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DIPARTIMENTO DI ECONOMIA POLITICA

JUAN GABRIEL BRIDA
MARTIN PUCHET ANYUL
LIONELLO F. PUNZO

Coding Economic Dynamics to Represent

Regime Dynamics

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Juan Gabriel Brida , Universidad de la Repùblica, Uruguay

Martin Puchet Anyul, UNAM, Mexico

Lionello F. Punzo, Università degli Studi di Siena, Italy

Abstract

In this paper a new approach to the analysis of the dynamics of economies is presented; applications to time series will also be suggested. In such applications, computational experiments may play a central role to provide a different heuristics and to explore data information. The approach is based upon ideas emerging in the literature on complex and chaotic dynamics, which imply that one can no longer rely on the fine description of classical dynamical systems: the state space structure breaks down, and instead of simple orbits, we should be satisfied with a description based upon symbolic trajectories, each symbol being associated with a partition of the original state space. Such partition can be induced by the introduction of the qualitative notion of regimes and of regime dynamics as a dynamics allowing for regime shifts. It can otherwise be suggested by preliminary data screening. Starting from a pre-set model, a regime is a set of dynamical paths generated by the same "canonical model" with parameters. By identifying bifurcation values in the parameter space, one can classify a finite collection of realizations of such canonical model. A symbolic dynamical model reproduces dynamics across such sets of realisations, and can be tested against available empirical data. A preliminary exploration of some simple models yielding a finite (low-)number of regimes with a complex cross-regimes dynamics is presented, to motivate the move towards a discrete space dynamics and as a step towards the building of a general approach to multidimensional dynamical models.

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Introduction

One of the achievements of recent macroeconomic literature is its new awareness that even the simplest (in particular, macrodynamic) model can display a great variety of hitherto unexpected behaviours. When this is not the case, it is obtained by the surreptitious insertion of quite restrictive assumptions. Hence, it is now typical to allow, in models of the later generations, at least for the possibility of multiple and/or indeterminate equilibria, and this fact opens up new vistas, e.g. over the debates in economic policy. This move can be interpreted as a very preliminary step in the direction of internalizing into current modeling technology an intuitive notion that belongs to the jargon of the empirically-oriented economist. This is the notion of "regime".

In this paper we make the next move: allowing a regime to be in general a whole set in state space rather than a single (equilibrium) state, and considering explicitly dynamics across such sets (rather than across equilibria[1]). The project was introduced in Böhm and Punzo (1992) and other works, as the theoretical counter-part of a fact-finding and fact-collecting research on comparative structural dynamics. Day ((1994), (1995) and other related works) formulates a similar framework for very long run (one-dimensional) dynamics and calls it multi-phase dynamics. We prefer to call such dynamics, cross-regime dynamics or for short regime dynamics, in order to stress the fact that in general, in our framework, it involves more than one variable. To keep track of it, we need to code classical point-dynamics and then reproduce it as a sequence or a string of dated symbols, whereby each symbol reports the regime membership of the corresponding state. With a unique regime, this dynamics is trivially repetitive; the same is true, with a word of caution, when everything takes place (sometimes, "eventually") in one and the same regime, though a few more are available. So, the whole story has interest when an economy's path traverses repeatedly regimes. The largest part of conventional economic dynamics[2] rests on the implicit assumption that this never happens ("stability principle"), or is irrelevant ("the principle of regime uniqueness").

We introduce a no-nonsense definition of regime (Section 4), which pins down what is essential, in our view, of the many existing, informal definitions (reviewed in Sections 2 and 3). Such definition turns out to lead to quite interesting new vistas over old (and less old) issues. This in particular, relates to the interpretation of observed irregular behaviours, and the assessment of some key policy issues. The corresponding macrodynamics is formalized by a mathematical model built upon the twofold idea of regime and cross regime dynamics whose evolution is represented by a *coded dynamics***.** The latter is related (and partly overlaps) with the more formal, mathematical branch called *symbolic dynamics*. Such proximity often permits the use of formal techniques that are well established in the mathematician's tool box (as illustrated in Sections 6-8). The economist's motivation for coding dynamics is however partly different: It lays also with the need of handling noisy time series information, and deriving "predictions" by synthesizing their information contents. It is therefore equally related with certain recent advances in information theory, as well as with the efforts to put to some use what we know of chaotic and complex dynamics. While coding is an entirely new technological idea (and for obvious reasons), the idea of a qualitative dynamics that moves across regimes is not. It was born in the theory of business cycles, and in Frisch's classical macrodynamics, where the regimes are identified with the phases of the cycle. The illustrations of our approach (in Sections 3-8) in this paper are all taken from this piece of history. Alternative definitions appear in the literature.

1: Preparing the ground.

The everyday definition is intuitive as is expected to be, and likewise suggestive. Intuitively speaking, a dynamic regime is a qualitative behaviour that can be usefully distinguished from other dynamical behaviours (called, likewise, regimes). One essential ingredient of the definition can be grasped from an example. Economists often talk of high inflation or of (un)sustainable growth. "High inflation" is always either the highest inflation compatible with some other economic variables values, or it is high compared to the values of the latter. It refers to a situation where an economy state can only be characterized

[1] See poverty trap and other related literature about low and high equilibria. This literature normally describes dynamics across equilibria as policy driven, hence exogenous. In this paper it is in principle endogenous.

