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Abstract -This paper explores general equilibrium asset pricing implications in a two-period model in which the production side explicitely describes the thermodynamic process unavoidably connected with production. We show that steady state of the production process, i.e. thermodynamic equilibrium, has a one to one correspondence with the absence of arbitrage possibilities. This provides an alternative definition of the absence of arbitrage.

Jel Classification: D5, G1, R3

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1 Introduction

This paper explores the definition of absence of arbitrage possibilities and equilibrium for an economy with complete capital markets in which production depends on time. As it has been forcefully argued by Gerogescu Roegen (1971) in his critique of the Neoclassical description of the production process through the familiar production function, the omission of the time dependency results in a very peculiar description of reality. His starting point is provided by the observation that an irreversible thermodynamic process is at the basis of every real world transformation. In this light, the Neoclassical description of the physical production process omits fundamental features like irreversible qualitative change and entropy creation. This has to do with the omitted role of time in the production process. In his analysis the omission of time may only be justified if the "factory system" is in a steady state (Georgescu Roegen (1971), p.238), which is not the case for the vast majority of production processes.

We highlight here some implications for general equilibrium analysis of a fully realistic technological description where production has not reached a steady state, which accomodates qualitative change and the role of time. This is achieved by adopting as production process a basic thermodynamic process that has not yet reached a steady state. [1] The technological description of the single good produced in this economy is that of "hot water", and it is chosen to unveil, in the simplest possible way, the thermodynamic aspect of the production process. Although the choice of the hot fluid production process may not seem of general interest, it is indeed the process behind the production of many goods, including the commodity with the widest possible use in the economic system, i.e. electric energy produced through thermal plants. Still the technological description adopted here is just a simple example of a thermodynamic characterization of the production process and leaves open for further research the identification of more articulate and useful thermodynamic descriptions of the production side.

We argue that we can have an equilibrium in this economy, i.e. in its contingent claims markets, if and only if a termodynamic equilibrium in the production side, i.e. no time dependency, has been achieved. The need for simultaneous equilibrium of the production and exchange side of our simple economy is obviously a feature of general equilibrium. However, the notion of equilibrium in the production side of the

[1] It is well known in the finance literature that equilibrium asset pricing conditions, in continuous time, may be represented with the same type of diffusion equations that describe heat dynamics. The Black and Scholes (1973) result is a case in point: their equation for the price of a call option is solved noting that it essentially the same as the unidimensional heat transfer equation (Black and Scholes, (1973), p.644). This fact is interpreted as a mere coincidence that completely different phenomena, like a no arbitrage condition in financial markets and the temperature field in a uniform mean, may be described using the same mathematics. We show in this paper that, in economies with a realistic production side as well as financial markets, there exist instead a deep interaction between financial and thermodynamic equilibrium which may be exploited to characterize the financial notion of absence of arbitrage. We may however start from the less radical consideration that a basic input to every production process is one of many forms of energy, which may always be reconducted to thermal energy in a (non ideal) Carnot cycle.x

economy we focus on is different from the usual producer's equilibrium. We focus in depth on a specific technological aspect of the production process, namely the use of a specific form of energy as input of the production process. For this particular input, departure from equilibrium is formally defined as a time-varying temperature field. This provides a strong characterization of arbitrage which is different from the traditional one.

In the present analysis, we go beyond simply considering thermodynamic relations as a pure formal analogy to economic relations, like in Lisman (1949) insightful thermodynamic interpretation of the budget constraints theory of Davis (1941). [2] However, we also clearly depart from the approach postulating an identity between energy and value.

After defining formally the notion of thermodynamic equilibrium, we show that this is a necessary and sufficient condition for the absence of arbitrage.

The plan of the paper is as follows: in Section 2 the thermodynamic production technology adopted in the model is presented and the definition of thermodynamic equilibrium of the production side is given. In Section 3 the equilibrium of the economy is derived. In Section 4 it is shown that the absence of arbitrage possibilities in the financial market of this economy is equivalent to the notion of thermodynamic equilibrium presented in Section 2. The intuition behind the formal result is also provided. Section 5 concludes the paper providing ideas for the extension and application of the result.

