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On the Absolute and Relative Differentials Orderings and Their Representations

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ON THE ABSOLUTE AND RELATIVE DIFFERENTIALS ORDERINGS AND THEIR REPRESENTATIONS

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ABSTRACT. In some situations, Lorenz ordering (LO) appears to be a disputable inequality criterion. Apparently, there exist some "paradoxical situations" where the Lorenz criterion fails to rank alternative income distributions. Two sub-orderings of LO, the *absolute differentials ordering* (ADO) and the *relative differentials ordering* (RDO), have been introduced and studied as suitable alternative inequality criteria. The classes of functions consistent with such rankings are identified.

JEL classification: D31, D63, I31.

1. INTRODUCTION

In his seminal paper, Atkinson [2] provided an elegant justification for the use of Lorenz curves as a means of measuring income inequality within the utilitarian framework. Since the publication of this paper, the Lorenz criterion has been represented as one of the fundamental tools for drawing conclusions about inequality of income distributions. This is because such a criterion it is attractive under a normative and descriptive point of view.

More recently, several theorists have noted the existence of situations where Lorenz ordering (LO) fails to be a suitable criterion. They have noticed there exist some situations where an income distribution dominates another distribution in the sense of Lorenz, even if the latter is obtained by means of a sequence of regressive transfers, namely a transfer of money from relatively poorer people to richer ones. "Pathological" situations like this have been observed by Moyes [12] and, in the literature of decision under risk, by Quiggin [13]. In order to avoid such "failures" in ranking income distributions, Chateauneuf [4] has introduced two alternative inequality criteria which check the change in inequality within groups of the population.

These two alternative criteria to LO have been denominated by Moyes [12] *absolute differentials ordering* (ADO) and *relative differentials ordering* (RDO). These two partial orderings first appeared in the research of Marshall, Olkin and Proschan [9] in the field of the *Theory of Majorization.*

Our concern is to analyze ADO and RDO and establishing their properties and characteristics. We show that ADO and RDO have a heuristic appeal as LO. We characterize their relative classes of indices, strengthening the Pigou-Dalton Principle. In fact, introducing two subclasses of the class of all Pigou-Dalton transfers called *transfers about a threshold* (see also Castagnoli and Muliere [3]), and *RDO*

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criterion for transfers, we show that we can avoid disputable situations like the one mentioned above and, at the same time, characterize the class of functions (i.e. the so-called *order-preserving functions*), which are consistent with ADO and RDO.

In the case of transfers about a threshold, we consider a sort of *poverty line*, about which transfers take place. According to Castagnoli and Muliere [3], we restrict the class of Pigou-Dalton transfers to the subset of transfers which take money from individuals above the fixed poverty line and give money to some people below this threshold, with the relative positions of individuals remaining unchanged. In this way we find the class of order-preserving functions consistent with ADO. Our result is very close to that of Castagnoli and Muliere, yet we rely on a lemma due to Marshall and Olkin [8]. The assumption of differentiability of the ordering preserving function is, in fact, not required. The proof is more intuitive, as it only checks local condition on two elements of the income distribution to infer the global behavior of the class of indices consistent with ADO.

As progressive taxation reduces the inequality when the pre-tax distribution is fixed, we can interpret a vector distribution y as the income distribution before tax and $x = f(y)$ as the same income distribution after taxation. The requirement of a progressive taxation, equivalent to $\frac{f(y)}{y}$ being non-decreasing, is usually perceived as reducing income inequality. Progressive taxation is a kind of redistribution of income comparable with a transfer that takes money from the rich and gives it to the poor. In this way, we obtain a criterion that ranks vector distributions after an income redistribution, via progressive taxation, occurs. We call this kind of redistribution process an *RDO criterion for transfer.* Studying this subclass of Pigou-Dalton transfers, we obtain a complete characterization of the class of functions consistent with RDO. Our approach is more general than that presented by Mosler and Muliere [11] as we include what Mosler and Muliere call the class of *transfers next to a threshold*, namely the transfers that may result in crossing the poverty line by the donor or the recipient. Moreover, our proof is a simpler version of Mosler and Muliere's. We apply for a more intuitive result, due to Marshall and Olkin [8] about star-shaped function, than the result, utilized by Mosler and Muliere and due to Landsberger and Meilijson [7].

