Effectivity Functions, Opportunity Rankings, and Generalized Desirability Relations

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Abstract

Generalized individual desirability relations are defined relying on a) effectivity functions (EFs) b) Galois lattices of EFs and c) opportunity rankings as defined on EFs. It is argued that such desirability relations enable an enlargement of the scope of game-theoretic approaches to the analysis of power allocation far beyond the narrow domain of voting procedures

1 Introduction

This paper is devoted to a presentation of the full potential of effectivity functions as a tool for game-theoretic modelling of individual power. Indeed, an *effectivity function (EF)* describes - for each coalition S in N- the set of subsets of the outcome set X within which S can 'force' the final outcome by means of some coordinate action of its members. Hence, an EF is a remarkably detailed and flexible representation of the a priori decision power allocation among individual players and coalitions as induced by the relevant game rules. However, since their introduction in the game-theoretic literature in the early '80s (see e.g. Moulin,Peleg(1982), Moulin(1983), Peleg(1984), Abdou,Keiding(1991)) EFs have been largely ignored in ongoing debates concerning measurement of 'power' in (voting) games and social situations. In our view, this is most regrettable, and the present paper's main aim is to provide some arguments which suggest that EF-models should be more widely considered.

As a matter of fact, the standard game-theoretic analysis of individual power rests upon (individual) desirability relations and power indices. While there is still a nonnegligible amount of controversy concerning the proper relationship between power indices and individual desirability relations (e.g. should the former be monotonic w.r.t. the latter?), both notions – as usually defined– refer to an underlying *simple game (SG)* which amounts to a list of 'winning' or 'all-powerful' coalitions out of a player set N (see e.g. Felsenthal,Machover(1998) for a recent and strongly opinionated critical review of power indices, and Taylor,Zwicker (1999) for a general, extensive treatment of simple games). Of course, simple games are an eminently tractable and elegant outcome-free construct, but *precisely for that reason* the scope of a simple-game-theoretic analysis of individual (and coalitional) power is definitely too restricted to be fully satisfactory. In our view, one basic limitation of the (standard) simple-game-theoretic approach is its virtually exclusive concern with voting-like interactive decision procedures (let's call Voting Domain (VD) this implicit restriction). Also, simple games are essentially confined to those voting schemes which are *neutral* w.r.t. outcomes and do *not* endow players with any *limited* veto power (let's denote Neutral-No-Limited-Veto (NNLV) this feature of the simple-game-theoretic approach).

Now, VD makes it uneasy for simple-game-theoretic models –except under trivial cases i.e.when there exists a 'dictator'– to deal with those bilateral 'power' relationships between players which are related to 'bossiness' i.e. the ability of a player (the 'boss') to affect the outcome in a way which is significant for another player *without* being affected himself. This is indeed no minor limitation, since arguably in common parlance 'power' relationships refer –more often than not– precisely to such 'bossy' relationships between players as opposed to –say– 'influence-rankings' of players in public decisions as represented by individual desirability relations of simple games or by values of power indices. To put it simply, and generally speaking, standard simple game-theoretic seem to be confined to analyzing the *comparative* 'has more power than'-relation among players (and coalitions) as opposed to the more commonly used 'has power over'-relation.

As for NNLV, it implies that the simple-game-theoretic approach is in fact confined to weighted majority and similar voting games since it is poorly adapted to the task of describing and analyzing (individual and/or coalitional) 'power' in general game forms whenever the latter allocate veto power among players and coalitions in a non-trivial way i.e. *not* in an 'all-or-none' manner. In particular, the simple-game theoretic approach is virtually silent about power allocation in an environment with private goods (unless one is prepared to represent such an environment by a unanimity simple game to the effect of licencing the highly implausible conclusion that the agents enjoy equal decision power whatever their respective initial (positive) endowments!).

Moreover, under VD and NNLV a coalition's decision power is either full or nil hence in particular only (two) inclusion-comparable sets are involved. As a consequence, there is just *one* natural way to define individual and coalitional desirability relations, namely in terms of set-inclusion (there are no proper extensions of set-inclusion to choose from, except for the trivial one i.e. universal indifference). Of course, the latter circumstance may be regarded as a bonus, but it should be clear by now that it does not come for free since it results precisely fron the heavy limitations which are embodied in VD and NNLV.

We claim that in fact EFs offer a convenient and promising escape route from some of the foregoing strictures. In fact, an EF framework suggests *many* approaches to the task of classifying and ranking individual (and coalitional) power in interaction structures as modelled by means of general game forms.

First, it is well-known–but worth emphasizing– that EF-models *allow a satisfactory treatment of decision mechanisms in the Voting Domain which do not satisfy NNLV.* Also, it can be shown that the usual individual (and coalitional) desirability relations carry over in natural ways to EFs and the larger class of interaction structures they model. In particular, it can be easily shown that in an EF-framework *transitive individual desirability relations [and 'normal'(in Taylor-Zwicker's sense) classes of coalitional desirability relations] obtain* in a quite straightforward manner. Moreover, by enabling a detailed representation of interaction structures featuring limited veto power, EFs provide the right environment for a proper treatment of *both 'bossy' and 'nonbossy' power relations among players within the same theoretical setting.* Indeed, 'bossy' i.e. asymmetric power relations among players can now be defined in a most natural way by positing that player i has power over player j if i can contribute or prevent some 'valuable opportunities' for j but not viceversa. Furthermore, such an EF-setting allows meaningful talk about 'fairness' of power allocations in standard economic environments with private goods *without having to rely on detailed information about individual preferences (as opposed to, say, computation of Shapley values of NTU-games of private good economies).*

However, it transpires that *several* non-trivial extensions of set-inclusion are available in this broader setting, hence specifying some particular *opportunity ranking* is now required in order to provide precise meaning to the notion of 'valuable opportunities'. Some available options will be discussed below: in particular, it will be shown that opportunity preorders induce *transitive individual desirability relations.*

Furthermore, one may observe that by definition EFs depend on the description of the outcome space, but –arguably– to an exceedingly large extent. Indeed, the present author has recently suggested that *Galois lattices of effectivity functions* might provide an useful further classification of individual and coalitional power (see Vannucci (1999): this is so because in a sense such a classification only relies on 'intrinsically' significant aspects of the outcome space. As it happens, *Galois lattices of EFs –via the notion of individual and coalitional rank functions –suggest another natural extensions of desirability relations* from simple games to *any game form (or correspondence).* Again, further extended individual desirability relations obtain by composing *EFs* and *opportunity rankings,* and looking at their Galois lattices. Clearly enough, within an EF-framework the supply of possibly significant individual desirability relations is quite rich.