[2] Except for neo-Austrian theory, see Amendola, Gaffard, and Punzo (1998) where the notion of regime dynamics is married to a neo-Austrian framework.

by a vector of variables and a single scalar may be not descriptive[3]. Thus, a regime of high inflation is a constellation of (state)vectors characterizing an economy, where one entry is the inflation rate (and values of all other variables are compatible with it) but it is "too high", say, compared with some external parameter values. Generalizing, a regime is a set of vectors where variables take up values that are "compatible" with one another, or follow a certain functional model. We never really talk of extreme values of anyone variable (e.g. "the highest inflation rate"), though the definition implies their presence if a regime has to have a frontier with other regimes (a "high inflation regime" implies the at least notional presence of a "low" one). Such type of borders are always difficult both to localize and to handle, and we may forget them for the time being.

Extremal values, however, i.e. values on the economics' frontier, play a key role in modeling and they are in a sense easier to understand than values that are simply compatible, as said above. They, normally, can be associated with equilibria, and regimes then are automatically equilibria states. Equilibria (or, likewise, disequilibria) do not necessarily coincide with regimes in the sense proposed by us, though there is of course a relationship between them. The two cannot be usefully distinguished in those specimens of dynamics, "simple dynamics", where a unique equilibrium is also a global attractor (in the conventional sense).

Hereafter, we explicitly consider the possibility that an "economy" display dynamic behaviours that are qualitatively different in a sense that is defined precisely in Section 2. Hence, a regime cannot be identified with any specific equilibrium state. The appropriate model for this type of dynamics would need to be different, depending upon some parameters and/or on values of certain (key) state variables. Seen in these terms, the *phase portrait* of an ideal economy should be represented by a collection of regimes as models, each model being in a sense a local representation of this dynamics that may account for only part of its virtual history. An actual or model simulated history is a specific realization of the collection of available regimes, a time sequence of paths through regimes with their timing and duration in some conventional clock. A specific economy can go through the collection as a whole or in part, over its history. Regime changing corresponds obviously to a form of structural change, and structural change can repeat over the history of anyone economy. The realization of the collection with the concatenation of realized models can be assigned to nonlinearities that capture the discrete nature of the jumps (regime switches).

One such framework may account for and rationalize the evidently different dynamics that has been observed across countries, regions, sectors, and any combination of them, as it has emerged as a "stylized fact" in the recent literature on growth empirics, regional divergence, forms of capitalism, etc. In this light, the much debated issue of "convergence" across countries becomes a matter of the presence of a unique "regime", where regime is a "growth model"[4], and divergence exhibits instead the plurality of models and the various ways into which they can be assembled in actual historical experiences [5].

The collection is therefore best conceived of as a menu or dynamical manifold over a set of dynamic models[6]. Formalizing the corresponding dynamics is formalizing a state space over regimes, geometrically discrete. All dynamical terms have to be re-defined in reference to regimes, beginning with the notion of equilibrium regime and that of regime attractors. Regime switch is a discontinuous change from one regime to another, or a discrete jump from one state to another state, if a state is associated with a regime. Regimes are assigned natural numbers or symbols from an adequate alphabet. To do this we have to go symbolic. The (tentative) use of techniques that go under the name of symbolic dynamics makes the fundamental difference in our approach from the conventional approach in state space where state variables are real numbers. Coding implies a simplification of the description of the actual dynamics, for the sequence of actual states is lost. It allows us, on the other hand, to take into account the fact that history does not repeat, neither for a

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[3] A model for a regime typically is a multivariate model. Therefore, one-dimensional dynamics is ill suited to handle regimes, though it can be done.

[4] See Punzo (1997).

[5] Growth empirics is largely about falsifying the Solow neoclassical model, assuming there is only one model to choose from, see Barro and Sala-i-Martin (1995). Here the point is that there may be different models "available at the same time".

[6] We use growth models when we use growth rates; instead it is standard in BC theories to use level variables. The two come together in the theory of growth cycles.

single country (region, sector, and the like) nor across countries. A country's history tends to have its own sequence of structural changes, according to an internal timing. Different economies may display different histories as realizations of the same collection. But obviously one cannot exclude the limiting case of a country remaining forever (for long periods) with the same regime, or just crossing one way from one to another. (Much of the latter has to do with the notion of "transition" that is often used nowadays for some countries.)

The issue is, now, to define the menu or collection of regimes, hence the qualitative criterion to distinguish dynamical behaviours.

2: Regimes: a review of definitions.

A regime is defined, hereafter, as a pair (f_R, R) where R is a set in a partition of the state space Σ induced by a dynamical model written in conventional form, and f_R is its rule or dynamic model. (Of course, when R=Σ, we are in the standard one regime situation.) For the moment, there no need to impose any other restriction but in this paper we will consider models in discrete time, for convenience as well as to keep with a forgotten tradition[7].