This paper consists of four sections. In section 2, we explain our basic definitions and crucial concepts analytically as well as intuitively and show how to avoid the so-called LO paradoxes. Section 3 is devoted to identifying the classes of indices consistent with ADO and RDO. Finally, some remarks on issues related to the use of partial orderings conclude the paper.

2. Lorenz ordering and alternative inequality criteria

In his pioneering work on economic inequality, Atkinson [2] discloses the theoretical analogy between the measurement of income disparity in a welfaristic evaluation context and the measurement of risk in decision under uncertainty. He shows that, given two income distributions with equal mean, *second-degree stochastic dominance* (ssd) is equivalent to the Lorenz dominance criterion. This approach evaluates inequality, using an income vector as determining social welfare. It relies on the *normative* and *descriptive* appeal of Lorenz ordering (LO) and it has been the basis of a lot of research of considerable interest.

Nonetheless, several theorists have recently noted the existence of situations in which the Lorenz order fails to be a suitable inequality criterion as it involves paradoxes in ranking different income distributions.

Let us suppose that the initial income distribution among four individuals is represented by the vector $y = (10, 15, 20, 25)$. If we compare y with the distribution $x = (11, 14, 21, 24)$, we see that the Lorenz curve associated with x lies above that y. This should mean that the distribution x is more "spread out" or more even than distribution y. However, x can be obtained from y by a progressive transfer (individual y_4 has given money to the poorest individual y_1) as well as a regressive transfer (y_2 gave money to y_3). Moreover, if we simply consider subgroup $\{2, 3\}$, inequality increases as we pass from situation y to situation x. The income distribution we obtain is obviously more polarized as inequality is increased between the groups $\{1, 2\}$ and $\{3, 4\}$. In other words, the Lorenz criterion does not take into account the differences of income among individuals belonging to a given income distribution. In order to measure inequality of income or wealth and to compare two different distributions as one being more equal than the other, the LO appears to be too weak as a criterion.

Considering such an example, a conclusion that several scholars have put forward is that there is room for alternative inequality criteria. In fact, two different inequality criteria are proposed by Chateauneuf [4], according to Moyes [12] and Quiggin [13], in order to avoid paradoxes as the one mentioned above: the *absolute differentials ordering* (ADO) and the *relative differentials ordering* (RDO).

ADO and RDO have been then suggested as suitable inequality criteria.

To see why let us begin by defining LO, ADO and RDO.

2.1. **Notation and Definitions.** We consider a given finite population $N = \{1, \dots, i, \dots, n\}$ of individuals. An *income distribution*

$$y = (y_1, ..., y_i, ..., y_n)$$

is a finite collection of positive real numbers such that $y_1 \leq \ldots \leq y_i \leq \ldots \leq y_n$ and $\sum_i y_i = T$. y_i is interpreted as the income of individual i in the population N. The set of all income distributions for the population N is denoted by

$$\aleph = \left\{ y \in \Re^n : y_1 \leq \ldots \leq y_i \leq \ldots \leq y_n \text{ and } \sum_i y_i = T \right\}.$$

An *inequality criterion* $\preceq$ is an ordering on $\Re^n$ and can be identified with a subset $\preceq$ of $\aleph \times \aleph$[1]: when the two income distributions x, y in $\aleph$ satisfy $x \preceq y$, we shall say that y is more unequal than x.

The three inequality criteria considered in this work are defined below.

Definition 1 (Inequality Criteria). *Given two income distributions $x, y \in \aleph$, we say that:*

1. *x is less unequal than y in the sense of Lorenz, denoted $x \preceq_{LO} y$, whenever*

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i, \quad \textit{for all } k \in \{1, \dots, n-1\};$$

[1] Strictly speaking, it is a preordering on $\Re^n$: that is a reflexive and transitive relation on $\Re^n$

2. *x is less unequal than y for the absolute differentials ordering (ADO), denoted $x \preceq_{ADO} y$, if*

$$x_{i+1} - x_i \leq y_{i+1} - y_i \quad \forall i \in \{1, ..., n-1\};$$

3. *x is less unequal than y, with $x, y \in \Re^n_{++}$, for the relative differentials ordering (RDO), denoted $x \preceq_{RDO} y$, if:*

$$\frac{x_{i+1}}{x_i} \leq \frac{y_{i+1}}{y_i} \quad \forall i \in \{1, ..., n-1\}.$$

The meaning of ADO and RDO therefore is that if the differences (ratios), between y_i and x_i increase, then, step by step, the value of y_i has to increase much more relatively to that of x_i. The components of y_i are then less "spread out" than the ones of x_i, i.e. there exists less disparity between rich and poor in the distribution x than in y.