2 The Production Process

We consider an economy evolving in the time interval $[-1, 1]$, populated by individuals who consume a single good at time $\tau = 0$ and $\tau = 1$. Conditional on the state of the world at time $\tau = 0$, the future state of the economy is uncertain and may evolve at time $\tau = 1$ into one of two possible states. We describe next the technology employed for the production of the single good, which is based, in this example, on the use of thermal energy.

2.1 Production Technology

Every individual produces the single good according to the same technology. Production of the good is achieved by heating a fluid, which is available in nature at no cost in unlimited quantity, until it reaches a temperature greater or equal to k. Some amount of thermal energy is freely available in nature for this purpose, and it has no other economic use. The minimum temperature k is a technological requirement. The fluid, which is available in nature at a temperature, T_0, lower than k, is useless until is heated. A source of heat at temperature T_1 is exogenously available to achieve this task. We may call this highly technological good *hot water* .

[2]See Mirowski (1988) for a review of energy based economic models. Mirowski generally terms *neo-simulators* followers of such approach, who "*regard the physics merely as a metaphorical resource..*".

We may now describe the production process more formally. We assume that the heating of the fluid occurs through conduction only, and is a non stationary process (it depends on time) involving the production of entropy. Given $x \geq 0$ the amount of the good (heated fluid) produced, production must satisfy the implicit constraint

$$T(t, x, T_0, T_1) \geq k,$$

where t is the amount of time during which heating occurs. The function $T(t, x, T_0, T_1)$ describes the temperature of the marginal unit produced and it is positive, monotonically decreasing in x and has continuous first and second derivatives. Temperature $T(t, x, T_0, T_1)$ satisfies Fourier's equation

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left[k_E \frac{\partial T}{\partial x} \right], \tag{1}$$

where k_E is the thermal conductivity with respect to energy, c is the fluid specific temperature coefficient of energy per unit of mass and ρ denotes mass density. For simplicity, we may refer to the solution with $\frac{k_E}{\rho c} = \sigma^2$; given $z = x/(2\sigma\sqrt{t})$,

$$T(t, z, T_0, T_1) = T_0 + (T_1 - T_0)[1 - erf(z)] \ .$$

A *thermodynamic equilibrium* is achieved if temperature does not vary with time, that is, may be defined by

$$\frac{\partial T}{\partial t} = 0. \tag{2}$$

Although this technological description is quite general, for convenience, we can visualize this production process as the heating of a fluid, initially at temperature T_0, contained in a cylindric capacitor with section of unit area, with perfectly insulated walls, the upper surface of which is subject to exogenous heating at a higher temperature T_1. Given $x \geq 0$ the depth at which the temperature is measured inside the cylinder, its distribution is described by the function $T(t, x, T_0, T_1)$. [3]

2.2 Two-period production and investment

After having described in the preceeding section the physical aspects of production technology, we now turn to the description of the production - investment choice of agents in the economy. In our two-period economy, producers must make consumption good available at two different dates, $\tau = 0$ and $\tau = 1$. For completeness, we start by describing the production process prior to the consumption date $\tau = 0$. At time $\tau = -1$, while the fluid inside the container has the initial temperature T_0, the upper surface is brought at the temperature $T_1 > T_0$. After one period, at time 0, it is therefore available for consumption any quantity x_0 which satisfies:

[3] This technological description highlights the qualitative change that inputs of production may go through to become outputs. Time here is a factor of production. As clearly highlighted by Georgescu Roegen (1971, Chapter IX), there is a subtle difference between a model of production involving qualitative change and a KLEM neoclassical production function.