To be interesting, a partition should slice the state space into (at least) two or more nonempty sets R_j, regimes, and cover the whole of the state space. The presence of multiple regimes provides alternatives: a variety of regimes is available to construct one's own history. In this case, it can be quite rich, for we have a twofold dynamics, one within a given regime and one across regimes. Their mixing can produce anything we like, or we observe[8]. Conventional dynamics focuses upon the former only, overlooking or under-estimating the latter. Our coded (regime) dynamics focuses upon the latter, and thus they are in principle complementary. In this new light, debates over policy issues may also have a broader scope: targeting is targeting one feasible region, of viable values of a whole constellation of variables, rather than say the over ambitious fine tuning.

The key to the definition above is in that multiple regimes go together with multiple dynamic models in a specific way. Often economists use the notion of regime in much looser, though intuitive, sense and with a number of different meanings reflecting different criteria to distinguish qualitatively different behaviours. Correspondingly, there are distinct procedures to introducing regimes. Clearly, the mathematical formulation that can be suited to induce a multiplicity of regimes is a kind of hyper-model, with local realizations that are themselves models. The hyper-model can be the outcome of a "general theory" (i.e. derived from first principles, as in the current fashion to theorizing) or be the result of generalizations of observed regularities. Its local realizations can often be approximated by (or simply thought of as) linear formulations, linear models. In this lucky case, the hyper-model can be either reduced to a finite set of realizations differing from one another by virtue of the values of some parameters (a technique close to bifurcation analysis). Or else, it can be re-constructed by piecing together locally estimated models into a suitable architecture. Either way, however, the key is that there are distinct models of dynamic behaviours and that their parameter values, if they can be defined, vary depending upon the set in the partition of the state space where we are.

It is to explain this that we review some elementary models in the classical tradition of Macrodynamics, models to which otherwise no particular analytical value is attached here. The advantage of taking a single basic framework (the interaction between multiplier and accelerator mechanisms) is in the easy comparative reference. However, this choice is to an extent forced upon us. The idea of a sequence of regime(s) can in fact be found, explicitly, in Hicks's approach to the theory of business cycle, and in fact it can only emerge in an analytical framework focussing on disequilibrium, rather than equilibrium, dynamics. This is what classical Macrodynamics precisely tries to do, in contrast with the current equilibrium approach to the business cycle. However, historically, the move to BC models formulated in continuous time has obscured the original intuition of economic dynamics as a discrete set of local behaviours, and the study of latter have come to be identified with topological dynamics. This work in a sense goes back to an old

[7] The next move should be regimes in continuous time models, see *infra* for some observations on this difficult task.
[8] In fact, even chaotic behaviour, see below, and this does not depend on the number of regimes, i.e. of the alternatives! In other words, unpredictability does not depend upon the fine structure of state space.

tradition to recover certain key ideas upon which to build a new kind of qualitative dynamics, complementary to the standard one.

We said that there are alternative strategies to constructing "models of local behaviour". Basically, however, they can be introduced either starting from models that economic theory tells us to go and try out against reality, or via "observation and induction"[9]. In the former case, in a sense we break a given model down into a set of local realizations, thus treating it as what we called a hyper-model[10]. In the sequel of this paper, we basically follow such procedure. The alternative consists in "estimating local models "and then re-construct the hyper-model that could correspond.

3: Regimes as mutually excluding alternatives.

Regime often is only a synonym for certain dynamic behaviours (instead of whole models), and is used to identify them. Normally they are generated via one of three (bifurcation, equilibrium and modal) criteria. Ours is a fourth, distinct criterion.

Typical is the use of regime as growth or oscillation. This is the simplest. One can do an exercise with a model that exhibits different dynamics depending upon the values of some parameters. With a linear model this is quite easy, for parameters are at the same time factors in coefficients of a characteristic equation, hence they determine its roots which are the system dynamic modes[11]. For example, in a well-known economic dynamic model[12] there are two coefficients, the average propensity to consume and the "accelerator" or capital output ratio, hence a vector $\mathbf{z} = [c, k]^T$ which belongs to the rectangular region C x K in R^2, with $C = (0,1)$ and $K= (1,.)$. The model is known to be able to yield either growth or oscillations, depending on where the actual coefficient vector $\mathbf{z}$ lies. This is so because both the sets I^C and I^R of vectors $\mathbf{z}$ such that $I^C \cup I^R = C\text{x}K$, (with I^C set of vectors yielding complex (for I^R real, respectively) roots) are nonempty. Here, two regimes, one of oscillations (business cycle frequencies) and the other of growth[13], can thus be introduced by a partition of the parameter space, by a bifurcation criterion.

Whether the predicted dynamic regime[14] is of the one type or the other, does not depend upon where in state space it is taking place (where e.g. lies its initial state, or the state at some date τ). It depends, instead, upon where we assume to be in the space of parameter values. This criterion, therefore, does not partition the state space[15]. The reason is that each model is local in the parameter space, but otherwise dynamics is globally the same. By contrast, in our definition above, initial conditions and/or the sequence of states (the history) do matter in determining dynamic behaviour: in this context, this can be represented by saying that our parameters values are state dependent, at least in principle. To see the difference, we can use a coding technique that identifies the pieces of state space where different dynamics may take place. This is however a case where the model either yields oscillations or exclusively growth over the whole of its state space; hence in our definition above, $R = \Sigma$, and oscillations or growth take place on the whole of the system state space.