The following questions now arise: "Why did the attention of theorists move to these two orderings?"; "Do ADO and RDO really have a heuristic appeal?"; "Why should we adopt these alternative criteria to LO?".

The rest of the paper is dedicated to attempting answers to these questions.

2.2. **Alternative inequality criteria for solving the paradoxes concerning LO.** Chateauneuf [4] claims that the use of Lorenz dominance, as a criterion to evaluate income disparity among individuals, leads to true paradoxes.

In fact, let us assume, as he does, quoting Moyes [12], that the initial income distribution y among 4 individuals is equal to $y = (10, 15, 20, 25)$. The vector $x = (11, 14, 21, 24)$, obtained from y through several transfers of shares of income among individuals, Lorenz dominates y. Nevertheless, if we call h a vector of transfers such that $\sum_{i=1}^{4} h_i = 0$, with h_i representing the share of income that the *ith* individual receives (or transfers) from (to) another individual j, we can conclude that x is obtained from y by the following transfers: $h_2 = -1$ and $h_3 = 1$: a relatively poorer person has given one unit of her income to a relatively richer one. Such a transfer should increase inequality. LO instead shows that the obtained vector x presents less disparity with respect to y. The same thing happens if we consider the vector $x^1 = (14, 14, 21, 21)$ that dominates y in sense of Lorenz, yet is obtained from y by imposing individual 2 to give one unit of income to individual 3, i.e. $h_2 = -1$ and $h_3 = 1$. Analogously for the vector $x^2 = (10, 16, 19, 25)$, which once again Lorenz dominates y, yet with $h_1 = 0$, $h_2 = 1$, $h_3 = -1$ and $h_4 = 0$, a richer person has obtained an increase of income while the poorest person has not, furthermore a relatively poorer person has suffered a reduction of income while the richest one has not.

If we consider such examples, the Lorenz criterion appears to be a too weak criterion. Chateauneuf [4] concluded that we have to look for some alternative tools for comparing the inequality between different income distributional profiles. Then, in order to avoid the disputable situations quoted above, he proposed to use ADO and RDO as suitable alternatives to LO. These two orderings are sub-orderings of LO.[2] They do not analyze the income distribution as a "*whole*", but estimate inequality by comparing alternative individual income profiles. ADO and RDO capture more information, relative to any distribution, than LO does. Indeed, comparing, for example, y with x, LO says that x Lorenz dominates y, while ADO

[2]This means that ADO and RDO implies LO, whereas the contrary does not hold.

says that by passing from y to x, inequality could increase, at least, if we compare the income of the second group of individuals in the distribution with that of the third one (i.e., $y_3 - y_2 = 20 - 15 = 5 < x_3 - x_2 = 21 - 14 = 7$).

Having shown why several theorists have recently analyzed ADO and RDO, we want now to answer to the question: "What is the heuristic value of these two inequality criteria?".

To see why ADO is a suitable inequality criterion, let us consider two vectors of income profiles $x, y \in \aleph$, that are ADO-ranked, and set $x_i = y_i + h_i \quad \forall i \in \{1, ..., n\}$, where h_i represents a vector of elementary transfers among individuals such that $\sum_{i=1}^{n} h_i = 0$. It is clear to observe that x can be obtained from y by transferring amounts of income from richer to poorer people. In other words, it is possible to show that $x \leq_{ADO} y$ if and only if $h_{i+1} \leq h_i \quad \forall i \in \{1, ..., n-1\}$ with $\sum_{i=1}^{n} h_i = 0$.

This seems a good requirement to expect from an inequality criterion. In fact, it is possible to obtain, from a given income distribution vector, another one which shows less inequality, simply by transferring shares of income from a richer individual to a poorer one in an absolute progressive way.