$$T(1, x_0, T_0, T_1) \geq k \ . \tag{3}$$

Denoting $\bar{x}_0$ the production for which (3) holds as an equality, at time $\tau = 0$ the quantity $x_0 < \bar{x}_0$ is instantaneously removed from the container, leaving inside the residual quantity of the produced good $\bar{x}_0 - x_0$. This implies that (3) holds as an inequality. The amount of the consumption good $\bar{x}_0 - x_0$ available at time 0 represents investment. What is invested here, i.e. set aside for future use, is clearly an amount of thermal energy. Figure 1 illustrates the production process in the time interval $[-1, 0]$ (Period I).
After the removal of the produced quantity x_0 in $\tau = 0$, the remaining fluid (residual) is further heated until time $\tau = 1$; the initial temperature of the fluid in the period $(0, 1]$ (Period II) depends on the amount of heated fluid removed at time $\tau = 0$. We may therefore write that, at the beginning of the second period, $T_0 = T_0^{'}(x_0)$; given however the monotonic relation between x_0 and its temperature $T(1, x_0, T_0, T_1)$, we may equivalently define $T_0 = T_0^{''}(T(1, x_0, T_0, T_1))$. [4] Given this initial condition of the fluid, the production constraint for period II production, x_1, may be written:

$$T(1, x_1, T_0^{'}(x_0), T_1) \geq k \ . \tag{4}$$

At the end of period II it is rational to use all the available production for consumption, and the constraint (4) holds as an equality.

2.3 Uncertainty

Assume now that the initial temperature T_1 for period II is uncertain and becomes known immediately after $\tau = 0$, after the choice x_0 is made. the initial temperature T_1 represents the state of nature for the second period, and may take two possible values: T_1^u with probability p, or T_1^d with probability $1-p$. This produces a different technological constraint depending on the state of nature.
Denote x_1^u production at the end of period II and state u and x_1^d production at the end of period II and state d. This yields the two state dependent technological constraints:

$$T(1, x_1^u, T_0^{'}(x_0), T_1^u) \geq k \ , \tag{5}$$

$$T(1, x_1^d, T_0^{'}(x_0), T_1^d) \geq k \ . \tag{6}$$

[4] We leave the function $T_0'(x_0)$ unspecified. Any function such that the implicit function defined in $(0, \bar{x}_0)$ by (4) holding as an equality is concave would do. We may imagine for simplicity that immediately after time 0 the initial temperature of the fluid instantaneously attains a spatially uniform value (average), which depends on x_0. Or, we may consider, with greater complications, that the initial temperature of the fluid is not spatially uniform, although it will be fully defined by $T_0'(x_0)$

In order to simplify notation, set

$$\begin{aligned} T(x_0) &= T(1, x_0, T_0, T_1), \\ T(x_1^s, x_0) &= T(1, x_1^s, T_0^{'}(x_0), T_1^s). \end{aligned}$$

Figure 2 describes the production process between time $\tau = 0$ and time $\tau = 1$ assuming the generic state s occurred.

3 Equilibrium with complete markets

We now briefly describe equilibrium under complete markets. Each of the individuals, denoted $j = 1, \ldots, N$, maximizes expected utility defined over consumption at time $\tau = 0$ and at time $\tau = 1$ in the generic state s, denoted C_0^j, C_{1s}^j respectively. When consumption at time $\tau = 0$ is used as numeraire, the first order conditions of the consumer's problem define state prices that we denote P_u, P_d. [5] We assume that each consumer is also a producer, and that state prices that clear the markets exist.[6] We now describe in greater detail producer's equilibrium.

3.1 Producer's optimum

Consider now the productive sector. Each producer faces the problem:

$$\begin{aligned} \max_{x_0, x_1^u, x_1^d} \quad & x_0 + P_u x_1^u + P_d x_1^d \ , \\ sub \quad & T(x_0) \geq k \ , \\ & T(x_1^u, x_0) \geq k \ , \\ & T(x_1^d, x_0) \geq k \quad . \end{aligned}$$

Assume regularity conditions such that an internal solution $0 < x_0 < \bar{x}_0$ obtains at time $\tau = 0$ and only the last two constraints are binding. The first order conditions imply:

[5] Formally the problem is

$$Max \ pU^j(C_0^j, C_{1u}^j) + (1-p)U^j(C_0^j, C_{1d}^j)$$

subject to the constraint $C_0^j + P_u C_{1u}^j + P_d C_{1d}^j = x_0^j$, where $U^j(.,.)$ is the concave utility function of the individual j, C_0^j, C_{1s}^j denotes his consumption. P_s is the Arrow-Debreu price of state s and x_0 is the initial wealth at time 0. The first order conditions of the consumer's problem define the state prices P_u, P_d.