Let us introduce at this point notation that will turn up more useful later. There are where two regimes to talk about, defined on the basis of a pre-determined criterion. Calling A the regime of oscillations, and B the regime of growth, one can re-construct trajectories as infinite strings of symbols, using only A or B. One simply inserts an A or else a B in a place in the string whenever the path is a oscillatory path, or respectively a growth path. This is the preliminary step to what we call coded dynamics, that for the moment looks like a simple re-coding of trajectories: from the infinite sequence $\{\ldots, x_0, x_1, \ldots, x_k, \ldots\}$ through initial condition x_0, where member x_j is a vector of real numbers, to a symbolic string: say, A in place of x_j if x_j lies

[9] The former approach was taken in Böhm and Punzo (1992).
[10] Whether we can do it or not, it depends on the model; likewise, what we get out of this logical operation depends also upon the model, too.
[11] Hence "qualitatively different" dynamics may refer to stability properties or alternatively to the distinction between monotonic or oscillatory motions. In economics, the former can often be associated with growth, and the latter with oscillations of different periods (short and long cycles).
[12] Samuelson (1939).
[13] Real roots are positive.
[14] Prediction is here akin to computer computation.
[15] Because it cannot: only a nonlinear model may be a candidate to do this.

in A (the definition will be made more formal a little later). In this case, we observe either infinite strings of A or of B, say

$$\text{A.AAAA} = (\text{A})^{\infty}$$

or

$$\text{B.BBBB} = (\text{B})^{\infty}$$

(with the convention that A means a sequence of regimes, i.e. an "orbit", that begins in regime A). What we cannot observe is a combination of the two. Moreover, which one will be observed will depend upon parameter values, not on state space[16]. Coding dynamics for the above model is not of much use, however, for this takes place always generically within the same regime, whatever that may be. Other cases, where still one and the same symbol appears in any string, can be more interesting[17]. They result from the adoption of an alternative criterion to introduce regimes, one which is related with the distinction between equilibrium/disequilibrium states. We are getting closer to our definition, for at least we are talking about a partition of the state space.

Harrod's version of the previous basic model[18] exemplifies a special dynamics where this new criterion can be used in its extreme version. Recalling that the marginal propensity to save is equal to s = (1-c), then parameter intervals C and S are equal, and Harrod's model predicts a solution path of the form

$$Y_t = (1 + s/k)^t \, Y_0$$

that solves

$$Y_t = (1 + s/k)Y_{t-1} = (1 + g_W)Y_{t-1}$$

which is the reduced form of the structural model

$$\begin{cases} S_t = sY_{t-1} \\ I_t = k(Y_t - Y_{t-1}) = k\Delta Y_t \\ I_t = S_t \end{cases}$$

In contrast with the previous case, growth follows here whatever the vector (s, k) in the parameter space C x K, and $g_w = s/k$ is the corresponding "warranted rate of growth".

The bifurcation criterion cannot be used to generate a regime classification, in other words. Here, in fact, the model yields a unique regime in the modal sense as used above: the growth regime is independent of parameter values (i.e. it is generic). Moreover, there is only equilibrium dynamics. The solution path connects a monotonic sequence within the state space, with the particularity of connecting only equilibrium states. It also coincides with the whole state space (in this case the nonnegative real half-line of values of Y_t, to the right of any initial value Y_0).

The model and the unique regime coincide. Finally, as within the regime there can only be equilibrium, the coded sequence will be now only of the type, say, A.AAAAAA…= (A)$^{\infty}$: we start from a point in and stay within the same regime forever, and always in equilibrium, Thus, it also implies a very simple dynamics within A.

[16] In other words, (A)$^{\infty}$ will be associated with all $\mathbf{z} \in I^C$, while (B)$^{\infty}$ for all $\mathbf{z} \in I^R$.

[17] This shows that the observation of a string is not sufficient to recover the model that may have generated it.

[18] We define the basic model as one with a multiplier and accelerator mechanisms are coupled together. Note that while in the Samuelson model the accelerator yields a second order difference equation, in Harrod's case we have first order. It is for this reason that linear dynamics in the former case can yield oscillatory motions, that are not available in the latter one.

The simplicity (or complexity) of the Harrodian model lies in such coincidence of three logically distinct notions: the model, the regime(s) and the dynamics within the regime. As there cannot be disequilibrium behaviour, coding of sequences not only does not add anything, it also looks trivial. More interesting models can be constructed by removing one or the other assumption, linearity and/or dimension one.

We postpone the former exercise, to do something else that recalls the modal criterion that failed before, when coupled with bifurcation analysis. This clarifies the role of linearity.

In fact, the mere introduction of a bigger dimension (while retaining linearity) gives us regimes as distinct dynamic modes associated now with a partition of the state space[19]. Dimension vindicates at least partially the modal criterion that earlier did not prove successful: we need to separate it from bifurcation analysis that pertains to parameter space (the modal distinction is naturally in state space). It can be shown [20] that a Leontief-like model dynamized with a simple lag yields an equation that, setting the impulse function identically equal to zero, can be considered as mathematically identical to Harrod's, except for dimension.

To recall, this takes the form[21]

$$x_t = Ax_{t-1,}$$

where A is a positive matrix.