Analogously for RDO, if, given two vectors $x, y \in \Re^n$, we interpret y_i as the individual income before tax and $x_i = f(y_i)$ as the same income vector after taxation[3], we see that $x \leq_{RDO} y$ is in accordance with progressive income taxation, if RDO is equivalent to $\frac{f(y)}{y}$ being non-decreasing on $(0, +\infty)$. In other words, if we suppose that x results from y through redistributive effects due to a progressive income tax, we can order the two vectors as RDO does. In such a case, even RDO will appear a suitable inequality criterion.

We now have all the elements for answering the last question posed before, namely: "Should we adopt ADO and RDO as alternative inequality criteria to LO in order to solve the quoted paradoxes?".

2.3. **An alternative solution to Lorenz order's paradoxes.** We have examined the suitable characteristics of ADO and RDO, candidated to represent an alternative to Lorenz ordering. We now want to investigate how to solve or avoid the paradoxical situations in measuring the inequality associated to different income distributions.

Critique of LO typically lies on the fact that it does not show all the information related to a given distribution. The Lorenz criterion is a tool that ranks income vectors as a "*whole*" and does not take into account all changes that could occur in every quantile of the distribution.

Indeed, if we consider a population of n individuals whose income profile is $y = (y_1, y_2, ..., y_n)$ with the elements ordered in an increasing ordering such that $y_i \leq y_j$ if $i \leq j$ and a vector x that represents a reallocation of the total income $\sum_i y_i$, such that $y \succeq_{LO} x$, then we can think that $x_i = y_i + h_i$, with $i \in \{1, ...n\}$ and $\sum_i h_i = 0$. Let us assume that there exist in y two individuals i, j whose incomes are y_i, y_j with $y_i \leq y_j$ and suppose transferring from i to j an amount of money such that $h_i < 0$ and $h_j > 0$. In such a case, if $x \preceq_{LO} y$ occurs, we have a disputable ordering, at least when focusing on individuals i and j. Such a reasoning is correct. However, the fact that x Lorenz dominates y simply means that the first distribution is more spread out with respect to the second one: the Lorenz dominance criterion does not tell who gives to whom. LO is a synthetic measure,

[3]In general, we can imagine f as any function reducing inequality, through a transformation $y_i = x_i + h_i$ with $\sum_i h_i = 0$.

a partial order that treats the vector distributions as a "whole" and does not care about inequality relative to each individual. Moreover, if we should take all types of income transfers into account, we cannot really use the LO criterion anymore, when the distribution x is derived from y through two elementary transfers, one positive and the other negative[4], as the Lorenz curves cross once. Actually, some paradoxes, in which composite transfers are admitted[5], violate what Shorrocks and Foster [18] define a *favorable composite transfer* (FACT), i.e. a kind of *mean-variance preserving transfer* that decreases the inequality of the distribution.

The issue of paradoxes is now evident.

If the vector of admissible transfers h is (precisely) defined, we may avoid some disputable rankings between different vector distributions. In fact, if we only admit transfers of the Pigou-Dalton type, then, in the case of reallocation (y, x) as defined above, x is a vector where the richest individual (the fourth) has transferred one unit of his income to the third (20 becomes 21), while the second one has given 1 money to the first (that goes from 10 to 11). This is perfectly coherent with the LO.

It is now evident that if we define the admissible transfers in a suitably restricted way, we rule out the pathological situations mentioned above.

3. The classes of indices consistent with ADO and RDO

Having shown a way to avoid what Chateauneuf [4] calls "paradoxical situations" in ranking income distributions of LO, we are now interested in characterizing the classes of functions consistent with absolute and relative differentials orderings.

According to Castagnoli and Muliere [3], it is sometimes better to strengthen the Pigou-Dalton Principle of Transfers in order to make clear how to reduce inequality of a given distribution.

Let us reproduce, once again, our four persons society and the two distributions $y = (10, 15, 20, 25)$ and $x = (11, 14, 21, 24)$. If we consider the subgroup of individuals $\{1, 2\}$ (i.e. those who have income 11 and 14 in x), then inequality decreases when we move from y to x. Nonetheless, inequality increases within group $\{2, 3\}$ (i.e. 14, 21), making the distribution x less attractive, at least from the point of view of individuals in such a position.

