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Choices under Risk and Uncertainty with Windfall Gains and Catastrophic Losses

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Abstract - This paper investigates a decision-making process involving both risk and uncertainty. The decision-maker is supposed to split events between 'familiar', and 'unfamiliar' ones, and she/he is assumed to behave differently with respect to them. In particular, it is showed that the DM overweighs unfamiliar gains and losses in his/her expected utility, formalized by means of the Choquet expected utility functional. If a specific subset of capacities is considered a further representation of the CEU is obtained in which the whole weight of uncertainty is placed on the windfall gain and the catastrophic loss.

Keywords: Knightian uncertainty, risk, capacity, Choquet expected utility.

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Expected utility theory combines linearity in probabilities and a utility function, which is either concave or convex if a decision-maker is risk averse or seeking. However, maximization of the expected utility as a criterion of choice among alternatives involving risk, fails to explain decision makers (DM)'s behavior when they face problems of *risk* and *uncertainty*, where the latter means that the DM finds him/herself in an "unfamiliar" world in which he/she cannot rely on a unique prior distribution.

Standard representation theorems used to formalize the utility of an action (technically, act) may not be fully satisfactory because either they misrepresent catastrophic and windfall events or they are insensitive to small probability outcomes.[i] It has been agreed that “people are limited in their ability to comprehend and evaluate probabilities of extreme outcomes, highly unlikely events may be ignored or else over estimated and difference between high probability and certainty is either neglected or exaggerated” (Kahneman and Tversky, 1979, p.282).

In two seminal papers Friedman and Savage (1948) and Markowitz (1952) proposed a S-shaped utility function for rationalizing the purchase of both insurance and lottery tickets, based on the idea that people have a reference point (*status quo*) with respect to which they compare relative gains and losses, exhibiting both risk loving and aversion.[ii]

This paper generalizes, in the mark of the Choquet Expected utility (CEU), the idea that a decision maker might have a reference point with respect to which he/she compares his outcomes (as it is in Wakker and Tversky (1993)). We will show that whenever a DM is facing a decision problem which encompasses both customary outcomes (such as, for instance, relatively small gains and losses) and less familiar ones ("extremal" consequences, e.g. "huge" gains or losses) we can represent his/her preferences by means of a specific utility functional. In this functional familiar outcomes are weighted according to their linear expectation and unfamiliar ones are compounded such as to imply an over/underestimation of the expected utility, depending on their relative "distance" from the reference point (gain/loss).

Restricting attention to a specific attitude towards uncertainty (i.e. a specific sub-set of "capacities"), we will show that the CEU functional provides a simple, intuitive representation of the decision problem, that corresponds to a linear combination of the expected utility and the utility of the most extreme outcome, the "windfall" gain or the "catastrophic" loss. The paper is organized as it follows. The next section explains how the CEU model can formalize choices involving *risk* and *uncertainty*, pointing out decision problems involving both familiar and unfamiliar outcomes and summarizing our outcomes in two representation theorems. Conclusive remarks follow in Section two. Proofs are collected in appendix.

1. KNIGHTIAN UNCERTAINTY AND CAPACITIES

Like in prospect theory, it is assumed that the decision-maker takes into account changes in wealth or welfare. We consider real outcomes, for the sake of simplicity, and assume that either the current income or wealth (*status quo*) are good proxies to define the reference point. Relative outcomes are defined as gains and losses from the reference point. We can distinguish among three classes of outcomes: customary, unfamiliar and positive, unfamiliar and negative. By customary outcomes (familiar world) we mean outcomes not ‘very far’ from the reference point. These outcomes are related to the subset of events the probability of which is considered by the decision-maker to be ‘sufficiently high’ and reliable, on the basis of her/his own personalized life experience. In the unfamiliar world we represent outcomes, which might be positive or negative with respect to the reference point, but that are related to events the probability of which is perceived to be ‘low’ and not fully reliable.[iii]

The decision-maker is assumed to face "ambiguity" or "Knightian" uncertainty[iv], which is represented by means of a non-additive measure or capacity on the set of all events. Uncertainty can arise from attitude towards ambiguous events, omitted states of the world and misspecification of the space state (e.g. Ellsberg paradox).