In fact, growth factor $(1 + g_w)$ can be interpreted as the one-dimensional (or scalar) matrix A of the (Leontief) adjustment model above. Hence, it is its unique eigenvalue, that is both real and nonnegative, given the restrictions upon the structural parameters, s and k. In the general n-dimensional case, however, the set of eigenvalues of the dynamic matrix A play the role of generalized warranted rates of expansion (contraction, respectively). But, obviously, we encounter a new possibility, that could not arise before. Some eigenvalues can now turn up to be complex, and they will generically be complex except for the dominant eigenvalue that is assured to be nonnegative by virtue of the nonnegativeness of the dynamic matrix.[22] Therefore, *in general* or for generic initial conditions, dynamics will be in an "oscillatory regime"[23], though it may be coupled with at least one monotonic expansionary path (the unique growth regime that is seen emerging alone in the one-dimensional case).

The example illustrates the kind of dynamic variety that can be obtained from a linear model when it is multi-dimensional. This is the modal regime criterion fully blown, for the resulting regimes do not depend upon particular values of the n^2 coefficients a_{ij} of the dynamical matrix A[24]**.** Here, the two (oscillations and growth) modal regimes are associated with certain linear subspaces and the related, restricted dynamics. To see this, using principal coordinate axes we may decompose the n-dimensional state manifold, here coinciding with $R^n{}_+$, into the direct sum $R^n{}_+ = \Sigma^R \cup \Sigma^C$, where Σ^R is the linear subspace of monotonic and Σ^C, the subspace of oscillatory dynamics. Both of them are generally non empty sets[25]. Selecting initial conditions appropriately, one can generate either monotonic expansion alone (contraction, if this is the case[26]), <u>or alternatively</u>, only oscillations. This corresponds to having two local models i.e. (M_C, Σ^C) and (M_R, Σ^R), where each model is obtained as the original M restricted on initial conditions lying in Σ^C, or Σ^R,

19 This is clearly a n-dimensional version of the modal criterion to introduce regimes, that was discussed earlier.
20 This is done in Punzo (1988). See also Punzo (1995).
21 From: $x_t = Ax_{t-1} + y(t)$, setting $y(t) \equiv 0$.
22 Remember the set of theorems that go under the name of Perron and Frobenius. They apply for we have taken as dynamic matrix the input output coefficient matrix, some problems would arise in the proper Leontief dynamic model, whose discussion is beyond the scope of the present illustration of a principle. Of course, the unique eigenvalue of a scalar matrix is also its dominant eigenvalue.
23 Made up of up to k = (n-1)/2 distinct oscillatory modes. This observation was originally made by R.M. Goodwin (1949); when nobody knew either the theorems on nonnegative matrices nor the notion of generic property, that is so basic to topological theory of models.
24 But only on their being nonnegative.
25 Surely, $\Sigma^R \neq \emptyset$, while generically $\Sigma^C \neq \emptyset$ and $0 \leq k \leq (n-1)/2$, if $\dim(\Sigma^C) = k$, for the above reasoning.
26 Negative eigenvalues or larger than unity, depending on the dynamic formulation.

respectively. Coding the corresponding dynamics, we may construct orbits as infinite strings inserting a symbol A whenever they refer to a state $x_\tau \in \Sigma^C$, and B if $x_\tau \in \Sigma^R$, for some τ. Again, the sequences so obtained are such that, if they start with an initial A (or B) as x_0 in Σ^C (respectively, $x_0 \in \Sigma^R$), they will show the same symbol forever (both backward and forward). A path started within one subspace interpreted as a "modal regime" *will not escape* into a different regime. Thus, coded dynamic strings can be either of the form A.AAAAA, or of the form B.BBBB. But, now, in contrast with the previous example, we can see in principle one or the other type of orbits, or even both of them, provided we have paths starting in different regimes.

What cannot be seen, are paths taking from one to the other regime, however. All this is to say that linear models of sufficiently great dimension are indeed able to display different qualitative behaviours over different pieces of their state space, i.e. regimes in the modal sense. In this, they hold, though in a weaker form, one key property (dependence upon initial conditions) of our regime dynamics. However, there, qualitatively dynamics does not really depend upon initial conditions. In general, it will be oscillatory, the probability of (observing) pure monotonic dynamics, i.e. (balanced) growth as a mobile equilibrium, being practically irrelevant. This can be expressed by saying that: as dynamics will take place in the cross product of the oscillatory by the monotonic linear subspaces, it cannot be effectively coded (by a finite alphabet)[27].

The definition of regimes on the basis of a principal axes partition (modal regimes), with local models being restrictions of a linear model, is different from ours[28]. Moreover, it is appropriate only to linear models. In linear models modal regimes can arise if there is a sufficient dimension, while in multi-regime dynamics, as defined above, dimension plays no essential role (and therefore decomposability).[29] Hence, the reference hyper-model should be nonlinear (while it may as well be one-dimensional[30]).

4: Going nonlinear: step one.

Linear models are constructed as global models, to chart dynamics in a un-differentiated state space. A global model leaves no room for other representations, by definition. They are the *prototype* of single-regime models. Only models that are conceived from the outset as local representations of a globally non-linear one may have such property. They are then local in the state space sense. If they are parameterized, parameter values must be dependent on state location. This is the key form of nonlinearity, the nonlinearity that may explain (the possibility of) regime switches. This is the discontinuous dynamics across regimes that is the new form of dynamics we are really after. We can build a road to it from the single state-variable equation above. We have seen that dimension is neither sufficient nor necessary to our definition.