The present paper is largely devoted to a short presentation of the generalizations of (individual) desirability relations mentioned above (and to discussion of a few elementary properties of theirs, e.g. transitivity). To repeat, I submit that such generalized desirability relations fulfill an useful dual role by

i) bridging the gap between game-theoretic notions of individual decision power and common parlance of (typically asymmetric) power relationships among agents in general (non-voting) social settings,

while at the same time

ii) enabling a meaningful analysis of power allocation in environments with private goods, thereby enlarging the scope for meaningful discussions of 'fairness' in power allocation to general social situations far beyond the restricted if classic domain of voting procedures.

2 Effectivity functions and desirability relations

Let (N, X) be a pair of non-empty sets (the sets of players and outcomes, respectively; we also assume $\#N \geq 2$ and $\#X \geq 2$ in order to avoid trivialities). A *(monotonic) simple game* on N is a set pair $G = (N, W)$, $W \subseteq P(N)$, such that $S \in W$ and $S \subseteq T$ entail $T \in W$ (W is said to be non-trivial iff it is non-empty) . The coalitions belonging to W are meant to represent the *winning* or all-powerful ones. The *individual desirability relation* $\succsim_G$*of a simple game G=(N,W)* is defined as follows : for any $i, j \in N$,

$i \succsim_G j$ iff for any $S \subseteq N$ s.t. $S \cap \{i, j\} = \emptyset : [S \cup \{j\} \in W$ only if $S \cup \{i\} \in W]$.

An *effectivity function* (EF) on (N, X) is a function $E : P(N) \to P(P(X))$ such that :

$EF1)$ $E(N) \supseteq P(X) \backslash \{\emptyset\}$; $EF2)$ $E(\emptyset) = \emptyset$; $EF3)$ $X \in E(S)$ for any S, $\emptyset \neq S \subseteq N$.

Moreover, E is a *well-behaved* EF if

$EF4)$ $\emptyset \notin E(S)$ for any S , $\emptyset \subset S \subseteq N$ is also satisfied.

An EF E on (N, X) is *monotonic* if for any $S, T \subseteq N$ and any $A, B \subseteq X$

$[A \in E(S)$ and $S \subseteq T$ entail $A \in E(T)]$ and

$[A \in E(S)$ and $A \subseteq B$ entail $B \in E(S)]$.

A monotonic EF E on (N, X) is *regular* if $\emptyset \neq A \in E(S)$ entails $X \backslash A \notin E(N \backslash S)$ for any $S \subseteq N$ and $B \subseteq X$, and *maximal* if $A \notin E(S)$ entails $(X \backslash A) \in E(N \backslash S)$ for any $\emptyset \neq S \subseteq N$ and $\emptyset \neq A \subseteq X$. Moreover, an EF E on (N, X) is *superadditive* if for any $S, T \subseteq N$ and $A, B \subseteq X$, $A \in E(S)$, $B \in E(T)$ and $S \cap T = \emptyset$ entail $A \cap B \in E(S \cup T)$.

Finally, an EF E on (N, X) is *simple* if there exists an order filter W of $(P(N), \supseteq)$ (namely a non-empty set $W \subseteq P(N)$ s.t. for any $S \subseteq T \subseteq N$ if $S \in W$ then $T \in W$) such that for any $S \subseteq N$, $A \subseteq X$, $A \in E(S)$ if and only if either $A = X$ and $S \neq \emptyset$ or $A \neq \emptyset$ and $S \in W$. Indeed, simple EFs amount to (non-trivial) simple games as endowed with a fixed outcome set.

Desirability relations provide the most widely studied formal counterpart to the notion of individual and coalitional "influence" in decision-making. In words, player (coalition) A is "at least as much desirable as" player (coalition) B if for any coalition C which is disjoint from both A and B, the coalition of A and C turns out to be at least as much "powerful" as the coalition of B and C (see e.g. Taylor,Zwicker(1999) for a thorough treatment of desirability relations in simple games).

In the extant literature, desirability relations have been defined and studied for simple games only. However, they can be easily extended to EFs in a fairly straightforward way. Indeed, the *individual desirability relation* $\succsim_E$of an EF E on (N, X) may be defined as follows : for any $i, j \in N$

$i \succsim_E j$ iff $E_i(S) \supseteq E_j(S)$ for every $S \subseteq N$ such that $S \cap \{i, j\} = \emptyset$,

(where for any $h \in N$, $E_h(.)$ denotes the *h-reduced EF* of E which is in turn defined as follows:

$E_h(S) = E(S \cup \{h\})$ for any $S \subseteq N$ such that $h \notin S \neq \emptyset$, and $E_h(S) = E(S)$ otherwise; it is easily checked that E_h is indeed an EF on (N, X) ; it should be noticed that I write E_i for $E_{\{i\}}$with a slight abuse of notation).

Clearly enough, desirability relations as defined above embody a dominance principle whose significance relies on the natural presumption that –for any coalition– having a larger image (w.r.t. set inclusion) under a certain EF is better. As it happens, such a notion can be furtherly generalized through the notion of an *opportunity ranking.* Here, an *opportunity ranking (OR)* on a basic outcome set simply denotes a *preordered set* $(P(X), \succcurlyeq)$ such that $\succcurlyeq$ extends $\supseteq$ (namely, $\succcurlyeq$ is reflexive, transitive, and $A \succcurlyeq B$ whenever $A \supseteq B$). Such a notion adds a further twist to the notion of an EF-based desirability relation in that it allows to express a notion of *comparative relevance* concerning outcome subsets and the attached decision power.