The decision-maker has well-defined risk and uncertainty attitude. Unlike standard models, the capacity is strictly non-additive on unfamiliar events and linear on events related to customary outcomes. As a result, the decision-maker gives different decision weight to events closer to his experienced ordinary world than "extreme" ones. In other words, we study a decision-maker perceiving genuine uncertainty with respect to unfamiliar losses and gains, being uncertainty neutral across the customary outcomes.[v]

Let us consider this more formally. Let $\Omega=\{\omega_1,\ldots,\omega_n\}$ be a non empty finite set of states of the world and let $S=2^{\Omega}$ be the set of all events. An act (or "prospect") $f\in F$ is a function assigning a consequence to each state, $f:\Omega\to C$, being $C \subseteq \Re$ the set of consequences. The decision-maker chooses acts in the set F. A set-function $v:S\to\Re_+$ is a capacity, or a non-additive measure, if it satisfies the following conditions:

i) $v(\varnothing)=0,\ v(\Omega)=1$ (normalization);

ii) $\forall A,B\in S$ such that $A\subset B$, $v(A)\leq v(B)$ (monotonicity).

A capacity is convex (concave) if for all $A,B\in S$ such that $A\cup B\in S$ and $A\cap B\in S$ $v(A\cup B)+v(A\cap B)\geq (\leq) v(A)+v(B)$. It is superadditive (subadditive) if $\mu(A\cup B)\geq(\leq)v(A)+v(B)$ for all $A,B\in S$ such that $A\cup B\in S$ and $A\cap B=\varnothing$. A capacity is additive (is a probability) if for all $A,B\in S$, $v(A\cup B) + v(A\cap B) = v(A) + v(B)$.

Assume $u:C\to\Re$ is an utility function, let v be a capacity on S, and $f\in F$ an act. The Choquet expected utility (CEU) is the following functional:

$$\text{CEU} \equiv \int_{\Omega} u(f(\omega))\mathrm{d}v(\omega) = \sum_{k=1}^{n} u(f(\omega_k))[v(\bigcup_{j=1}^{k}(\omega_j)) - v(\bigcup_{j=1}^{k-1}(\omega_j))] \quad (1)$$

where $u(f(\omega_n)) \geq ... \geq u(f(\omega_0))$ and, for notational simplicity, $v(\omega_0) = 0$.

The DM's preferences can be represented by a unique capacity v and a unique (up to a positive affine transformation) utility function $u(\cdot)$[vi].

The convexity (concavity) of the capacity, that implies superadditivity (subadditivity) of the Choquet integral, captures the decision-maker’s attitude towards uncertainty[vii], as can be easily seen extending the CEU representation in (1). As a consequence, a pessimistic (uncertainty averse) DM over-weights the worst outcome, on the contrary an optimistic (uncertainty loving) decision-maker over-weights the best consequence.

Define two acts $x, y \in F$ as comonotonic[viii], if there are no $\omega_1, \omega_2 \in \Omega$, so that $x(\omega_1) > x(\omega_2)$ and $y(\omega_1) < y(\omega_2)$, and denote by $X \subseteq F$ the subset of all comonotonic acts.

We suppose that, for each $x \in X$, consequences are ordered from the best outcome to the worst one, that is, $x_1^+ \geq ... \geq x_n^-$ (where the uppercase positive or negative sign stands for "gain" or "loss", with respect to the initial wealth, respectively), and the set of consequences is partitioned in three closed intervals, such that $[x_1^+, x_i^+]$ is the subset of unfamiliar gains, $[x_{i+1}^+, x_j^-]$ is the subset of customary outcomes (both losses and gains, thus encompassing the reference point) and $[x_{j+1}^-, x_n^-]$ is the subset of unfamiliar losses. More formally, for each x there exist three subsets of $\Omega = \{\{\omega_1,...,\omega_i\}, \{\omega_{i+1},...,\omega_j\}, \{\omega_{j+1},...,\omega_n\}\}$, namely $\Omega^+ = \{\omega_1,...,\omega_i\}$, $\Omega^{+-} = \{\omega_{i+1},...,\omega_j\}$, $\Omega^- = \{\omega_{j+1},...,\omega_n\}$, such that $(\Omega^+ \cap \Omega^-) = (\Omega^+ \cap \Omega^{+-}) = (\Omega^{+-} \cap \Omega^-) = \varnothing$; thus $\{\Omega^+, \Omega^{+-}, \Omega^-\}$ is a partition of Ω. For each $x \in X$, Ω^- represents the set of unfamiliar negative "states", i.e. states that imply "unusual" losses (since $x^-:\Omega^- \rightarrow [x_{j+1}^-, x_n^-]$