Let us thus go along to introduce two (co-existing) state dependent regimes, but in steps. The Harrod equation predicts dynamics that can again be intuitively coded by a single letter, A, whatever the initial condition, and therefore would look like a infinite string of As. (That dynamics be monotone as is the case there, or it is not, is immaterial once coded, see the remark above[31].)

For later uses, it is better to resort to a graphical representation, where the dynamic equation is represented in the plane (Y_{t-1}, Y_t), and the coefficient $(1+ g_w)$ is the (positive) slope of a line through the origin, cutting through first and third orthant. Embedding the previous one-dimensional state space into a larger space will exhibit useful properties of the solution path, as well as permitting us to (begin to) be a little more technical.

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[27] Or else, only special dynamics can be coded, yielding strings like the two above. In other words dynamics cannot generically take place on disjoint sets of the state space.

[28] It was discussed in Punzo (1995).

[29] Actually one dimension is always sufficient, though often it does not capture economic reality. In fact, decomposability is a feature of linear systems.

[30] To recall, dimension is tantamount for the number of independent state variables, this is a standard definition.

[31] We said that this approach to defining regimes is not based upon the distinction between oscillatory and monotonic modes. We shall see later that this applies also to more complex dynamical behaviours, e.g. chaotic dynamics.

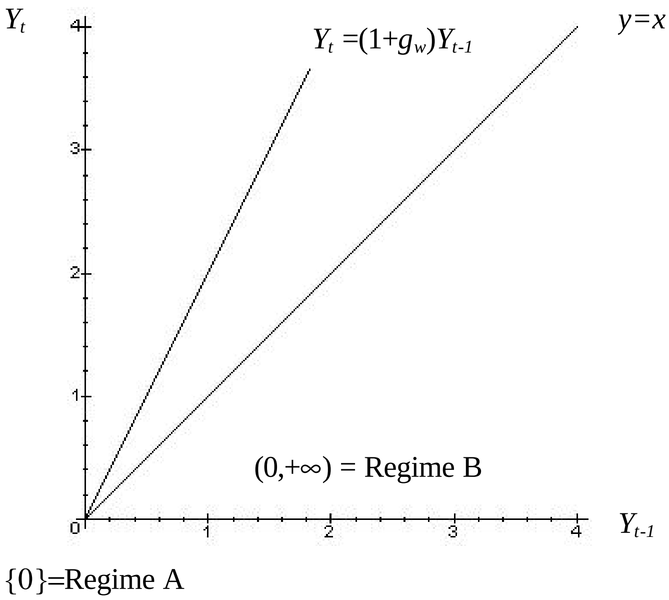

Figure 1: ***One dimensional half-line with an origin as Harrod's phase line.***

The equilibrium manifold is thus projected onto the origin of the plane, all growth paths been zoomed into it, independently of initial condition Y_0. To do this, we should re-define the state variable and introduce deviations from equilibrium: $y_t = Y_t - Y^*_t$, where Y^*_t is the solution to the original equation. Hence, the new equation in deviations from a given equilibrium[32]

$$y_t = (1+ s/k)\, y_{t-1} = j\,(y_{t-1}).$$

An equilibrium of Harrod equation above is a fixed point of map j: i.e. a value y_t such that $y_t = y_{t-1}$. Linearity of j(.) implies that there is only one such fixed point, i.e. $y_t = y_{t-1} = 0$, while elsewhere in R/{0} a nonzero deviation is monotonically increasing to ± ¥. This shows that the fixed point is an unstable equilibrium.

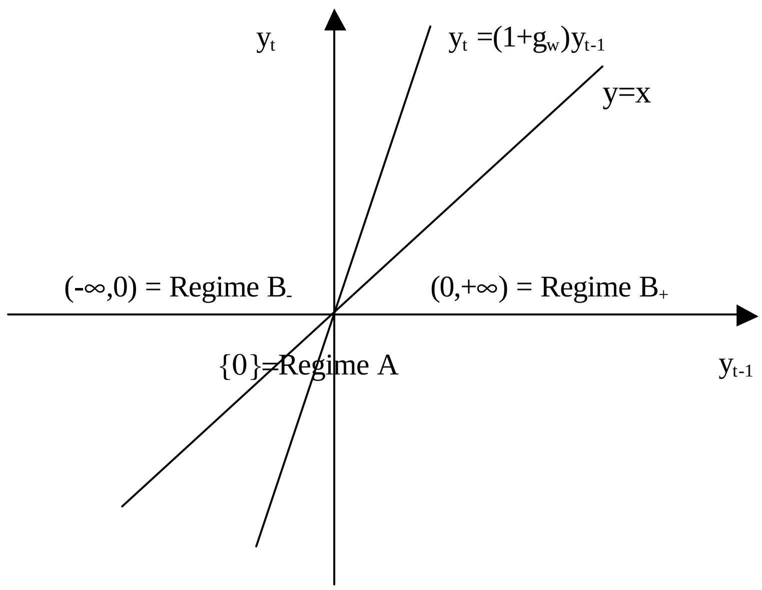

Figure 2: ***Harrod's model in deviations.***

We may now make use of the distinction between an "equilibrium regime" (the set zoomed into the zero of the plane) and the related "disequilibrium regime", i.e. elsewhere. But now a distinct model can now be associated with each dynamical regime[33]: this is a model in levels for equilibrium (over $S = R_+$ or else over {0} in Σ = R), a model in deviations for the disequilibrium regime (over the whole of Σ = R/{0} The

[32] That obviously depends upon the initial condition Y_0.
[33] This is the original model for equilibrium and the model in deviation for the latter.

possibility of two models did not appear explicitly in the original Harrod version (though it is hinted at), because there was no distinction between deviations and levels.