Then, in order to avoid unrankable situations like this, our proposal consists in strenghtening the Pigou-Dalton principle, fixing a threshold, a sort a poverty line around which transfers take place:

> "[...] *Every transfer from a person above the line to a person under the line is considered as decreasing or not increasing the inequality, provided that people are maintained in the original position with respect to the poverty line*" (Castagnoli and Muliere [3]).

In this way, we obtain, first of all, a clearer and unquestionable criterion to analyze the transfers and the inequality measures. Furthermore, fixing a threshold, we can avoid trivial distinctions between income vector distributions such as $y^1 = (10, 90, 1000, 1100)$ and $x^4 = (11, 89, 1000, 1100)$. Finally, by using the class of transfers about a threshold, we can characterize the functions preserving the absolute differentials ordering.

[4] A negative elementary tranfer is what Rothschild and Stiglitz [15] call a regressive transfer, which consists of tranferring a certain amount of money from a poor to a rich person.

[5] One positive and one negative.

For any given partial order $\preceq$ of a set $\Re^n$, real-valued functions φ defined on $\Re^n$ which satisfy $\varphi(x) \leq \varphi(y)$ whenever $x \preceq y$ are variously referred to as "*monotonic*", "*isotonic*" or "*order-preserving*". Many of the inequalities that arise from majorization can be obtained simply by identifying an appropriate order-preserving function. Our aim is to find inequalities of the form $\varphi(x) \leq \varphi(y)$, obtained using any order-preserving function φ for any vectors x and y such that $x \preceq_{ADO} y$ or $x \preceq_{RDO} y$.

We have seen that for $x, y \in \aleph$, ADO means that $(x_i - y_i)$ is non-increasing in i with $\sum_{i=1}^{n}(x_i - y_i) = 0$. These two conditions are equivalent to saying that $x_i = y_i + h_i$, i.e. there exists a vector $h \in \Re^n$, written as $h = [h_1, ..., h_n]$, whose first k elements (with $k \in \{1, n-1\}$), are not negative and the remaining $(n-k)$ are not positive such that $\sum_i h_i = 0$. At the same time, the conditions above imply that for some k, $k = 1, ..., n-1$, $y_k \leq x_k$ and for $j > k$, $y_j > x_j$. Such a restriction amounts to introducing an (implicit) threshold, a poverty line, with respect to which transfers take place.

We know that the Principle of Transfers of Pigou-Dalton establishes that if y_k is the income of individual k, $k = 1, ..., n$, if $y_i < y_j$, and if an amount δ of income is transferred from individual j to i, income inequality is diminished provided $\delta \leq y_j - y_i$. That means that:

Definition 2. *A vector y majorizes a vector x, denoted $x \preceq y$, if x can be derived from y by a finite number of transfers (each satisfying the restriction $\delta \leq y_j - y_i$)*

Let us denote by Υ_{PD}^n the set of Pigou-Dalton transfers, i.e. $(y, x) \in \Upsilon_{PD}^n$ if and only if x can be derived from y through transfers of Pigou-Dalton type. Moreover, suppose that the vector h of transfers belongs to the set:

$$H(x) = \left\{ h : h_1 \geq 0, h_2 \geq 0, ..., h_k \geq 0;\ h_{k+1} \leq 0, ...h_n \leq 0; \sum_i h_i = 0 \right\}$$

and given a k, let τ be a fixed threshold such that $y_k \leq \tau \leq y_{k+1}$. Then, we can represent the set of *transfers about the threshold* τ as follows:

$$\Upsilon_\tau^n = \bigcup_{n \geq 2} \left\{ (x, y) \in \Upsilon_{PD}^n : x_i \leq \tau \text{ if } x_i - y_i > 0;\ x_i \geq \tau \text{ if } x_i - y_i < 0 \right\}.$$

We are now looking for the functions that preserve the ordering of majorization given by Υ_τ^n[6].

Definition 3. *A real-valued function $\varphi : \Re^n \to \Re$ is an order-preserving function, satisfying the principle of transfers about a threshold τ, if $\varphi(y) \geq \varphi(x)$ whenever $(y, x) \in \Upsilon_\tau^n$.*

In particular, in order to determine the class of functions that satisfy the principle of transfers about τ and are consistent with ADO, we need the following definition:

[6] The general class of order-preserving functions is called Schur convex. Obviously, every Schur-convex function will be a function that satisfies the class of transfers about τ. What we are doing here is to restrict this class to the subclass of all functions preserving the absolute differentials ordering.