Thus, let $(P(X), \gtrsim)$ be an OR on X; then for any $S \subseteq N, A \subseteq X$:

$A \in (E \circ \gtrsim)(S)$ if and only if there exists $B \subseteq X$ such that $B \in E(S)$ and $B \sim A$ (where $\sim$ denotes the symmetric component of $\gtrsim$).

It is easily checked that the following fact holds true:

Claim 1 *Let E be an EF on (N, X) and $(P(X), \gtrsim)$ an OR. Then, $(E \circ \gtrsim)$ is a (possibly non well-behaved) EF.*

Proof. Observe that –by reflexivity of $\gtrsim$– $E \subseteq (E \circ \gtrsim)$. Hence $(E \circ \gtrsim)$ satisfies both EF1 and EF3. Moreover, $A \in (E \circ \gtrsim)(\emptyset)$ entails that there exists $B \sim A$ such that $B \in E(\emptyset)$, a contradiction since $E(\emptyset) = \emptyset$:thus, $(E \circ \gtrsim)$ also satisfies EF2. To see that $\emptyset \in (E \circ \gtrsim^*)(S)$ for some OR $(P(X), \gtrsim^*)$, just take a non-trivial set-filtral OR (i.e. $A \gtrsim^* B$ iff [either $A \supseteq B$ or $B \notin F$ where F is any order filter of $(P(X), \supseteq)$ such that $F \neq P(X)$]). $\square$

Then, one may rely on $(P(X), \gtrsim^*)$ in order to define a suitable OR $(P(P(X)), \gtrsim_{(\gtrsim^*)}$) (e.g. posit $\mathbf{A} \gtrsim_{(\gtrsim^*)} \mathbf{B}$ iff $\max_{\gtrsim^*} \mathbf{A} \gtrsim^* \max_{\gtrsim^*} \mathbf{B}$). A further interesting option consists in introducing an OR $(\widetilde{P}(P(X)), \gtrsim)$ directly (i.e. without any reference to an underlying OR $(P(X), \gtrsim^*)$ as mentioned above). In any case, given an OR $(P(P(X)), \gtrsim)$ one may define a new (individual) desirability relation as follows: for any $i, j \in N$

$i \succcurlyeq_{(E, \gtrsim)} j$ iff [for all $S \subseteq N$ such that $S \cap \{i, j\} = \emptyset : E_i(S) \gtrsim E_j(S)$].

Individual desirability relations of simple games are typically transitive. Indeed, transitivity of an individual desirability relation should be regarded as a remarkably nice property –if not the hallmark– of the former. It can be easily shown that both $\succcurlyeq_E$and $\succcurlyeq_{(E,\gtrsim)}$as defined above are in fact transitive, namely

Proposition 2 *Let E be an EF on (N,X),$(P(P(X)),\gtrsim)$ an OR, and $\succcurlyeq_E$,$\succcurlyeq_{(E,\gtrsim)}$as defined above. Then, $(N,\succcurlyeq_E)$ and $(N,\succcurlyeq_{(E,\gtrsim)})$ are preordered sets.*

Proof. Let us consider $(N,\succcurlyeq_{(E,\gtrsim)})$. Reflexivity follows trivially from the definition. Concerning transitivity, take $i,j,k \in N$ such that $i \succcurlyeq_{(E,\gtrsim)} j$ and $j \succcurlyeq_{(E,\gtrsim)} k$ i.e. for any $S,T \subseteq N$ with $S \cap \{i,j\} = \emptyset, T \cap \{j,k\} = \emptyset$, $E_i(S) \gtrsim E_j(S)$ and $E_j(T) \gtrsim E_k(T)$. Now, take any $U \subseteq N$ such that $U \cap \{i,k\} = \emptyset$. Two cases are to be distinguished: $j \notin U$ and $j \in U$. If $j \notin U$ then $U \cap \{i,j,k\} = \emptyset$ hence $E_j(U) \gtrsim E_k(U)$ and $E_i(U) \gtrsim E_j(U)$. Therefore, $E_i(U) \gtrsim E_k(U)$, a contradiction. If $j \in U$, consider $U' = U \setminus \{j\}$. Then, $U' \cup \{k\} \subseteq N \setminus \{i,j\}$, whence

$(*)E(U' \cup \{k\} \cup \{i\}) = E_i(U' \cup \{k\}) \gtrsim E_j(U' \cup \{k\}) = E(U' \cup \{k\} \cup \{j\}) = E_k(U' \cup \{j\})$.

Since $U' \cup \{i\} \subseteq N \setminus \{j,k\}$, it follows from $(*)$ that

$$E_i(U) = E_i(U' \cup \{j\}) = E(U' \cup \{j\} \cup \{i\}) = E(U' \cup \{i\} \cup \{j\}) =$$
$$= E_j(U' \cup \{i\}) \gtrsim E_k(U' \cup \{i\}) = E(U' \cup \{i\} \cup \{k\}) \gtrsim E_k(U' \cup \{j\}) = E_k(U).$$

The same argument also applies to $(N,\succcurlyeq_E)$ by taking $\gtrsim=\supseteq$ (just recall that –by definition– $\gtrsim$ denotes *any* preorder that includes $\supseteq$). $\square$

As mentioned in the Introduction above, EFs are arguably dependent on an excessive amount of details concerning the chosen description of the outcome space. This is indeed the main motivation leading to the introduction of another construct, namely *the Galois lattice(s) of an EF.* To start with, it should be noticed that : i) the set of all EFs on (N,X) is bijective to a set of binary relations on $(P(N),P(X))$, hence any EF on (N,X) can be equivalently regarded as a binary relation ; ii) therefore, the classic Birkhoff theorem on so called Galois connections applies. It follows that the functions $f_E : P(P(N)) \longrightarrow P(P(X))$, $g_E : P(P(X)) \longrightarrow P(P(N))$ as defined by the rules

