thus, WSI fails because for each $X' \subset X$ there exists $x \in X$ such that $X\backslash\{x\} \supseteq X'$ and $X\backslash\{x\} \succcurlyeq_F^t B$ for any $B \subseteq X$.

SI holds true if $F = \{X\}$:indeed, for any $A \subset X$, $X = A \cup (X\backslash A) \succ_F^t A$ such that $X' = X\backslash\{x\} \in F$; conversely, if $F \neq \{X\}$ then $X\backslash\{x\} \in F$ for some

$x \in X$, whence $X \backslash \{x\} \succcurlyeq_F^t A$ for any $A \subseteq X$, and SI fails.

WSE is satisfied whenever $F \neq P(X)$ because one may choose

$X' \in \max_{\supseteq} \{C \subseteq X : C \notin F \neq \emptyset\}$:

then for any $A \subseteq X$ such that $X' \subset A$, $A \succ_F^t X' = A \backslash (A \backslash X')$. If, however, $F = P(X)$ then obviously $\succcurlyeq_F^t = P(X) \times P(X)$ hence WSE fails. WPE is satisfied if $X \backslash \{x\} \notin F$ for some $x \in X$: indeed, under that hypothesis one may posit $X' = X \backslash \{x\}$, and observe that $X \succ_F^t X \backslash \{x\}$. Conversely, if $X \backslash \{x\} \in F$ for all $x \in X$ then for any $x \in X$, $X \backslash \{x\} \succcurlyeq_F^t X$, which contradicts WPE. Finally, if $F = P(X) \backslash \{\emptyset\}$ then SE holds: indeed, take any $A \neq \emptyset$. Then, posit $B = A$:clearly $A \in F$ while $\emptyset \notin F$ i.e.$A \succ_F^t \emptyset = A \backslash A$. Conversely, if $F \neq P(X) \backslash \{\emptyset\}$ we may assume w.l.o.g. that $F \subset P(X) \backslash \{\emptyset\}$ (otherwise $F = P(X)$ under which even WSE fails, as shown above). Hence, $A \notin F$ for some $A \neq \emptyset$: but then, $C \succcurlyeq_F^t A$ for any $C \in F$, which contradicts SE.

ii) First, notice that $A \succcurlyeq_F^{t*} B$ entails either $A \in F$ or $A \cup B \notin F$. Since $A \cup B \notin F$ entails $B \notin F$, it follows that– in any case– $A \succcurlyeq_F^t B$. In view of the lemma in Puppe(1996) already mentioned above(see the proof of the previous Proposition), it only remains to prove that $\succcurlyeq_F^{t*}$ is not transitive. Indeed, consider $x, y, z \in X$, $A = \{x\}, B = \{y\}, C = \{z\}$, and $F = \{D \subseteq X : D \supseteq \{x, z\}\}$. Then, $A \notin F, A \cup B \notin F$, $B \cup C \notin F$ while $A \cup C \in F$: it follows that $A \succcurlyeq_F^t A \cup B$, $B \succcurlyeq_F^t B \cup C$ but $A \cup C \succ_F^t A$ i.e. $\succcurlyeq_F^{t*}$is not transitive.

iii) For any $x, y \in X$, $\{x\} \succcurlyeq_F^t \{y\}$ iff either $\{x\} \in F$ or $\{y\} \notin F$, i.e. iff either $x \in G^1(F)$ or $y \notin G^1(F)$. Moreover, $\{x\} \succ_F^t \{y\}$ iff $[\{x\} \in F$ or $\{y\} \notin F]$ and not $[\{y\} \in F$ or $\{x\} \notin F]$ i.e. $x \in G^1(F)$ and $y \notin G^1(F)$.□

Proof of Claim 22. i) That $(P(X), \succcurlyeq_\#)$ is a total preorder follows from the basic properties of cardinal numbers. Moreover, for any $A, B \subseteq X$, $A \cup B \succcurlyeq_\# A$ because $\#A \cup B \geq \#A$. Also, it is immediately seen that for any $A \subset X$, $\emptyset \neq B \subseteq X$, and $x \in X \backslash A$, $y \in B$, $A \cup \{x\} \succ_\# A$ and $B \succ_\# B \backslash \{y\}$.