The half-line coded by A before in the Y-line is correspondently zoomed into the set $\{0\}$ which then can still be denoted with the same symbol. Growth Regime A and Harrod's equilibrium coincide as before. For states outside the equilibrium set, however, it is useful to introduce a new different symbol, say B, to identify the set of disequilibrium states as a distinct regime. As deviation value $y_t \in R$, R being the phase line, a partition can be $R = A \cup B$, with $B = B_- \cup B_+$, and this is a covering partition[34]. Looking at the dynamics as a ordered sequence of regimes, we know that starting in equilibrium like before, we stick to it. Else, starting or after having been shocked out of equilibrium, we will never return to it, the Harrodian growth regime not being an attractor[35]. Coding as before we again obtain either one of the two possible infinite strings that can be constructed with two regimes: i.e. $(A)^\infty$ or, alternatively, $(B)^\infty$, (for all strings starting in A or B, respectively). These are the only two classes of orbits that can be generated, by choosing initial states appropriately. We cannot construct strings with AB or BA as part components, in other words traverse orbits.

This minor re-formulation confirms what we know from the old version of the model, but it also adds something to our knowledge. By introducing a second regime, that is rendered possible by the embedding into a two-dimensional space, we see how getting outside regime A, i.e. <u>into B</u>, implies an increasing distancing from the unique equilibrium. (This is a non secondary aspect of the knife edge property of the model, that is shown by our coding trick.) It is useful to have a name for regimes where the system behaves like in such a B. They may be called[36] null regimes as they are regimes made up of transient states only. They are regimes where no systematic ("long run") behaviours can be observed, as they are globally unstable regimes: they fly away. They do not add much to the model's prediction, and in fact, it can be shown that their presence does not change coded dynamics (*see infra for an example*). However, they can help us understand the importance of the outward properties of a regime to determine (cross-)regime dynamics. To see this we go through an intermediate step, modifying slightly the above model, but this will alter within dynamics in a relevant way.

5: Going nonlinear: step two.

In the economists' intuition regimes are basically the same as models (though, perhaps, with some qualifications), and thus they are expected to comprise at least one equilibrium. This is a proof of internal consistency, but it also represents that type of long run behaviour an economist depends upon to derive her "predictions". The simple model above identifies equilibrium with a regime, but normally a regime may comprise one or more equilibria and something else within. Thus, identifying equilibrium and disequilibrium as two distinct regimes is little appropriate, outside that particular situation. On the other hand, regime B has neither an attractor nor even an equilibrium state. It is not usable in the above sense, which gives another reason to call it a null regime. This is a peculiar situation of a two-regime dynamics whose peculiarity can be removed easily, once we understand that it is the result of the linearity assumption: a set of states that is not in the basin of attraction of a unique equilibrium, can only be a null regime. If it is in its basin of attraction, it cannot span a regime independent from the equilibrium set.

Proceeding step by step, it is useful to remove first what is peculiar of regime A defined above. That is, that it is a <u>point set</u>, from which characteristics stems the "oddity" of a <u>generic</u> null regime[37]. Thus, let us first consider the following generalized version of the model, that introduces a ceiling to the investment function[38].

---

[34] See later the definition of a covering partition. In the graph we further distinguished $B = B_- \cup B_+$ in an obvious way, but it does not add much, as dynamics is the same on both of them, see below.
[35] This is the essence of the Harrodian instability, or at least one of its facets.
[36] Following Day (1995).
[37] Generic for almost all initial conditions fall into the null regime, a re-interpretation of Harrod's instability.
[38] It should be clear to the reader that we are doing the exercise of piece by piece constructing the Hicks and Goodwin models of nonlinear accelerators.

$$I_t = \begin{cases} k\Delta Y_t, \text{if } \Delta Y_t \le d \\ a, \text{if } \Delta Y_t > d \end{cases}$$

where a is a positive number.