$f_E(\mathbf{S}) = \{A \subseteq X : A \in E(S) \text{ for any } S \in \mathbf{S}\}$ for any $\mathbf{S} \subseteq P(N)$, and

$g_E(\mathbf{A}) = \{S \subseteq N : A \in E(S) \text{ for any } A \in \mathbf{A}\}$

enjoy the following list of properties:

a) the functions $K_E = g_E \circ f_E$ and $K_E^* = f_E \circ g_E$ are *closure operators* on $(P(N),\supseteq)$ and $(P(X),\supseteq)$, respectively (we recall here that a (Moore) closure operator on a preordered set $(Y,\geq)$ is a function $K : Y \to Y$ such that for any $y,z \in Y : K(y) \geq y$; $y \geq z$ entails $K(y) \geq K(z)$; $K(y) \geq K(K(y))$).

b) the corresponding *closure systems* - i.e. sets of closed sets - $\mathbf{C}(K_E) = \{\mathbf{S} \subseteq P(N) : \mathbf{S} = K_E(\mathbf{S})\}$, $\mathbf{C}(K_E^*) = \{\mathbf{A} \subseteq P(X) : \mathbf{A} = K_E^*(\mathbf{A})\}$ are (dually isomorphic) *complete lattices* under the join and meet operations defined as follows:

for any $\{\mathbf{S}_i\}_{i\in I} \subseteq \mathbf{C}(K_E)$, $\{\mathbf{A}_i\}_{i\in I} \subseteq \mathbf{C}(K_E^*)$,

$\vee_{i\in I}\mathbf{S}_i = K_E(\cup_{i\in I}\mathbf{S}_i)$, $\wedge_{i\in I}\mathbf{S}_i = \cap_{i\in I}\mathbf{S}_i$, $\vee_{i\in I}^*\mathbf{A}_i = K_E^*(\cup_{i\in I}\mathbf{A}_i)$, $\wedge_{i\in I}^*\mathbf{A}_i = \cap_{i\in I}\mathbf{A}_i$

(we recall that a lattice is a partially ordered set $(L, \geq)$ such that for any pair $\{x, y\} \subseteq L$, both a greatest lower bound (glb) -or meet- $\wedge\{x, y\}$and a lowest upper bound (lub) - or join - $\vee\{x, y\}$ exist; a lattice is *complete* if any subset of L has both a glb and a lub).

c) the lattices under b) are *dense,* i.e. have a unique atom and - if E is well-behaved- *co-dense*, i.e. have a unique co-atom (an *atom* of a lattice $(L, \geq)$ is a $\geq$-minimal non-bottom element of L, and a *co-atom* is-dually- a $\geq$-maximal non-top element of L).

The *Galois lattice* of an EF E is $\mathbf{L}(E) = (Iso\left[\mathbf{C}(K_E) \times \mathbf{C}(K_E^*)\right], \supseteq)$, where $Iso\left[\mathbf{C}(K_E) \times \mathbf{C}(K_E^*)\right]$ denotes the set of canonically isomorphic pairs of the closure systems of E,

and for any $\{(\mathbf{S}_i, \mathbf{A}_i)_{i \in I}\} \subseteq Iso\left[\mathbf{C}(K_E) \times \mathbf{C}(K_E^*)\right]$

$\bigvee_{i \in I}(\mathbf{S}_i, \mathbf{A}_i) = (K_E(\cup_{i \in I}\mathbf{S}_i), \cap_{i \in I}\mathbf{A}_i)$, $\bigwedge_{i \in I}(\mathbf{S}_i, \mathbf{A}_i) = (\cap_{i \in I}\mathbf{S}_i, K_E^*(\cup_{i \in I}\mathbf{A}_i))$.

Clearly enough, the Galois lattice $\mathbf{L}(E)$ (that is also sometimes called a *concept lattice)* is lattice-isomorphic to the closure systems of E. Hence, $\mathbf{L}(E)$ is complete, has a unique atom and, if E is well-behaved, a unique co-atom (see Vannucci(1999) for more details).

We are mainly interested in those EFs that can represent the decision power of coalitions under a certain decision mechanism, or *game correspondence.* A *game correspondence* on (N, X) is a correspondence $G : D \rightarrow\rightarrow X$ where $D \subseteq \prod_{i \in N} S_i$, and S_i is the set of "interactive behaviours" available to player $i \in N$. A *game form* is a single-valued game correspondence.

Now, the notion of decision power admits at least two distinct interpretations, namely "guaranteeing power" and "counteracting power" that in turn correspond to the ability to force maximin and minimax outcomes, respectively. Thus, the allocation of "guaranteeing power" under game correspondence G with domain D is represented by the $\alpha - EF$ of G - denoted by $E_\alpha(G)$- as defined by the following rule:

for any non-empty $S \subseteq N$,

$$(E_\alpha(G))(S) = \left\{ \begin{array}{c} A \subseteq X\text{: a } t^S \in \prod_{i \in S} S_i \text{ exists such that } (t^S, s^{N \setminus S}) \in D \text{ and} \\ G(t^S, s^{N \setminus S}) \subseteq A \\ \text{for any } s^{N \setminus S} \in \prod_{i \in N \setminus S} S_i, \end{array} \right\}.$$

Conversely, the allocation of "counteracting power" under game correspondence G with domain D is represented by the $\beta - EF$ of G, denoted by $E_\beta(G)$ and defined as follows :

for any non-empty $S \subseteq N$

$$(E_\beta(G))(S) = \left\{ \begin{array}{c} A \subseteq X : \text{ for any } s^{N \setminus S} \in \prod_{i \in N \setminus S} S_i \text{ some } t^S \in \prod_{i \in S} S_i \\ \text{exists such that } (t^S, s^{N \setminus S}) \in D \\ \text{and } G(t^S, s^{N \setminus S}) \subseteq A \end{array} \right\}.$$