[6] Assuming every consumer is also a producer, we have

$$\sum_{j=1}^{N} C_0^j = N x_0, \quad \sum_{j=1}^{N} C_{1u}^j = N x_1^u, \quad \sum_{j=1}^{N} C_{1d}^j = N x_1^d.$$

$$-dx_0 = P_u\, dx_1^u + P_d\, dx_1^d \ , \tag{7}$$

which represents a no-arbitrage condition. Using consumption at time 0 as numeraire, from the producer's first order condition (7) an arbitrage opportunity is the possibility to achieve a marginal improvement in the production plan such that

$$\begin{cases} dx_0 \geq 0 \\ dx_1^u \geq 0 \\ dx_1^d \geq 0 \end{cases}$$

with the inequality being strict in at least one case. That is, it is possible to increase marginally production in a time and state without giving up marginal amounts of production in another time and state. [7]

4 Thermodynamic interpretation of arbitrage

In this setting, we may give an additional interpretation to the equilibrium and the related absence of arbitrage opportunities. We show that an arbitrage possibility implies a thermodynamic disequilibrium and a consequent flux of thermal energy representing the arbitrage gain (loss), measured in terms of energy. Due to the thermodynamic disequilibrium created by the arbitrage possibility, the energy arbitrage gain automatically flows to the individual who realizes it. Under the assumptions contained in the previous sections, we are now ready to verify the following:

Proposition: no arbitrage possibilities exist if and only if thermodynamic equilibrium is achieved.

[7] Individuals in this economy use the same production technology and have the same endowment, so their optimal production plan (x_0, x_1^d, x_1^u) is the same, although their consumption plans may differ. Each individual cannot increase its production or consumption in a time and state without decreasing production or consumption at another time and state. If we define the excess demand for the consumption good of individual j at different times and in different states dC_0^j, dC_{1u}^j, dC_{1d}^j, and define

$$\mathbf{Y} = \begin{bmatrix} dx_1^u & dx_1^d \\ dC_{1u}^j & dC_{1d}^j \end{bmatrix}, \ \mathbf{V} = \begin{bmatrix} dx_0 \\ dC_0^j \end{bmatrix}.$$

An arbitrage possibility, which would prevent equilibrium, would be an allocation $\boldsymbol{\eta} = [\eta_1, \eta_2]$ in which:

$$\boldsymbol{\eta}\mathbf{Y} \geq \mathbf{0}, \text{ and } \boldsymbol{\eta}\mathbf{V} \leq 0$$

with the inequality being strict in at least one case. As it is well known, there are no arbitrage possibilities if and only if there exist two strictly positive constants, $\mathbf{P}' = [\pi_u, \pi_d]$, such that $\mathbf{YP} = -\mathbf{V}$ holds $\forall j$.

To make the point, consider the energy market. [8] This allows us perform a change of numeraire and use the thermal energy cost of the marginal unit produced instead of consumption good available at time 0. In a competitive market, marginal energy cost would be equal to marginal energy revenue and market price.