ii) For any $A, B \in P(X)$, $A \succcurlyeq_\#^* B$ iff $A \succcurlyeq_\# A \cup B$ i.e $\#A \geq \#A \cup B$ or equivalently $A \supseteq B$. Hence $(P(X), \succcurlyeq_\#^*)$ is transitive and thus–by the lemma mentioned above– a *unanimous* IPM OR.

iii) Clearly enough, for any $x, y \in X$, $\{x\} \sim_\# \{y\}$ whence the thesis.□

Proof of Proposition 27. i) Transitivity follows trivially from the definition, and from transitivity of $\supseteq$. Reflexivity also follows trivially from the definition, and from reflexivity of $\supseteq$.*Weak monotonicity* is also an immediate consequence of the definition, since $A \supseteq B$ obviously entails $co_{\geq} A \supseteq co_{\geq} B$. To see that WPI is satisfied, posit

$$E_{\geq}^{+}(X) = \{x \in X : \text{for any } z \in X, \textit{not } z > x \},$$
$$E_{\geq}^{-}(X) = \{x \in X : \text{for any } z \in X, \textit{ not } x > z \},$$

and notice that $\#[E_{\geq}^{+}(X) \cup E_{\geq}^{-}(X)] \geq 2$ since $\#X \geq 3$. Then, take a non-empty set S, $S \subset E_{\geq}^{+}(X) \cup E_{\geq}^{-}(X)$ and define $X' = S \cup [X \backslash (E_{\geq}^{+}(X) \cup E_{\geq}^{-}(X))]$: hence $X' \subset X$, and for any $A \subseteq X, X' \subseteq A \subset X$, and any $x \in X \backslash A$, $co_{\geq}(A \cup \{x\}) \supset co_{\geq} A$ (because $x \in co_{\geq} A$ iff there exist $y, z \in A$ such that $y > x > z$, a contradiction since $x \in E_{\geq}^{+}(X) \cup E_{\geq}^{-}(X)$), i.e. $A \cup \{x\} \succ_{(co_{\geq})} A$.

PE is checked as follows: for any $A \subseteq X, A \neq \emptyset$, choose $z \in \max_{\geq} A$ (such a z must exist since X is finite). Then, notice that $y \geq z$ for no $y \in X \setminus \{z\}$(by antisymmetry). It follows that $z \notin co_{\geq}(A \setminus \{z\})$, hence $co_{\geq}A \supset co_{\geq}(A \setminus \{z\})$ i.e. $A \succ_{co_{\geq}} A \setminus \{z\}$. In order to see that SI is *not* satisfied if $\geq$ includes a chain of size 3, take $x, y, z \in X$ such that $x > y > z$, then notice that $co_{\geq}(X \setminus \{y\}) = co_{\geq}X$ whence $X \setminus \{y\} \succcurlyeq_{co_{\geq}} X$ which contradicts SI.

ii) It can be shown that

(∗) for any $A, B \subseteq X$, $co_{\geq}A \supseteq co_{\geq}(A \cup B)$ iff $co_{\geq}(A) \supseteq co_{\geq}(B)$.

Indeed, assume $co_{\geq}(A) \supseteq co_{\geq}(B)$ and take any $z \in co_{\geq}(A \cup B)$, i.e. there exist $y, w \in A \cup B$ such that $y \geq z \geq w$. Generally speaking four cases are to be distinguished, namely : a) $\{y, w\} \subseteq A$: here clearly $z \in co_{\geq}(A)$ (by definition); b) $\{y, w\} \subseteq B$: in this case, $z \in co_{\geq}(B)$ hence it follows from our assumption that $z \in co_{\geq}(A)$ as well; c) $y \in A, w \in B$: in this case, $w \in co_{\geq}(B)$ hence–by assumption–$w \in co_{\geq}(A)$. It follows that $y \geq z \geq w'$ for some $w' \in A$, i.e. $z \in co_{\geq}(A)$; d) $y \in B, w \in A$: here, $y \in co_{\geq}(B) \subseteq co_{\geq}(A)$, whence $y' \geq z \geq w$ for some $y' \in A$. Again, it follows that $z \in co_{\geq}(A)$.

Conversely, suppose $co_{\geq}(A) \not\supseteq co_{\geq}(B)$ i.e. there exists $z \in co_{\geq}(B) \setminus co_{\geq}(A)$ hence a fortiori $z \in co_{\geq}(A \cup B) \setminus co_{\geq}(A)$, and $co_{\geq}(A) \not\supseteq co_{\geq}(A \cup B)$.