It is easily checked that $E_\alpha(G)$ is regular, $E_\beta(G)$ is maximal, and both of them are *monotonic* and - provided that G is non-empty valued- *well-behaved* . Also, it is well-known that superadditivity and monotonicity of an EF E imply that a game correspondence G exists such that $E = E_\alpha(G)$: see Moulin(1983), Peleg(1984), and Otten,Borm,Storcken,Tijs(1995)) . Indeed, monotonicity of $\alpha - EFs$ and $\beta - EFs$ of game correspondences is our main reason for confining the ensuing analysis to monotonic EFs (as mentioned previously). Furthermore, the foregoing distinction between $\alpha - EFs$ and $\beta - EFs$ brings us to the general notion of a *polarity operator* for EFs, implicitly defined as follows (see e.g. Abdou,Keiding(1991)): the *polar* E^* of a monotonic EF E on (N, X) is an EF on (N, X) such that :

i) $E^*(\emptyset) = \emptyset$ and ii) for any non-empty $S \subseteq N$ and $A \subseteq X$, $A \in E^*(S)$ if and only if $X \backslash A \notin E(N \backslash S)$.

(Notice that $E = E^*$ if and only if E is both regular and maximal).

It should be remarked here that opportunity rankings allow a further extension of the foregoing constructs by considering the resulting *composed EFs* $E \circ \succcurlyeq$ as defined above.

Remark 3 *In view of the foregoing observations it must be the case that* $\mathbf{L}(E) = \mathbf{1} \oplus B(\mathbf{L}(E)) \oplus \mathbf{1}$ *for some lattice* $B(\mathbf{L}(E))$ *if* E *is well-behaved , and* $\mathbf{L}(E) = \mathbf{1} \oplus B(\mathbf{L}(E))$ *otherwise (where* $\mathbf{1}$ *denotes the degenerate 1-element lattice, and* $\oplus$ *denotes the linear or ordinal sum operation: see e.g. Birkhoff(1967), or Davey,Priestley(1990)). In any case, we shall refer to the lattice* $B(\mathbf{L}(E))$ *as the bulk of* $\mathbf{L}(E)$ *.*

Thus, as mentioned above, Galois lattices of EFs - and their bulks- provide us with an algebraic invariant that allows some significant new classifications of game correspondences. (Of course, a game correspondence G is entitled to - at least- two Galois lattices, the α-Galois lattice and the β-Galois lattice, that correspond to $E_\alpha(G)$ and $E_\beta(G)$, respectively. We shall refer to *the* Galois lattice of game correspondence G when $E_\alpha(G) = E_\beta(G)$). One parameter of the Galois lattice $\mathbf{L}(E)$ is particularly relevant to the ranking of individual and coalitional power under EF E, namely its *length* . We recall here the relevant definition (see again Birkhoff(1967), or Davey,Priestley(1990)).

Definition 4 *The length* $l(L)$ *of a lattice* L *is the least upper bound of the set of lengths of chains included in* L *(a chain is a totally ordered set; the length of a chain of* $k+1$ *elements is* k*).*

A most useful notion of *rank* for coalitions (and issues) can be introduced relying on the *length* $l(\mathbf{L}(E))$ of the Galois lattice of an EF E as defined above.

Definition 5 *Let* E *be an EF on* (N, X)*. The height* $h_E(x)$ *of* $x = (\mathbf{C}, \mathbf{C}') \in \mathbf{L}(E)$ *is the least upper bound of the set of lengths of chains in* $\mathbf{L}(E)$ *having* x *as their maximum. The rank* $r_E(S)$ *of a coalition* $S \subseteq N$ *is the height* $h_E(x)$ *of the highest* $x = (\mathbf{C}, \mathbf{C}') \in \mathbf{L}(E)$ *such that* $S \in \mathbf{C}$ *(a dual definition obtains for an issue* $A \subseteq X$ *).*

Relying on such a notion of rank, several further extended desirability relations may be introduced in a most natural way. Such relations seem to be particularly apt to scrutinize those aspects of decision power which are related to *positional* considerations. In particular, let us define the basic Galois-latticial individual desirability relation of an EF, namely $(N, \succcurlyeq_{G(E)})$ where for any $i, j \in N$

$i \succcurlyeq_{G(E)} j$ iff [for all $S \subseteq N$ such that $S \cap \{i,j\} = \emptyset : r_{E_i}(S) \geq r_{E_j}(S)$]

(of course an OR-augmented EF $E' = E \circ \gtrsim$ may also be considered in that con||nection).

Remark 6 *It is still to be clarified under which conditions Galois-rank based desirability relations of an EF are transitive.*

Thus, we have by now a quite long list of individual desirability relations based upon EFs which may deployed in order to enlighten several facets of individual decision-making power and influence in game situations. The following section will provide some examples of interaction structures whereby some EF-based individual desirability relations seem to offer useful tools for representation and analysis.

3 Some examples

This section is devoted to a short presentation of a few relevant examples whereby EFs and EF-related notions as defined above seem to exhibit a definite comparative advantage. In particular, we consider : i) Constitutional Effectivity Functions; ii) Authority Relationships under Incomplete Contracting; and iii) Economies with Private Goods.

3.1 Constitutional Effectivity Functions

Modern representative democracies rely on governance structures whose architectures may vary in many relevant respects. The most significant distinction is perhaps the one between *parliamentary* and *presidential* systems. Indeed, under *parliamentary* government forms executive-termination can be prompted by a non-confidence vote on the part of the legislature. By contrast, under *presidential* systems the executive is *not* subject to non-confidence votes and -as a result- some degree of separation of powers between legislature and executive typically obtains. More often than not, under presidential systems the head of executive is directly appointed by means of general elections, while the opposite is the case with parliamentary systems. However, direct election of the premier in *parliamentary* systems has been recently enacted (or considered) in order to enhance stability in multiparty environments (see e.g. Israel, Italy; such systems are variously referred to as *neo-parliamentary or mixed*). It is our contention that the foregoing government and their core-stability properties forms can be aptly represented and analyzed in terms of EFs (as opposed to simple games). Let us then single out for discussion two somehow polarly opposite types of government forms with a directly elected head of executive, namely a) the EF of a presidential system with perfect separation of powers and b) the EF of a parliamentary system with a directly elected premier and a fixed majority (see also Vannucci(2000a, 2000b)).