Adopting a classical thermodynamic approach, from Fourier'law (see for example Fuchs (1996), p.314) the instantaneous flux of thermal energy through the fluid at each depth is:

$$I_E(x) = -K_E \frac{\partial T(x)}{\partial x} \tag{8}$$

where $K_E > 0$ is the temperature coefficient of energy. Consider the producer's problem under the new numeraire:

$$\max_{x_0, x_1^u, x_1^d} \quad x_0 \ I_E(x_0) + P_u \ x_1^u \ I_E(x_0) + P_d \ x_1^d \ I_E(x_0),$$

$$\begin{aligned} sub \quad & T(x_0) \geq k \ , \\ & T(x_1^u, x_0) \geq k \ , \\ & T(x_1^d, x_0) \geq k \quad . \end{aligned}$$

The producer's first order conditions under the new numeraire require:

$$-dx_0 = P_u dx_1^u + P_d dx_1^d - \frac{K_E(x_0 + P_u x_1^u + P_d x_1^d)}{I(x_0)} \frac{\partial^2 T(x_0)}{\partial x_0^2} \ dx_0. \tag{9}$$

Comparison with (7) immediately reveals that no arbitrage possibilities will exist if the second order partial derivative appearing in the producer's first order conditions under the new numeraire is equal to zero. From the heat equation (1) it is both necessary and sufficient that

$$\frac{\partial T(x)}{\partial t} = 0, \tag{10}$$

[8] The delivery of thermal energy in this market is straightforward. To see this, recall that under the production technology adopted there exists a monotonically decreasing relation between the temperature of the marginal unit produced and production. Consider for simplicity two producers, A and B. Producer A achieves a marginal in crease in production, to be delivered against thermal energy. If producer B does not do the same, $x^A = x^B + dx$. Since temperature is monotonically decreasing in production, $x^A > x^B$ at the margin implies $T(x^A) < T(x^B)$, and a unidirectional temperature gradient is therefore created between the fluid of A and the fluid of B.
This gradient allows for the automatic transfer of thermal energy from B to A: it is sufficient the containers (productive plants) are linked through a perfect thermic conductor to deliver energy. Denoting Δy the distance between the two containers, making this distance infinitesimal the temperature gradient between the two containers will be

$$-\frac{dT}{dy} = \lim_{\Delta y \to 0} -\frac{T(x^B + \Delta y) - T(x^B)}{\Delta y} = \lim_{\Delta x \to 0} -\frac{T(x^B + \Delta x) - T(x^B)}{\Delta x} = -\frac{dT}{dx}$$

that is, for marginal difference between the level of production of two producers, the absolute value of temperature gradient between the two containers at the margin is equal to the gradient inside the container measured in the x direction. See Figures 3 and 4.

which is the definition of thermodynamic equilibrium (2), to have

$$\frac{\partial^2 T(x)}{\partial x^2} = 0.$$

Therefore, thermodynamic equilibrium is sufficient for the absence of arbitrage.

On the other hand, the change of numeraire should be irrelevant for the equilibrium allocation, so for the no arbitrage condition (7) to hold it is clearly necessary that (10) holds.

4.1 Interpretation

If the production process is not in a steady state at time $\tau = 0$, the change of numeraire is not irrelevant. In fact under the new energy numeraire there will be an increase in the value of the firm as a consequence of the increase of the energy content of the marginal unit produced at time $\tau = 0$, that is $I_E(x_0)$.

Clearly, before steady state of the production process is reached, production will not be entirely efficient, as some excessive amount of thermal energy will be used to bring the initial layers of fluid at a temperature greater than the "efficient" temperature k. If $x_0 > 0$, the marginal unit of the produced good available at time $\tau = 0$ will have an enegy content, which becomes its market price, that is lower than the average energy content of the quantity x_0. In a steady state economy production would be more efficient. However this cannot be achieved in finite time.

If production is not in steady state, to maximize the value of the firm the producer will set at the highest possible level the energy price $I_E(x_0)$. If $I_E(x_0)$ is monotonically decreasing and concave in x_0, for any positive prices P_s this will lead to the corner solution $x_0 = 0$, that is no production at time $\tau = 0$, and the receipt of an energy amount $I_E(0) \sum_s P_s x_1^s$. Hence, if (10) does not hold for every x, under the new numeraire markets will not clear.