As a consequence of (∗), for any $A, B \subseteq X$: $A \succcurlyeq^{*}_{co_{\geq}} B$ iff $A \succcurlyeq_{co_{\geq}} B$. Therefore, $(P(X), \succcurlyeq_{co_{\geq}})$ is indeed a $fixed-base\ unanimous\ IPM$ OR.

iii) It is immediately checked that for any $x, y \in X$, $co_{\geq}\{x\} \supseteq co_{\geq}\{y\}$ iff $x = y$ hence the thesis follows.□

Proof of Proposition 28. i) *Reflexivity* is obvious. Moreover, take $A, B, C \subseteq X$ such that $A \succcurlyeq_{(co_{\geq}, F)} B$, and $B \succcurlyeq_{(co_{\geq}, F)} C$: if $co_{\geq}A \supseteq co_{\geq}B$, and $co_{\geq}B \supseteq co_{\geq}C$ then clearly $co_{\geq}A \supseteq co_{\geq}C$ i.e. $A \succcurlyeq_{(co_{\geq}, F)} C$; if $co_{\geq}C \notin F$, then $A \succcurlyeq_{(co_{\geq}, F)} C$ by definition; if $co_{\geq}B \notin F$ and $co_{\geq}B \supseteq co_{\geq}C$ then $co_{\geq}C \notin F$ whence again $A \succcurlyeq_{(co_{\geq}, F)} C$. Thus, *transitivity* holds.

In order to check *Weak Monotonicity,* just observe that for any $A, B \subseteq X$, $co_{\geq}(A \cup B) \supseteq co_{\geq}A$.

To check WPI , define $E^{+}_{\geq}(X), E^{-}_{\geq}(X)$ as in the proof of the foregoing Proposition, and

$$X' = \max_{\supseteq} \left\{ \begin{array}{c} Y \subseteq X : \text{there exists } S \text{ such that} \\ \emptyset \neq S \subset E_{\geq}(X) = E^{+}_{\geq}(X) \cup E^{-}_{\geq}(X), \\ Y = S \cup [X \setminus (E_{\geq}(X))], \text{ and } co_{\geq}Y \notin F \end{array} \right\}.$$

(If no such Y exists, then posit $X' = S \cup [X \setminus (E_{\geq}(X))]$ for some S, $\emptyset \neq S \subset E_{\geq}(X)$: since $co_{\geq}X' \in F$, one may conclude that $A \cup \{x\} \succ_{(co_{\geq}, F)} A$ from the fact that $co_{\geq}A \cup \{x\} \supset co_{\geq}A \in F$ for any A, $X \supset A \supseteq X'$, and any $x \in X \setminus A$). Now, for any A such that $X \supset A \supseteq X'$, and for any $x \in X \setminus A$, one may conclude again that $co_{\geq}(A \cup \{x\}) \supset co_{\geq}A$ (by the same argument already provided in the proof of the foregoing Proposition). Since by definition $co_{\geq}(A \cup \{x\}) \in F$, it follows that $A \cup \{x\} \succ_{(co_{\geq}, F)} A$.

To check WPE, consider $\mathbf{Y} = \{Y \subseteq X : Y \subseteq X \setminus (E_{\geq}(X)), co_{\geq}Y \notin F\}$ and take $X^{*} \in \max_{\supseteq} \mathbf{Y}$ if $\mathbf{Y} \neq \emptyset$. Then, posit $X' = X^{*} \cup \{x_0\}$ for some $x_0 \in E_{\geq}(X)$ if $\mathbf{Y} \neq \emptyset$, and $X' = X \setminus (E_{\geq}(X))$ if $\mathbf{Y} = \emptyset$. Now, choose any A such that $X' \subset A \subseteq X$: if $X' = X^{*} \cup \{x_0\}$ then by definition $co_{\geq}(A) \in F$, and

$co_{\geq}A \supset co_{\geq}(A\backslash\{x_0\})$(recall that $x_0 \in E_{\geq}(X)$), whence $A \succ_{(co_{\geq},F)} (A\backslash\{x_0\})$; if $X' = X\backslash(E_{\geq}(X))$ then for any $x \in A\backslash X' \subseteq E_{\geq}(X)$, $co_{\geq}A \supset co_{\geq}(A\backslash\{x\})$ and –by definition–$co_{\geq}A \in F$, whence again $A \succ_{(co_{\geq},F)} A\backslash\{x\}$.