Definition 7 *(The EF of a presidential system with perfect separation of powers) Let 0^* denote the elected president of the executive, and $N = \{1, .., n\}$ the set of parties- or voting blocs- of a legislature of size h. The parties have weights- or number of seats- w_i, $i = 1, .., n$. We also suppose that the weight profile $\mathbf{w} = (w_i)_{i\in N}$ is strong (i.e. for any $S \subseteq N$ either $\sum_{i\in S} w_i \geq \lfloor \frac{h}{2} \rfloor + 1$ or $\sum_{i\in N\setminus S} w_i \geq \lfloor \frac{h}{2} \rfloor + 1$). Moreover, we assume a sharp distinction between the respective "jurisdictions" of the executive and legislature. Therefore, the outcome space is $X = Y \times Z$, where Y denotes the "jurisdiction" of the executive, and Z the "jurisdiction" of the legislature. Then, the EF $E^{PS}(\mathbf{w})$ of a presidential system with perfect separation of powers and weight profile $\mathbf{w}$ is defined by the following rule: for any $S \subseteq N \cup \{0^*\}$, $A \subseteq X$, $A \in (E^{PS}(\mathbf{w}))(S)$ if and only if one of the conditions i)-iv) listed below is satisfied :*
i) $A \neq \emptyset$, $0^* \in S$ and $\sum_{i\in S} w_i \geq \lfloor \frac{h}{2} \rfloor + 1$;
ii) $A \supseteq \{y\} \times Z$ for some $y \in Y$, and $0^* \in S$;
iii) $A \supseteq Y \times \{z\}$ for some $z \in Z$, and $\sum_{i\in S} w_i \geq \lfloor \frac{h}{2} \rfloor + 1$;
iv) $A = X$ and $S \neq \emptyset$.

Definition 8 *(The EF of a parliamentary system with a directly elected premier and a fixed majority) Let 0^* denote the elected premier, $N = \{1, .., n\}$ the set of parties -or voting blocs- of a legislature of size h (whose allocation of seats is represented by a n-dimensional strong weight profile $\mathbf{w}$ as under the previous definition), $M = M(\mathbf{w}) \subseteq N$ a (possibly minimal) majority coalition, X the outcome set, and $x^* \in X$ a "deadlock" outcome that corresponds to legislature-termination, i.e. new elections. Then, the EF $E^{PA}(\mathbf{w}, M)$ of a parliamentary system with directly elected premier and fixed majority $M = M(\mathbf{w})$ at weight profile $\mathbf{w}$ is defined by the following rule: for any $S \subseteq N \cup \{0^*\}$, $A \subseteq X$, $A \in (E^{PA}(\mathbf{w}, M))(S)$ if and only if one of the clauses i)-iii) described below is satisfied :*
i) $A \neq \emptyset$ and $S \supseteq M \cup \{0^*\}$;
ii) $x^* \in X$ and $S \cap (M \cup \{0^*\}) \neq \emptyset$;
iii) $A = X$ and $S \neq \emptyset$.

Observe that for any $\mathbf{w}$, $E^{PS}(\mathbf{w})$ and $E^{PA}(\mathbf{w}, M)$ share the same set of *winning* i.e. all-powerful coalitions. Hence, a simple game-theoretic approach is plainly unable to distinguish among the foregoing government forms. By contrast, a few basic differences between $E^{PS}(.)$ and $E^{PA}(., .)$ concerning the allocation of decision power are easily depicted relying on the EF-machinery and related notions, as testified by validity of the following (see Vannucci(1999)):

Claim 9 *Let $E^{PS}(\mathbf{w})$ and $E^{PA}(\mathbf{w}, M)$ be the presidential and parliamentary EFs as defined above. Then,*

i) $B(\mathbf{L}(E^{PS}(\mathbf{w}))) = \mathbf{2}^2$;
ii) $B(\mathbf{L}(E^{PA}(\mathbf{w}, M))) = \mathbf{3}$ (where $\mathbf{3}$ denotes the three-sized chain).Thus, as expected, $E^{PS}(\mathbf{w})$ exhibits a specialized pattern of power allocation, whereas under $E^{PA}(\mathbf{w}, M)$ power allocation is unspecialized (i.e. there are no equally Galois-ranked coalitions with different decision power).

In my view, such an example on Constitutional EFs confirms that *within Voting Domains that do not satisfy the Neutral-No-Limited-Veto restriction* EF-models and related notions may indeed offer a quite distinctive contribution.

3.2 Incomplete Contracting and Authority Relationships

Incomplete contracting refers to situations whereby some relevant variables are (typically) ex-post observable by the parties but nonverifiable in a court (for whatever reason, including of course inability of the parties to foresee some relevant contingencies and/or condition contractual claims on them). Incomplete contracting is currently seen by and large as the main rationale underlying the observed boundaries of the firms and the widespread existence of *authority relationships* between employers and employees within firms. An *authority relationship* can be regarded as the quintessential template of a "bossy" decision mechanism, in that the Employer/Principal pays a constant wage to the Employee/Agent, and is endowed with the power or authority to assign tasks to the latter (within a contractually defined range). To see that such an arrangement may indeed count as a significant example of a "bossy" mechanism as defined in the Introduction, just consider the case where the Employer is indifferent between certain tasks while the Employee is not: under such circumstances, the Employee is able to add (or detract) valuable opportunities (and ultimately affect the well-being of the Employee) at no cost for herself. Moreover, a version of this story *which relies on opportunity rankings as opposed to preferences* can be offered in a quite straightforward way. This can be made more precise by means of the following model.