So, either the no arbitrage condition defined under the new numeraire holds, or temperature varies through time, i.e. there is neither thermodynamic equilibrium nor economic equilibrium. Only thermodynamic production processes that are already in steady state at time $\tau = 0$ are compatible with economic equilibrium and absence of arbitrage possibilities.

5 Conclusions

This paper provides a view as to why the mathematical models associated with thermodynamic phenomena may also be used in the description of financial equilibrium and absence of arbitrage possibilities. By recognizing that a thermodynamic process is at the heart of a production process, it is shown a one to one correspondence between thermodynamic equilibium of the production process and the equilibrium of the financial market in which contingent claims on production are exchanged. Thermodynamic and economic equilibrium are not only isomorphic, a lesson which is already evident from the use of the same mathematics in physics and economics, but rather they are *connected*. In the present context, this is highlighted by assuming an amount of (thermal) energy as numeraire, namely the amount needed to produce in the most efficient way a unit of a specific good.

The connection with thermodynamic equilibrium adds a dimension to the formal analysis of economic equilibrium with time dependent production. In particular, if steady state of the termodynamic production process is not achieved, by switching to an energy numeraire, we may exploit an arbitrage possibility which prevents equilibrium.

The result presented in this paper may be straightforwardly extended to multiple goods and multiple periods (or continuous time economies). A simple way to extend this setting to multiple goods is to assume that each good is a different fluid heated at the same temperature greater or equal to k. A different fluid is characterized by a different diffusion coefficient σ. If we exclude the possibility of joint production process, each good is produced by a distinct capacitor (process).

If we take the view that a description of the production process in which production depends on time is indeed realistic, some mechanism which prevents arbitrary switches of numeraire is required to achieve equilibrium. Forcing the use of a monetary numeraire through cash in advance constraints may be a way to guarantee an equilibrium.

Figure 1

At time $\tau = -1$ the fluid in the container is at the temperature T_0. The upper surface is brought to the temperature $T_1 > T_0$. After a unit of time the fluid temperature in relation to its depth is $T(x)$, which is lower the deeper inside the container.

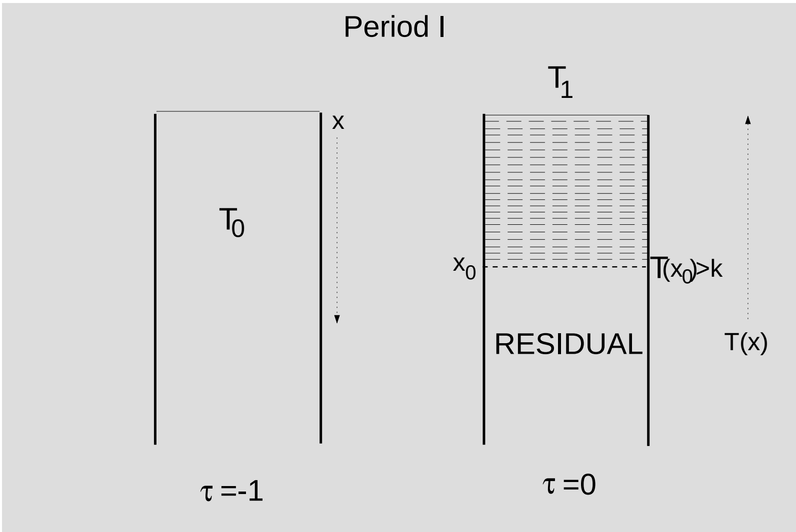


Figure 2

At time $\tau = 0$, after the removal of the produced amount x_0, the residual is at a (non uniform) higher temperature $T'(x_0) > T_0$. Further heating until time $\tau = 1$ from the upper surface at the new temperature T_1^s results in the produced amount x_1^s, which has a marginal temperature equal to k.