Next, suppose that $(X, \geq)$ does not include a chain of size 3. Then, by definition $X = E_{\geq}(X)$. Hence, by definition, for any $A \subset X$, $co_{\geq}(A \cup X\backslash A) = co_{\geq}X \supset co_{\geq}A$, and SI holds. Conversely, suppose that $(X, \geq)$ does include a chain of size 3 (i.e $x > y > z$ for some $x, y, z \in X$). Then, choose $A = X\backslash\{y\}$: obviously, $co_{\geq}A = X \in F$ hence $A \succcurlyeq_{(co_{\geq},F)} B$ for any $B \subseteq X$, and SI fails.

Now, suppose that $(X, \geq)$ does not include a chain of size 3 (hence $X = E_{\geq}(X)$) and there exists $x^* \in X$ such that $\{x^*\} = co_{\geq}\{x^*\} \in F$. Then, for any $A \subset X$ two cases are to be distinguished: a)$x^* \in A$,b)$x^* \notin A$.If $x^* \in A$ then for any $x \in X\backslash A$, $\emptyset \neq co_{\geq}(A \cup \{x\}) \supset co_{\geq}A$, and $co_{\geq}(A \cup \{x\}) \in F$, hence $A \cup \{x\} \succ_{(co_{\geq},F)} A$; if $x^* \notin A$ then $co_{\geq}(A \cup \{x^*\}) \supset co_{\geq}(A)$ and–again–$co_{\geq}(A \cup \{x^*\}) \in F$ hence $A \cup \{x^*\} \succ_{(co_{\geq},F)} A$. It follows that PI is satisfied. Conversely, suppose that $(X, \geq)$ does include a chain of size 3, i.e. there exist $x, y, z \in X$ such that $x > y > z$. Then, take $A = X\backslash\{y\}$ and observe that $co_{\geq}(A) = X \succcurlyeq_{(co_{\geq},F)} B$ for any $B \subseteq X$, which contradicts PI. Also, suppose that for each $x \in X$, $\{x\} \notin F$. Then, observe that $\emptyset \succcurlyeq_{(co_{\geq},F)} \{x\}$ for any $x \in X$, which also contradicts PI.

If $F \supseteq \{B \subseteq X : B = \{x\}$ for some $x \in X\}$ then for any $A \neq \emptyset$, and any $x \in A$, $co_{\geq}A \supseteq co_{\geq}\{x\} = \{x\}$ and $co_{\geq}A \in F$. It follows that $A \succ_{(co_{\geq},F)} \emptyset = A\backslash A$, i.e. SE holds. Conversely, assume that there exists $x \in X$ such that $\{x\} = co_{\geq}\{x\} \notin F$. Then, $B \succcurlyeq_{(co_{\geq},F)} \{x\}$ for any $B \subseteq X$ hence SE fails.

If $F \supseteq \{B \subseteq X : B = \{x\}$ for some $x \in X\}$ and $(X, \geq)$ does not include a chain of size 3, then for any $A \subseteq X$ such that $A \neq \emptyset$ and any $x \in A$, $co_{\geq}(A) \supset co_{\geq}(A\backslash\{x\})$ (because $X = E_{\geq}(X)$), $co_{\geq}A \in F$ (because $co_{\geq}A \supseteq co_{\geq}\{x\} = \{x\} \in F$): it follows that $A \succ_{(co_{\geq},F)} A\backslash\{x\}$, hence PE holds. Conversely, assume that $(X, \geq)$ does include a chain of size 3, i.e. there exist $y, z, w \in X$ such that $y > z > w$ then $X\backslash(E_{\geq}(X)) \neq \emptyset$. Thus, for any $x \in (X\backslash(E_{\geq}(X))$, $co_{\geq}(X\backslash(E_{\geq}(X) \cup \{x\})) = co_{\geq}(X\backslash(E_{\geq}(X)))$, hence $X\backslash(E_{\geq}(X) \cup \{x\}) \sim_{(co_{\geq},F)} (X\backslash(E_{\geq}(X)))$, which contradicts PE.