holds true); Ω^+ is the set of unfamiliar positive ones ($x^+:\Omega^+ \rightarrow [x_1^+, x_i^+]$) and Ω^{+-} the set of familiar consequences ($x^{+-}:\Omega^{+-} \rightarrow [x_{i+1}^+, x_j^-]$).

We assume that the decision-maker is pessimistic on unfamiliar losses, neutral on customary outcomes and optimistic on unfamiliar gains. The DM's attitude towards uncertainty is represented by a capacity v which is non-additive on S, in particular, it is locally concave on $[x_1^+, x_i^+]$, locally convex on $[x_{j+1}^-, x_n^-]$ and is both concave and convex on $[x_{i+1}^+, x_j^-]$, i.e., it is additive on customary consequences.

We show now that a DM whose attitude towards risk and uncertainty is represented by the capacity specified above will express his/her preferences by means of a CEU which coincide with an EU functional for customary outcomes, but in which unfamiliar outcomes (both gains and losses) are overweighted with respect to the weight that would have been attached had these had been familiar outcomes:

Theorem 1. Assume the DM can rank his/her outcomes according to the following partition C = $[x_1^+, x_i^+] \cup [x_{i+1}^+, x_j^-] \cup [x_{j+1}^-, x_n^-]$, and that a CEU exists[ix] where v is a convex/concave capacity defined over 2^Ω that is concave on $[x_1^+, x_i^+] \cup [x_{i+1}^+, x_j^-]$ and convex on $[x_{i+1}^+, x_j^-] \cup [x_{j+1}^-, x_n^-]$. Then the DM's CEU can be represented as:

$$\sum_{m=1}^{i} u(x_m) \left[v\left(\bigcup_{r=m}^{n}(\omega_r)\right) - v\left(\bigcup_{r=m+1}^{n}(\omega_r)\right)\right] + \sum_{m=i+1}^{j} u(x_m)\pi(\omega_m) + \sum_{m=j+1}^{n} u(x_m) \left[v\left(\bigcup_{r=1}^{m}(\omega_r)\right) - v\left(\bigcup_{r=1}^{m-1}(\omega_r)\right)\right]$$

where π is an additive probability and v is convex.

Proof. In Appendix.

The representation of the DM's CEU in Theorem 1 shows that the DM will evaluate uncertain outcomes having as a reference the expected utility that he/she can construct on the set of customary consequences, overweighing those consequences that he/she conceives as particularly positive or negative. This is coherent with the definition of such a DM as being optimist on gains and pessimist on losses. In fact, in the CEU representation of Theorem 1, both the l.h.s. and the r.h.s. (the unfamiliar gains and the catastrophic losses) are overweighed w.r.t. the weight that would have been attached by an additive measure (i.e. if gains or losses would have not been "unusual" but on the contrary "ordinary" ones).

By restricting the set of convex capacity to a specific subset, called "simple" and the set of concave ones to the subset of "dual simple", we show that the CEU assumes a further simplified representation which is a linear combination of expected utility and the most unfamiliar, "extreme", outcomes, in this sense really "windfall" or "catastrophic" losses. Formally, assume a windfall gain is the best possible gain among the set of unfamiliar positive outcomes, i.e. is $\overline{x} = \{\max x |\ x \in [x_1^+, x_i^+]\}$, and define a catastrophic loss as the worst possible loss among the set of unfamiliar losses, i.e. $\underline{x} = \{\min x |\ x \in [x_{j+1}^-, x_n^-]\}$

Let v be a simple capacity, such that

$$v(A) = \gamma\,\pi(A)\ \forall\ A \subset \Omega,\ \mathrm{v}(\Omega) = 1 \tag{2}$$

where $\pi(\cdot)$ is an additive probability distribution, and $\gamma \in [0,1]$.