Definition 4. *An index $\varphi : \Re^n \to \Re$ is called additive if*

$$\varphi(y_1, ..., y_n) = \sum_{i=1}^{n} f(y_i). \tag{3.1}$$

and the following lemma[7]:

Lemma 1. *Let φ be a real-valued function defined on $\Re^n$. Then*

$$x \prec y \text{ on } \Re^n \text{ implies } \varphi(x) \leq \varphi(y)$$

if and only if, for all $z \in \Re^n$ and $k = 1, ..., n-1$

$$\varphi(z_1, ..., z_{k-1}, z_k + \epsilon, z_{k+1} - \epsilon, z_{k+2}, ..., z_n) \tag{3.2}$$

is decreasing in ϵ over the region

$$0 \leq \epsilon \leq \min[z_k - z_{k-1}, z_{k+2} - z_{k+1}], \quad k = 1, ..., n-2$$

$$0 \leq \epsilon \leq z_{n-1} - z_{n-2}, \qquad k = n-1$$

We are now ready to state the following:

Theorem 1. *Let $\tau \in \Re_+$ be a given threshold. Then, a non-increasing additive function $\varphi : \Re^n \to \Re$ satisfies the Υ_τ^n-principle of transfers if and only if*

$$\max_{y_i < \tau} f(x_i - y_i) \leq \min_{y_j > \tau} f(x_j - y_j) \quad \textit{for all } y \in \aleph.$$

Proof. We know that every Pigou-Dalton transfer can be decomposed into a finite number of elementary transfers[8], i.e. transfers that occur just between two individuals.[9] Since Υ_τ^n is a subclass of Pigou-Dalton transfers, we only have to consider the case $n = 2$ [10]

Then, $(y, x) \in \Upsilon_\tau^n$ is equivalent to the existence of some i, j, with $i < j$, and an $h \in \Re^n$ such that $h_i = -h_j$ and $h_k = 0$ for $k \neq i, j$. If $x_i = y_i + h_i \leq \min\{y_{i+1}, \tau\}$, $x_j = y_j - h_j \geq \max\{y_{j-1}, \tau\}$, then condition (3.2) in Lemma 1 holds if and only if

$$\max_{y_i < \tau}[f(y_i + h_i) - f(y_i)] \leq \min_{y_j > \tau}[f(y_j + h_j) - f(y_j)]$$

for all i, j and h and all $y \in \Re^n$.

This is equivalent to:

$$\max_{y_i < \tau} f(x_i - y_i) \leq \min_{y_j > \tau} f(x_j - y_j).$$

∎

[7] See Marshall and Olkin [8], Lemma 3.A.2 pg. 55.

[8] See Hardy, Littlewood and Pòlya [6] pg.47.

[9] In such a case h has only two non-zero entries.

[10] See Marshall and Olkin [8], chapter 3, section B pg.21-22.

We have considered two income classes, below and above a given threshold, in order to characterize the class of inequality indices consistent with ADO, namely the class of functions that is nondecreasing with every transfer about τ. A threshold can then be interpreted as a poverty line; each transfer from a person above the line to a person under the line is considered as increasing social welfare.

Let us turn now to the class of functions that preserves the *relative differentials ordering*.

By definition, x is majorized by y according to RDO if $\frac{x_{i+1}}{x_i} \leq \frac{y_{i+1}}{y_i}$ for all $i \in \{1, n-1\}$. This is tantamount to $\frac{y_i}{x_i} \leq \frac{y_{i+1}}{x_{i+1}}$, which means that the ratio between the components of x and y must be non-decreasing for all i. Saying that x is more spread-out than y means that x could be obtained from y through a redistribution of income due, for example, to a progressive taxation of individuals in the distribution y. Hence, it is straightforward to interpret y_i as the income before tax and $x_i = f(y_i)$ as the same income after taxation. For the moment, let us suppose that the function $f : [0, \infty) \to \Re_+$ is an inequality reducing transformation. The ordering $x \preceq_{RDO} y$ is then in accordance with a progressive income taxation, if RDO is equivalent to $\frac{f(y)}{y}$ being non-decreasing on $(0, +\infty)$.