ii) Let $A, B \subseteq X$: then, by definition, $A \succcurlyeq^*_{(co_{\geq},F)} B$ iff either $co_{\geq}(A) \supseteq co_{\geq}(A \cup B)$ or $co_{\geq}(A \cup B) \notin F$. Now, –as we have already seen above in the proof of the foregoing proposition– $co_{\geq}(A) \supseteq co_{\geq}(A \cup B)$ iff $co_{\geq}(A) \supseteq co_{\geq}(B)$, while $co_{\geq}(A \cup B) \notin F$ clearly entails $co_{\geq}(B)$: therefore, $\succcurlyeq^*_{(co_{\geq},F)} \subseteq \succcurlyeq_{(co_{\geq},F)}$. Moreover, let us consider an order filter F of $(P(X), \supseteq)$ such that for some $A, B, C \subseteq X$: $co_{\geq}(A \cup B) \notin F, co_{\geq}(B \cup C) \notin F, co_{\geq}(A) \not\supseteq co_{\geq}(C), co_{\geq}(A \cup C) \in F$ (e.g. take $(X, \geq)$ and $x, y \in X$ such that $\{x, y\} \subseteq E_{\geq}(X)$, $F = \{D \supseteq X : D \supseteq \{x, y\}\}$ where $x, y \in X, x \neq y$, and posit $A = \{x\}, C = \{y\}, B = \{z\}$, where $z \in X\backslash\{x, y\}$). Then, by definition, $A \succcurlyeq^*_{(co_{\geq},F)} B$, $B \succcurlyeq^*_{(co_{\geq},F)} C$, but $not(A \succcurlyeq^*_{(co_{\geq},F)} C)$. Thus, $\succcurlyeq^*_{(co_{\geq},F)}$is not transitive and–as a consequence, by Puppe's lemma as already mentioned above– $(P(X), \succcurlyeq^*_{(co_{\geq},F)})$ is not a $fixed-base\ unanimous\ IPM$ OR.

iii) For any $x, y \in X$, $x \succsim_{\succcurlyeq_{(co_{\geq},F)}} y$ iff $\{x\} \succcurlyeq_{(co_{\geq},F)} \{y\}$ i.e. iff [either $\{x\} = co_{\geq}\{x\} \supseteq co_{\geq}\{y\} = \{y\}$ or $\{y\} \notin F$] or equivalently iff [either $x = y$ or

$\{y\} \notin F$]. In particular, it follows that $x \succ_{(co_{\geq},F)} y$ iff $\{x\} \in F$ and $\{y\} \notin F$.□

Proof of Proposition 32. i) Notice that $w(\geq)$ may be regarded as a nonnegative integer-valued function $w(\geq) : P(X) \to Z_{+}$. Hence $\succcurlyeq_{w(\geq)}$is, by definition, a linear order. Moreover, *Weak monotonicity* also holds true by definition (because the largest $\geq -$antichain in $A \cup B$ is obviously at least as large as the largest $\geq -$antichain in A). To check that SE holds, observe that $w(\geq)(\emptyset) = 0$ and $w(\geq)(A) \geq 1$ for any $A \neq \emptyset$: thus, for any $A \neq \emptyset$, $A \succ_{w(\geq)} \emptyset$.

Let us now assume that $(X, \geq)$ has a unique antichain A^* of maximum size (hence, in particular, $\#A^* > 1$). Next, consider the minimum cardinality 'gap' between A^* and other antichains of $(X, \geq)$ i.e.

$g(A^*) = \min \{\#A^* - \#A : \ A \neq A^*, \text{ and } A \text{ is an antichain of } (X, \geq) \}$,