Let e_1, e_2 denote the Employer and the Employee, respectively. As usual, we take the Employee's wage w to be bounded below by an exogenously determined reservation wage w^*(indeed, we may posit without any significant loss of generality $w = w^*$) . Then, let $\mathbf{T} = \{1, .., T\}$ denote the (finite) set of tasks. Each task t is characterized by a (finite) action set $A^t = \left\{a_1^t, .., a_{n_t}^t\right\}$, a (finite) set of possible task-specific-outcomes $Y^t = \left\{y_1^t, .., y_{m_t}^t\right\}$, and a (finite) stochastic technology $p^t = (p_1^t = (p_{11}^t, .., p_{1m_t}^t), .., p_{n_t}^t = (p_{n_t m_t}^t, .., p_{n_t m_t}^t))$ where p_{ij}^t denotes the probability of t-outcome y_j^t if e_2 chooses action a_i^t, $i = 1, .., n_t, j = 1, .., m_t$ (hence in particular –for any i, j– $p_{ij}^t \geq 0$ and $\sum_j p_{ij}^t = 1$; moreover, there is an obvious one-to-one correspondence between actions a_i^t and probability distributions p_i^t). Thus, the –contractually specified– set of *feasible* task-assignments is a family $\Omega \subseteq P(\mathbf{T})$ of subsets of $\mathbf{T}$. An EF E° can be attached in a most natural way to such an interaction structure as follows. The player set is $N = \{e_1, e_2\}$, and the outcome set is $X = \Omega \times F$, where

$F = \left\{ f \in [\prod_{t \in \mathbf{T}} A^t]^{\Omega} : \text{for any } C \in \Omega, \, \#f(C) = \#C \text{ and } f(C) \cap A^t \neq \emptyset \text{ iff } t \in C \right\}$.

Then, $E^{\circ}(N) = P(X) \setminus \{\emptyset\}$, $E^{\circ}(\{e_1\}) = \{Y \subseteq X : Y \supseteq \{\omega\} \times F \text{ for some } \omega \in \Omega\}$,

$E^{\circ}(\{e_2\}) = \{Y \subseteq X : Y \supseteq \Omega \times \{f\} \text{ for some } f \in F\}$, and $E^{\circ}(\emptyset) = \emptyset$.

Clearly, $e_1 \parallel_{E^{\circ}} e_2$ i.e. not $e_1 \succcurlyeq_{E^{\circ}} e_2$ and not $e_2 \succcurlyeq_{E^{\circ}} e_1$(by definition of E°). However, such an EF model is also consistent with the following admittedly extreme scenario. Let us assume that all the tasks are technology-equivalent i.e. for any $s, t \in T$, $p^s = p^t$ (which entails $\#Y^s = \#Y^t$, and $\#A^s = \#A^t$), *and that actions are observable*- a most characteristic if strong assumption of the extant incomplete contracting literature– so that one may also assume without much loss of generality $\#A^t = \#A^s = 1$. Next, consider an opportunity ranking $(P(X), \gtrsim)$ such that for some $S, S' \subseteq \Omega, F' \subseteq F : (S \times F') \gtrsim (S' \times F')$ and *not* $(S' \times F') \gtrsim (S \times F')$ while $(S \times F') \sim (S \times F'')$ for any $S \subseteq \Omega, F', F'' \subseteq F$ (notice that under the foregoing assumptions once the Employer has chosen the tasks the Employee can only perform *the* corresponding actions resulting in equivalent probability distributions).

Then, $(E^{\circ} \circ \gtrsim)(\{e_1\}) =$

$\{Z \subseteq X : \text{there exist } Y \subseteq X, \, \omega \in \Omega \text{ such that } Z \sim Y \supseteq \{\omega\} \times F\}$

$\supseteq \{Z \supseteq X : Z \sim S \times F, \emptyset \neq S \subseteq \Omega\} = \{Z \subseteq X : Z \sim S \times F' \neq \emptyset, \, S \subseteq \Omega, \, F' \subseteq F\} =$ $(E^{\circ} \circ \gtrsim)(N)$.

Hence, $(E^{\circ} \circ \gtrsim)(\{e_1\}) = (E^{\circ} \circ \gtrsim)(N) \supset$

$\supset \{Z \subseteq X : \text{there exist } Y \subseteq X, f \in F \text{ such that } Z \sim Y \supseteq \Omega \times \{f\}\} =$

$= (E^{\circ} \circ \gtrsim)(\{e_2\}$.

It follows that $[e_1 \succcurlyeq_{E^{\circ} \circ \gtrsim} e_2 \,]$ and *not* $[\, e_2 \succcurlyeq_{E^{\circ} \circ \gtrsim} e_1]$.

Thus, we have been eventually able to single out a *bilateral asymmetric* power relationship arising from desirability relations of OR-augmented EFs. In my view, the somehow concocted nature of this example partly testifies to the genuine difficulty to disentangle truly asymmetric, hierarchical power relationships within bilateral interaction structures (a point vividly made a long time ago by philosopher G.W.F.Hegel in his famous *Phænomenologie des Geistes(1807)*). On the other hand, such an example also confirms the remarkable breadth and flexibility of EF models of decision power.

3.3 The Walrasian Correspondence for Private-Good Economies

The notion of an effectivity function, and computation of its Galois lattice(s) are easily extendable to solution concepts and correspondences. This is so because whenever the "objects" to be "solved" include a description of non-verifiable individual characteristics (e.g. preferences), the latter can be regarded as the output of strategic behaviour. As a result, the solution concept under consideration can be aptly interpreted as a revelation-game correspondence. In particular, the Galois lattice of such a solution correspondence provides, once again, a succinct description of the structure of coalitional power when the actual behaviour of players is well predicted by the given solution concept (hence, the coalitional power discussed here is of a *conditional* sort) . This subsection is devoted to an application of those ideas to some domains of pure exchange private-good economies. In particular, we shall focus on the Walrasian (equilibrium) correspondence.

To begin with, a few basic definitions are to be recalled. A *pure exchange private-good-economy* is a tuple $\mathbf{e} = (N, (X_i)_{i\in N}\ , (\succcurlyeq_i)_{i\in N}\ , (\omega_i)_{i\in N}\ )$ where for each agent $i \in N$, $X_i = \mathbb{R}^k_+$ is her consumption set whose dimension k denotes the number of available private goods, $\succcurlyeq_i$is the total preference preorder of i on the allocation space $X = \prod_{i\in N} X_i$, and ω_i is her endowment. The usual selfishness, monotonicity, continuity, and convexity restrictions on each preference preorder $\succcurlyeq_i$are also assumed.