See that a simple capacity is convex. The parameter γ assumes a natural interpretation in terms of DM's degree of confidence on his/her probabilistic assessment (Dow and Werlag (1992), Marinacci (2000)). Thus, $\gamma = 1$ means full confidence. i.e. no-uncertainty, and the capacity reduces to the additive probability $\pi(\cdot)$, while $\gamma = 0$ means full uncertainty i.e. no positive weights are placed to any set but the whole state-space.

A dual simple capacity is the following capacity $v'(\cdot)$:

$v'(A) = 1 - \gamma\pi(\Omega \backslash A)\ \forall\ A \subset \Omega,\ v'(\Omega) = 1$ (3)

The dual simple capacity is concave, and the parameter γ expresses the player's degree of confidence.

Suppose that the DM is pessimist on unfamiliar losses, and express it by means of a simple capacity $v(\cdot)$. Similarly, he/she is optimist on unfamiliar gains, and the concave capacity on unfamiliar gains is a dual simple one. Formally, the DM's capacity on 2^{Ω} is the following:

$$v(A) = \begin{cases} \gamma\pi(A)\forall A \subseteq \Omega^{-} \\ \pi(A)\forall A \subseteq \Omega^{+-} \\ 1-\gamma\pi(A^{c})\forall A \subseteq \Omega^{+} \end{cases} \quad (4)$$

As a result, the DM's CEU is a linear combination of expected utility, windfall gain and catastrophic loss according to the following theorem:

Theorem 2. Assume conditions of theorem 1 hold true and that DM attitude towards uncertainty is represented by the capacity defined in (4). Then his/her CEU is:

$\text{CEU} = \gamma \sum_{k=1}^{n} u(x_k)\pi(w_k) + (1-\gamma)[u(\overline{x}) + u(\underline{x})]$

Proof. In Appendix.

Theorem 2 points out that a DM who express his/her attitude towards uncertainty by means of a simple capacity will represent his/her preferences according to a functional which is a linear combination of the expected utility outcome over all gains and losses and the utility of the windfall gain and the catastrophic loss. More clearly, suppose that the DM has a reference point in the set of familiar outcomes with respect to which he/she normalize his utility, so that any gain is represented as a positive $\Re$, a negative one for losses. Then $u(x) \in \Re_+ \; \forall \; x \in [x_1^+, x_i^+]$, $u(x) \in \Re_- \; \forall \; x \in [x_{j+1}^-, x_n^-]$. In this case, the right hand side of the CEU representation in Theorem 2 shows that the DM balances the windfall gain and catastrophic loss that he is going to bear, For a given degree of confidence γ he/she is more willing to undertake an act that might lead to both windfall gain and catastrophic loss if the utility of the former is bigger than the negative utility of the latter, and *vice versa*. If the utility of the windfall gain equals the negative utility of the catastrophic loss the CEU shows that the DM would behave according to the expected utility weighted by his/her degree of confidence, still exhibiting a "cautious" attitude, due to the ambiguity.

2. CONCLUDING REMARKS.

The CEU representation in Theorem 1 is sufficiently general, since v is a unique capacity (see Schmeidler (1989), Chatoneauf (1991), Wakker and Twersky (1993)). This is not the case for Theorem 2, since it relies on the specific capacity defined in equation (4). However, simple (and dual simple) capacities are a particular subset of convex (or concave) capacities, characterized by an extremely intuitive structure which might be sufficient to express DM's attitude towards uncertainty for many decision problems. For instance, it can be shown that whenever the set of events 2^Ω can be partitioned into two mutually exclusive events (such as in the Ellesberg's two-color urn paradox) any convex (concave) capacity is simple (dual simple). For all those cases, Theorem 2 shows that it is possible to obtain a CEU representation in which the full weight of the uncertainty embedded into the "unfamiliar" world is placed on the most "extreme" outcomes, which are regarded as a truly "windfall" gain and a "catastrophic" loss. Moreover, within this framework, it becomes possible to justify the "precautionary principle" which is often appealed to in "real world" decision problem whenever a policy has to be implemented which entangles both more reliable ("risky") consequences and less know ("uncertain"), extreme outcomes.