choose $B \subset A^*$ such that $\#B = \#A^* - g(A^*)$, and posit $X' = X \backslash (A^* \backslash B)$. Then, by definition, for any A such that $X' \subseteq A \subset X$ and any $x \in A^* \backslash A$, $w(\geq)(A \cup \{x\}) > w(\geq)(A)$, i.e. $A \cup \{x\} \succ_{w(\geq)} A$ hence WPI holds. Similarly, for any A such that $X' \subset A \subseteq X$ and any $x \in A \backslash X'$, $w(\geq)(A) > w(\geq)(A \backslash \{x\})$ i.e. $A \succ_{w(\geq)} A \backslash \{x\}$ hence WPE is also satisfied. Conversely, suppose $(X, \geq)$ has (at least) two antichains A', A'' of maximum size, and consider any $X' \subset X$. First, observe that for any $x \in X$, $w(\geq)(X \backslash \{x\}) = w(\geq)(X)$ hence WPE fails. Also, if either $A' \subseteq X'$ or $A'' \subseteq X'$ then for any $B \subseteq X$, $w(\geq)(X') \geq w(\geq)(B)$, i.e. $X' \succcurlyeq_{w(\geq)} B$ and WSI fails. Otherwise, choose $y \in A' \backslash (X' \cup A'')$ and $z \in A'' \backslash (X' \cup A')$: then posit $A = X \backslash \{z\}$, and observe that $X' \subset A \subset X$ and $w(\geq)(A) = w(\geq)(X)$ i.e. $A \succcurlyeq_{w(\geq)} B$ for any $B \subseteq X$, which also contradicts WSI.

Next, suppose that $(X, \geq)$ is an antichain . Then, obviously, for any $A, A' \subseteq X$, $A \neq X$, $A' \neq \emptyset$, $x \in X \backslash A$, $y \in A'$, $w(\geq)(A \cup \{x\}) > w(\geq)(A)$ and $w(\geq)(A') > w(\geq)(A' \backslash \{y\})$ i.e. both PI and PE hold. Conversely, let $(X, \geq)$ be such that there exist $x, y \in X$ with $x > y$. Then there exists $z \in \{x, y\}$ such that $w(\geq)(X \backslash \{z\}) = w(\geq)(X) \geqq w(\geq)(X \backslash \{x, y\})$. It follows that for any $B \subseteq X$, $X \backslash \{z\} \succcurlyeq_{w(\geq)} B$ which violates SI. Moreover, $w(\geq)(\{x, y\}) = w(\geq)(\{x\}) = w(\geq)(\{y\})$ i.e. $\{x, y\} \sim_{w(\geq)} \{x\} \sim_{w(\geq)} \{y\}$ hence PE is violated as well.

ii) Notice that $A \succcurlyeq^*_{w(\geq)} B$ iff $w(\geq)(A) \geqq w(\geq)(A \cup B)$. Since clearly $w(\geq)(A \cup B) \geqq w(\geq)(B)$, it follows that $A \succcurlyeq_{w(\geq)} B$. Also, if $(X, \geq)$ is a chain then $w(\geq)(A) = 1$ for any $A \subseteq X$, hence by definition $\succcurlyeq^*_{w(\geq)}$is transitive. If $(X, \geq)$ is an antichain then, by definition $A \succcurlyeq^*_{w(\geq)} B$ iff $\#A \supseteq \#B$. In both cases, $(P(X), \succcurlyeq^*_{w(\geq)})$ is a *fixed-base unanimous IPM* OR. Suppose however $(X, \geq)$ is such that $x, y, z \in X$ are pairwise distinct elements of X, with $x > y$ and $z > y$ (or $y > x$ and $y > z$) but neither $x > z$ nor $z > x$. Then, $w(\geq)(\{x\}) = w(\geq)(\{x, y\}) = w(\geq)(\{z, y\}) = w(\geq)(\{z\}) = 1$, while $w(\geq)(\{x, z\}) = 2$: therefore $\{x\} \succcurlyeq^*_{w(\geq)} \{y\}$, $\{y\} \succcurlyeq^*_{w(\geq)} \{z\}$ and *not* $\{x\} \succcurlyeq^*_{w(\geq)} \{z\}$. It follows that in this case $\succcurlyeq^*_{w(\geq)}$is *not* transitive hence $(P(X), \succcurlyeq^*_{w(\geq)})$ is *not* a *fixed-base unanimous IPM* OR.

iii) By definition, for any $x, y \in X$, $x \precsim_{\succcurlyeq_{w(\geq)}} y$ iff $\{x\} \succcurlyeq_{w(\geq)} \{y\}$ i.e. iff $w(\geq)(\{x\}) \geqq w(\geq)(\{y\})$ which is obviously the case since $w(\geq)(\{z\}) = 1$ for all

$z \in X$. It follows that $\succsim_{\succcurlyeq_{w(\geq)}} = X \times X$, hence $\succ_{\succcurlyeq_{w(\geq)}} = \emptyset$.□

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