In this way, we have obtained a criterion that ranks the vector distributions after an income redistribution of type $x_i = y_i + h_i$, where h_i is a vector of transfers such that $\sum_{i=1}^{n} h_i = 0$. We collect the set of transfers described above as:

$$\Upsilon_\star^n = \bigcup_{n \geq 2} \left\{ \begin{array}{c} (y, x) \in \Upsilon_{PD}^n : \frac{f(y)}{y} \text{ non-decreasing on } (0, +\infty), \text{ with} \\ f : [0, \infty) \to \Re_+ \text{ and } x_i = f(y_i) = y_i + h_i \text{ and } \sum_{i=1}^{n} h_i = 0 \end{array} \right\}.$$

and we call it a *RDO criterion for transfers.*

Now, in order to determine the class of indices of inequality consistent with RDO and the kind of transfers explained above, we specialize the class of inequality reducing transformations through the notion of *starshaped* function:[11]

Definition 5. *Let* $\mathcal{A} \subset \Re_+$. *A function* $f : \mathcal{A} \to \Re$ *is starshaped above at* 0 *and supported if* f *is continuous and*

$$f(z)/z \quad \textit{is non-decreasing at all } z.$$

Then, we show that:

Theorem 2. *Let* φ *in (3.1) an additive function, then it satisfies the RDO criterion for transfers for all* n *if and only if* f *is continuous and starshaped above at* 0.

Proof. $\Rightarrow$ If $\varphi(z) = \sum_i f(z_i)$ satisfies the redistributive (RDO) criterion described above and there exist $p, q \in \Re^2$ with $p = \frac{m}{m+l} q$, i.e. $p \leq q$, and $l, m \in \mathcal{N}$ such that $l + m \leq n$, then $pl = (q - p) m$. Choose $\zeta > 0$, such that $p > \zeta$ and $\alpha = \zeta$, $\beta = q - \zeta$, $\gamma = p + \zeta$. Then $0 < \alpha < \gamma < \beta$ and $(\gamma - \alpha) l = (\beta - \gamma) m$.

If we consider $(y, x) = (y, y + h)$, with $y_i = \alpha$ when $i \in \{k - l + 1, ..., k\}$, $y_i = \beta$ when $i \in \{k + 1, ..., k + m\}$, $h_i = \gamma - y_i$ when $i \in \{k - l + 1, ..., k + m\}$, and $h_i = 0$ elsewhere, then $\sum_{i=1}^{n} h_i = (\gamma - \alpha) l + (\gamma - \beta) m = 0$, hence $(y, y + h)$ is a transfer. Moreover, (y, x) belongs to the kind of ordering explained above (i.e. RDO) and by assumption:

[11]See Landsberger and Mailijson [7] and Marshall and Olkin [8].

$$0 \geq \varphi(y+h) - \varphi(y) = (l+m) f(\gamma) - lf(\alpha) - mf(\beta)$$

is equal to $mf(\beta) - mf(\alpha) \geq (l+m)[f(\gamma) - f(\alpha)]$. If we divide this last inequality for $(\gamma - \alpha)(l+m) = (\beta - \alpha) m > 0$, we obtain:

$$\frac{f(\beta) - f(\alpha)}{\beta - \alpha} \geq \frac{f(\gamma) - f(\alpha)}{\gamma - \alpha}$$

approaching to the limit, $\zeta \to 0$, this yields:

$$\frac{f(q)}{q} \geq \frac{f(p)}{p}$$

which means that f is starshaped above at 0;

$\Leftarrow$ If f is starshaped it means that it is supported by a linear function at 0 of the form $f(z) \equiv f(0) + z\beta$ for all z, with some $\beta \in \Re$.

Since f is starshaped above at 0, f gives less weight to income changes in the lower part of the distribution than in the upper one. This implies that the expression

$$\varphi(x) - \varphi(y) = \sum_{i=1}^{n} [f(y_i + h_i) - f(y_i)] \tag{3.3}$$

is less or equal then 0.

Then, as $\sum_i h_i = 0$, the transition from $x_i = x_i + h_i$ to y_i is a *mean preserving spread.* This completes the proof. ∎

This class of additive inequality indices is given by the set of all real functions which are starshaped above at 0. It is larger than the class of Schur convex functions as the RDO principle of transfers is more restrictive than the Pigou-Dalton one. This implies that less pairs of distributions are comparable. But, at the same time, some index which is not consistent with the Pigou-Dalton principle of transfers, may take into account the differences between the different groups of individuals belonging to a given distribution and clarify the preferences questioned in several empirical studies (e.g. Amiel and Cowell [1]) about the transfers reducing inequality.