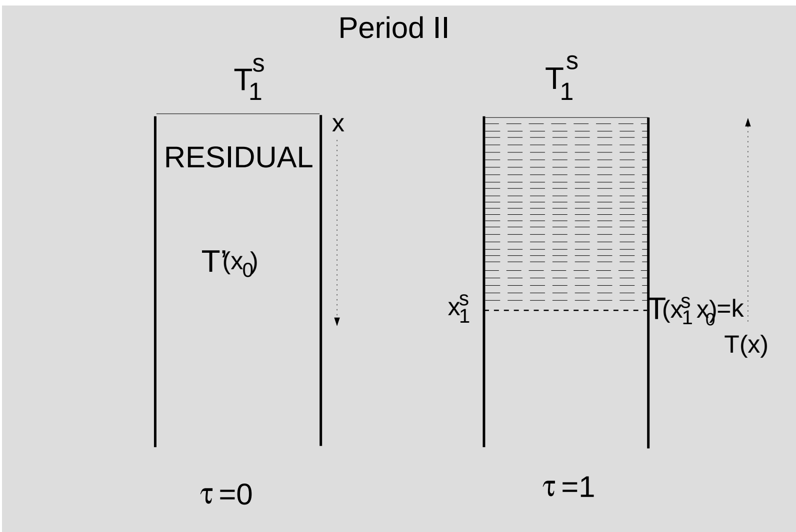


Figure 3

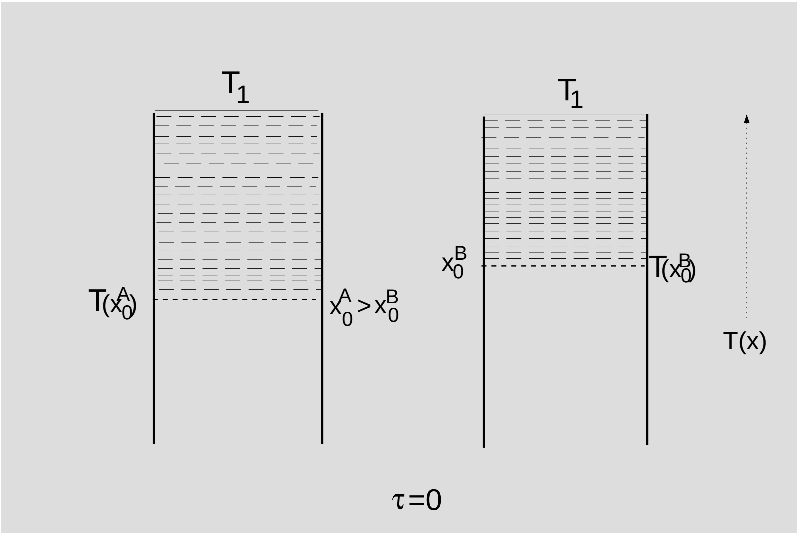
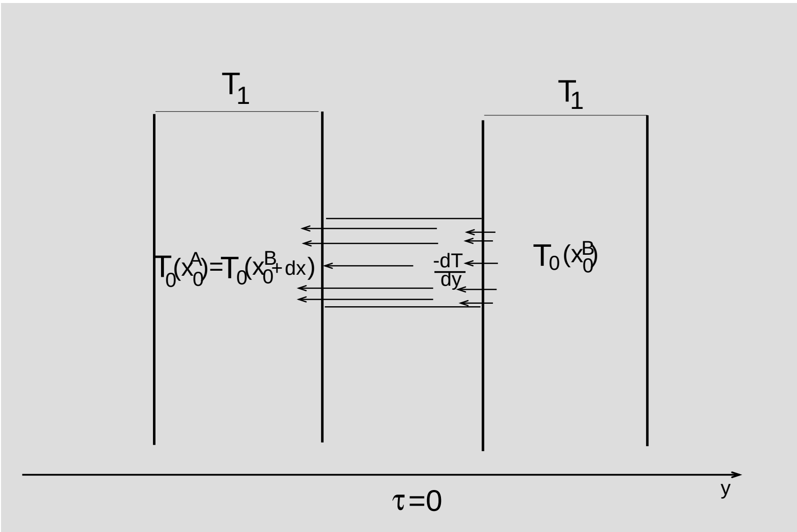


Figure 4



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