We denote by $\mathbb{E}(\boldsymbol{\omega})$ the set of all n-agent pure exchange private-good economies with endowment profile $\boldsymbol{\omega} = (\omega_i)_{i\in N}$. A *feasible allocation* of an economy $\mathbf{e} \in \mathbb{E}(\boldsymbol{\omega})$ is a profile of consumption programs $\mathbf{x} = (x_i)_{i\in N} \in X$ such that $\sum_{i\in N} x_i \leq \sum_{i\in N} \omega_i$. The set of feasible allocations of an economy $\mathbf{e} \in \mathbb{E}(\boldsymbol{\omega})$ will be denoted by $F(\boldsymbol{\omega})$. We are now ready to introduce the familiar Walrasian solution correspondence for private-good economies.

Definition 10 *(Walrasian correspondence on* $\mathbb{E}(\boldsymbol{\omega})$ *) A Walrasian equilibrium of an economy* $\mathbf{e} \in \mathbb{E}(\boldsymbol{\omega})$ *is a pair* $(\mathbf{p}^*, \mathbf{x}^*) \in \mathbb{R}^k_+ \times X$ *such that* $\mathbf{p}^* \neq \mathbf{0}$*,* $\mathbf{x}^* \in F(\boldsymbol{\omega})$ *and* $\mathbf{x}^* \succcurlyeq_i \mathbf{y}$ *for any* $i \in N$ *and any* $\mathbf{y} \in F(\boldsymbol{\omega})$ *with* $\mathbf{p}^* \cdot \mathbf{y}_i \leq \mathbf{p}^* \cdot \boldsymbol{\omega}_i$ *. A feasible allocation* $\mathbf{x} \in F(\boldsymbol{\omega})$ *is a Walrasian allocation of the economy* $\mathbf{e} \in \mathbb{E}(\boldsymbol{\omega})$ *if* $(\mathbf{p}, \mathbf{x})$ *is a Walrasian equilibrium of* $\mathbf{e}$ *for some non-null non-negative price vector* $\mathbf{p}$ *. The set of Walrasian allocations of an economy* $\mathbf{e} \in \mathbb{E}(\boldsymbol{\omega})$ *is denoted by* $w(\mathbf{e})$*. The Walrasian correspondence* $W(.)$ *is defined by the following rule :* $W(\mathbf{e}) = w(\mathbf{e})$ *. (Of course, it is well-known from equilibrium existence theorems that on the given standard domain of economies* $W(.)$ *is non-empty-valued).*

The Walrasian correspondence is usually meant to capture somehow the working of prices as coordination devices in a noncooperative environment. Since the typical stories underlying Walrasian equilibria treat players as price-takers, and the price system is taken to be one and the same for all of them, power relationships are usually regarded as simply alien from the perspective of Walrasian analysis. However, EF models and related desirability relations enable –in a most natural way– a meaningful discussion of power relationships even in such a standard general equilibrium setting with (only) private goods. To see this, take an endowment profile ω^* such that $\omega^*_i \gg \omega^*_j$ for some $i, j \in N$ i.e. $\omega^*_{ih} > \omega^*_{jh}$ for any good $h, h = 1, .., k$. Under such circumstances, one should like to be able to say that agent i is 'more powerful/influential' than agent j, quite independently of preferential characteristics. Apparently, desirability relations should provide –again– the right language to express such notion (while–to repeat– Shapley values of NTU games don't, since they obviously *do rely on detailed information concerning the prevailing preference profile on allocations*). But what desirability relations should be considered? Under the most natural interpretation, the simple game attached to a private-good economy is a *unanimity* simple game, not a promising starting point given the present aim of being able to discriminate between the decision power available to different players. Again, EF models come to the rescue providing a much more general and flexible language which helps us to express the foregoing notion of differential 'power' between agents i and j as mentioned above. To begin with, recall that the selfishness hypothesis, and the usual assumption of identical consumption sets (i.e. $X_i = \mathbb{R}^k_+$, $i = 1, .., n$) allow us to focus on $\mathbb{R}^k_+$. Now, consider the α-EF (see definition above) E^W of the Walrasian correspondence on $\mathbb{E}(\omega^*)$. By definition, for any coalition $S \subseteq N$ of agents, $E^W(S)$ denotes the set of feasible consumption programs at *some* Walrasian equilibrium price vector $\mathbf{p} = \mathbf{p}(\mathbf{e})$ of *some* economy $\mathbf{e} \in \mathbb{E}(\omega^*)$, given an endowment vector $\omega^*_S = \sum_{i \in S} \omega^*_i$. It follows that –for any $S \subseteq N$ such that $S \cap \{i, j\} = \emptyset$– $E^W_i(S) \supset E^W_j(S)$, whence $i \succ_{E^W} j$ (i.e. $i \succcurlyeq_{E^W} j$ and *not* $j \succcurlyeq_{E^W} i$) as required.

Once again, EF-related (generalized) desirabilities provide a language which allows us to import "power talk" into new and previously unreached realms.

4 Concluding Remarks

As mentioned in the Introduction, this paper has been mainly concerned with the problem of devising a language enabling significant discussions of (individual) decision power in a suitably comprehensive array of interaction structures. The main claim of the previous analysis is that effectivity functions (EFs) and related notions do indeed provide a remarkably apt language for such a "power talk" across different domains. In particular, the role of generalized EF-based desirability relations has been emphasized, showing that:

i) (at least) some EF-based individual desirability relations are in fact *transitive* (a nice typical property of their simple-game theoretic counterparts)

ii) various EF-based individual desirability relations effectively help us to express in a consistently precise manner strongly held intuitions on the allocation of decision power in several situations including government forms, authority relations, and private-good-economies.

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