4. Conclusion

In some situations, the Lorenz order involves paradoxes in ranking different income distributions. One proposal consists in using two different inequality criteria, namely ADO and RDO, in order to solve such "disputable situations". Here, we have tried to single out the suitable properties of absolute and relative differentials ordering. We found that a strenghtening of the Pigou-Dalton principle of transfers has entailed: a) a complete characterization of the class of functions consistent with ADO and RDO; and b) the disappearance of the above mentioned pathological situations.

The lesson to be drawn is the following: of course a partial ordering (such as LO, ADO and RDO), leaves some pairs unranked. Then, when choosing an inequality criterion, one has to pay attention to their limits and their overall normative and descriptive consistency. Finally, it could be useful to apply a combination of different principles of transfers in order to have more than one tool for equalizing income distributions.

References

[1] Amiel, Y., and F. Cowell (1992) - "Measurement of income inequality: experimental test by questionnaire" - *Journal of Public Economics* 47, 3-26.

[2] Atkinson, A.B. (1970) - "On the measurement of Inequality" - *Journal of Economic Theory* 2, 244-263.

[3] Castagnoli, E. and P. Muliere (1990) - "A note on inequality measures and Pigou-Dalton principle of transfers" - in *Studies in contemporary Economics* (Dagum, C. and M. Zenga eds.) Income and Wealth Distribution, Inequality and Poverty - Springer Verlag, Berlin.

[4] Chateauneuf, A. (1996) - "Decreasing inequality: an approach through non-additive models" - w.p.96.58 CERMSEM, Université de Paris I.

[5] Dasgupta, P., A.K. Sen, and D. Starret (1973) - "Notes on the measurement of inequality" - *Journal of Economic Theory*, 6, 180-187.

[6] Hardy, G.H., J.E. Littlewood, and G. Polya, (1934, 1952) - *Inequalities* - Cambridge University Press, London.

[7] Landsberger, M. and I. Meilijson, (1990) - "Lotteries, Insurance and Star-shaped Utility Functions" - *Journal of Economic Theory*, 52, 1-17.

[8] Marshall, A.W., and I. Olkin, (1979) - *Inequalities: Theory of Majorization and its Applications* - Academic Press: New York.

[9] Marshall, A.W., I. Olkin, and F. Proschan, (1967) - "Monotonicity of ratio of means and other applications of majorization" - in *Inequalities* (O. Shisha, ed.) pp.177-190. Academic Press, New York.

[10] Marshall, A.W., D.W. Walkup and R.J.B. Wets, (1967) - "Order-preserving functions: applications to majorization and order statistics" - *Pacific Journal of Mathematics*, vol.23, 569-585.

[11] Mosler, K. and P. Muliere, (1996) - "Inequality indices and the starshaped. principle of transfers" - *Statistical Papers*, 37, 343-364.

[12] Moyes, P. (1994) - "Inequality reducing and inequality preserving transformations of incomes: symmetric and individualistic transformations" - *Journal of Economic Theory*, 63, 271-298.

[13] Quiggin, J. (1988) - "A stronger characterization of increasing risk" - *Fourth International Conference on the Foundations and Applications of Utility, Risk and Decision Theory* - FUR 88, Budapest.

[14] Preston, I. (1990) - "Ratios, differences, and inequality indices" - Institute for Fiscal Studies Working Paper W90-9.

[15] Rothschild, M. and J.E. Stiglitz (1973) - "Some further results on the measurement of inequality" - *Journal of Economic Theory*, 6, 188-204.

[16] Sen, A.K. (1997) - *On Economic Inequality* - (Second eds), extended edition with J. Foster, Oxford: Clarendon Press.

[17] Sen, A.K (1980) - "Equality of What?" - in McMurrin, S. ed., *Tunner Lectures on Human Values*; vol.1, Cambridge: Cambridge University Press.

[18] Shorrocks, A.F. and J. Foster (1987) - "Transfer sensitive inequality measures" - *Review of Economic Studies*, 54, 485-497.

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