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4. APPENDIX

Proof of Theorem 1. According to equation (1), we can write the CEU as:

$$\sum_{m=1}^{n} u(x_m))[v(\bigcup_{r=1}^{m}(\omega_r)) - v(\bigcup_{r=1}^{m-1}(\omega_r))];$$

expanding it we have:

$u(x_1)v(\omega_1) + ... + u(x_i)[v(\omega_1 \cup ... \cup \omega_i) - v(\omega_1 \cup ... \cup \omega_{i-1})] + u(x_{i+1})[v(\omega_1 \cup ... \cup \omega_{i+1}) - v(\omega_1 \cup ... \cup \omega_i)] + ... + u(x_j)[v(\omega_1 \cup ... \cup \omega_j) - v(\omega_1 \cup ... \cup \omega_{j-1})] + u(x_{j+1})[v(\omega_1 \cup ... \cup \omega_{j+1}) - v(\omega_1 \cup ... \cup \omega_j)] + ... + u(x_n)[1 - v(\omega_1 \cup ... \cup \omega_{n-1})];$

recall that v is concave on Ω^+, additive on Ω^{+-} and convex on Ω^-. See that, for v additive, we can write $v(\omega_1 \cup ... \cup \omega_f) - v(\omega_1 \cup ... \cup \omega_{f-1}) = v(\omega_1 \cup ... \cup \omega_{f-1}) + v(\omega_f) - v(\omega_1 \cup ... \cup \omega_{f-1}) = v(\omega_f) = \pi(\omega_f)$, where $\pi(\cdot)$ is an additive probability (defined on the subsets of Ω^{+-}). Moreover, for v concave, there exists a dual capacity of v, defined as $v(A) = 1 - v(A^c)\ \forall\ A \subseteq C$, which is convex. Applying it to $v(B)$, $B \subseteq \Omega^+$ in the previous expansion and simplifying it we obtain:

$u(x_1)[1 - v(\omega_2 \cup ... \cup \omega_n)] + ... + u(x_i)[v(\omega_i \cup ... \cup \omega_n) - v(\omega_{i+1} \cup ... \cup \omega_n)] + u(x_{i+1})\pi(\omega_{i+1}) + ... + u(x_j)\pi(\omega_j) + u(x_{j+1})[v(\omega_1 \cup ... \cup \omega_{j+1}) - v(\omega_1 \cup ... \cup \omega_j)] + ... + u(x_n)[1 - v(\omega_1 \cup ... \cup \omega_{n-1})]$

which we can write in a more compact notation as:

$$\sum_{m=1}^{i} u(x_m)\ [v(\bigcup_{r=m}^{n}(\omega_r)) - v(\bigcup_{r=m+1}^{n}(\omega_r))] + \sum_{m=i+1}^{j} u(x_m)\pi(\omega_m) + \sum_{m=j+1}^{n} u(x_m)\ [v(\bigcup_{r=1}^{m}(\omega_r)) - v(\bigcup_{r=1}^{m-1}(\omega_r))]$$

Q.E.D.

Proof of Theorem 2. Let us re-write here the expanded CEU representation of Theorem 2:

$u(x_1)[1 - v(\omega_2 \cup ... \cup \omega_n)] + ... + u(x_i)[v(\omega_i \cup ... \cup \omega_n) - v(\omega_{i+1} \cup ... \cup \omega_n)] + \sum_{m=i+1}^{j} u(x_m)\pi(\omega_m) + u(x_{j+1})[v(\omega_1 \cup ... \cup \omega_{j+1}) - v(\omega_1 \cup ... \cup \omega_j)] + ... + u(x_n)[1 - v(\omega_1 \cup ... \cup \omega_{n-1})]$

Assume v is a simple capacity: $v = \gamma\pi$; substituting it into the previous expression we have:

$u(x_1)[1 - \gamma\pi(\omega_2 \cup ... \cup \omega_n)] + ... + u(x_i)[\gamma\pi(\omega_i \cup ... \cup \omega_n) - \gamma\pi(\omega_{i+1} \cup ... \cup \omega_n)] + \sum_{m=i+1}^{j} u(x_m)\pi(\omega_m) + u(x_{j+1})[\gamma\pi(\omega_1 \cup ... \cup \omega_{j+1}) - \gamma\pi(\omega_1 \cup ... \cup \omega_j)] + ... + u(x_n)[1 - \gamma\pi(\omega_1 \cup ... \cup \omega_{n-1})]$;

recalling that for additive probabilities it is true that $\pi(\omega_a \cup \omega_b) = \pi(\omega_a) + \pi(\omega_b)$. Applying it into the previous expression and simplifying it we obtain:

$u(x_1)[1 - \gamma + \gamma\pi(\omega_1)] + ... + u(x_i)[\gamma\pi(\omega_i)] + \sum_{m=i+1}^{j} u(x_m)\pi(\omega_m) + u(x_{j+1})[\gamma\pi(\omega_{j+1})] + ... + u(x_n)[1 - \gamma + \gamma\pi(\omega_n)]$;

where we made use of probabilities' additivity property, namely $\sum_{m=1}^{n-1} \pi(\omega_m) = 1 - \pi(\omega_n)$ and $\sum_{m=2}^{n} \pi(\omega_m) = 1 - \pi(\omega_1)$. Thus, we can write it in a compact notation as:

$$\gamma \sum_{m=1}^{i} u(x_m)\pi(\omega_m) + \sum_{m=i+1}^{j} u(x_m)\pi(\omega_m) + \gamma \sum_{m=j+1}^{n} u(x_m)\pi(\omega_m) + (1-\gamma)u(x_1) + (1-\gamma)u(x_n)$$

This can be written as:

$$\gamma \sum_{m=1}^{n} u(x_m)\pi(\omega_m) + (1-\gamma)[u(x_1) + u(x_n)] + (1-\gamma) \sum_{m=i+1}^{j} u(x_m)\pi(\omega_m)$$

See that $v(B) = \pi(B)\ \forall\ B \subseteq \Omega^{+-}$, since for $\pi(\cdot)$ additive it is true that $\gamma = 1$; it implies that $(1-\gamma) \sum_{m=i+1}^{j} u(x_m)\pi(\omega_m) = 0$. Moreover, from the definition of $\overline{x} = \{\max x |\ x \in [x_1^+, x_i^+]\}$, $\underline{x} = \{\min x |\ x \in [x_{j+1}^-, x_n^-]\}$, it follows that $\overline{x} = x_1$, $x_n = \underline{x}$. Since $u(\cdot)$ is a monotone positive function we can write $u(\bar{x}) = u(\max x | x \in [x_1^+, x_i^+]) = \max u(x); x \in [x_1^+, x_i^+]$ and $u(\underline{x}) = u(\min x | x \in [x_{j+1}^-, x_n^-]) = \min u(x)$. Therefore, the CEU becomes:

$$\gamma \sum_{m=1}^{n} u(x_m)\pi(\omega_m) + (1-\gamma)[u(\bar{x}) + u(\underline{x})]$$

Q.E.D.

[i] See Khaneman and Tversky (1992) and Chichilnisky (1996).

[ii] A lot of models have been introduced to explain the S-shaped utility function; amongst others Luce and Fishburn (1991), Wakker and Tversky (1993), Tversky and Wakker (1995).

[iii] Reliable or unreliable probabilities mean that they are sure or unsure in the sense of Savage (1954).

[iv] From now onward we will call it just "uncertainty", to avoid cumbersome repetitions.

[v] It has been reported some evidence that attitude toward ambiguity are roughly uncorrelated with attitudes toward risk.

[vi] Provided that a set of axiom is satisfied that replicates Savage's one but where "the sure thing" is replaced with a weakened one. See, for instance axioms A1-A5 in Chateauneuf (1991), where preferences are defined over non-negative, A-measurable functions on Ω (where A is a σ-algebra on S).

[vii] See Chateauneuf (1991) and Ghirardato and Marinacci (2000) for a formal characterization of a DM's pessimism or optimism in term of the convexity or concavity of the capacity.

[viii] Roughly speaking, two actions $x,y \in X$ are comonotonic if they induce the same favorable state ordering and the same permutation.

[ix] This is the same as stating that his/her preferences satisfies a set of axioms that guarantee that preferences can be represented by an utility function unique up to a positive affine transformation, integrated with respect to a unique capacity by means of the Choquet Integral. Again, see Schmeidler (1989), Chateauneuf (1991), Sarin and Wakker (1992).