Now, in the adjustment equation: $y_t = j(y_{t-1})$, function $j(.)$ becomes now piecewise linear, with one discontinuity

$$j(y_{t-1}) = \begin{cases} (1 + s/k) y_{t-1} \\ (1-s) y_{t-1} + a \end{cases}.$$

In the (y_t, y_{t-1})- plane the situation looks like this

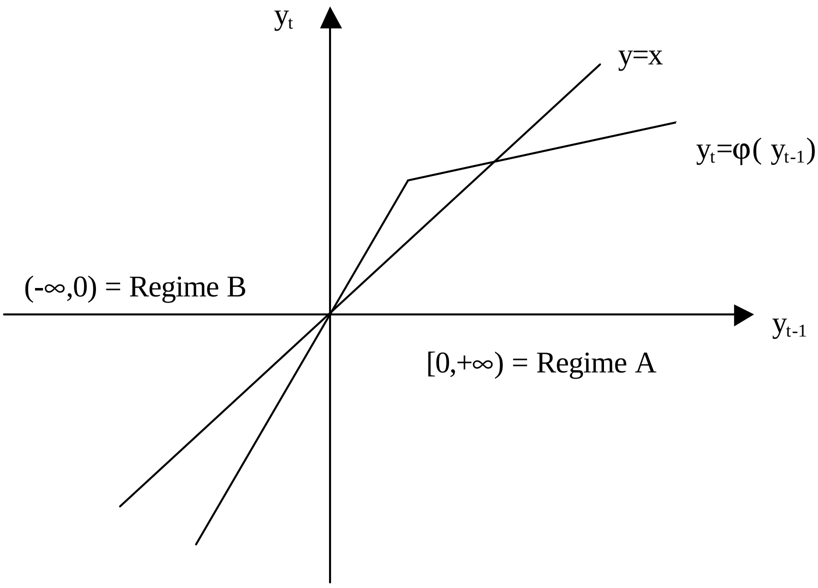

Figure 3: ***The piecewise linear model.***

Here, we finally see the regime A as the pair (M_{I+}, I_+), while regime B is the pair (M_{I-}, I_-), such that $R = I_+ \cup I_-$, and the two Ms models correspond to restrictions of the original model[39]. The original unique equilibrium 0 is now a member of regime A, which however has a richer dynamics. The introduction of a single nonlinearity has added one equilibrium to regime A, and re-defined regime B as the left-hand side of the phase line. The new B, like the old one, is a null regime; any sequence starting there will stay within but systematically flying away. Regime A has a better internal structure, now: its equilibrium set comprises the equilibrium of the previous model, which stays unstable, while a local attractor within the regime has been added. It satisfies the expectations of an economist of a model.

Coded dynamics remains however unaltered after the above modification. It will still be either a string $(A)^\infty$ or a string like $(B)^\infty$. What has changed is the dynamics internal to one of the two regimes, because its structure has changed. This regime A implies a partition of the right hand side phase line that decomposes into an equilibrium set, made up of two states, and the union of two open intervals. This union acts as the basin of attraction of one of the equilibria. But the coded dynamics does not show this. This can be seen as an information loss one incurs into, in general, when coding. Another way to put it, is that coding simplifies description of dynamics, hopefully to focus on some more macroscopic features.

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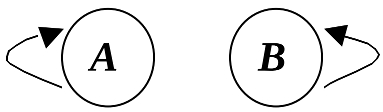
[39] It can be convenient to make $0 \in I_+$, making 0 the closure of the interval I_+, but this arbitrary at least to an extent. It corresponds to states with two identification tags.

It is to bring these facts to light the above examples were worked out. Not any non-linearity will do, if we want to establish the possibility of traverse paths which are the essence of regime dynamics.

6: Garden-variety regime dynamics.

The preceding section showed the most elementary possibilities that arise with two regimes associated with partitions of the state space, in that case a phase line. With only regimes A and B, we may have infinite strings with same symbols, i.e. $(A)^{\infty}$ or $(B)^{\infty}$, but behind them there are dynamic scenarios with more than one possibility to take into account.[40] This can be represented by a directed graph G, called the *transition graph* of the coded dynamics. For every path through this graph, there exists an orbit with itinerary satisfying the sequence of symbols determined by the path. Then, in this case the graph has two vertices labeled by A and B, one arrow from A to A and one arrow from B to B. Another way to represent the dynamics is via the transition matrix that represents the graph. That is a square {0,1} matrix whereby there is a 1 (or alternatively a 0) in the ij-entry whenever there is an arrow in the transition graph leading from the i-th to the j-th vertex. In the following figure we show the graph and the matrix.



$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Figure 4: Directed graph *G* and adjacency matrix *M* representing the coded dynamics of Harrod's model.

This is an alternative way of representing a "fully decomposable" dynamics: i.e. dynamics whereby starting in A (in B, respectively) implies staying there forever[41]. Clearly, with one of the two strings on hand, it is impossible to recover the model generating it uniquely, as the previous example of two structurally different versions of the same model should have convinced you[42].

With two regimes, however, one can see the possibility of more interesting cases: e.g. one can think of symbolic strings that contain blocks like AB and/or BA. For example, we can encounter models yielding dynamics that once coded looks like

$$A.BABAB\ldots = (AB)^{\infty},$$

or else

$$B.ABABA\ldots = (BA)^{\infty},$$

or they may contain either block after an initial transient period of a different dynamics within one of the two regimes, say $AAABAB = AA(AB)^{\infty}$. More complicated examples can be thought of, but we do not need them now. However, we can now begin thinking of and using a terminology that is better fit to coded dynamics.

Assume there is an accepted partition in regimes, i.e. in a finite set of pairs (M_j, R_j), where it is $j = 1, 2,\ldots, n$. It is coded with exactly j symbols, say the first 1 to j letters of the Latin alphabet, called a sub-alphabet. This coding converts any orbit $\{\ldots, x_0, x_1,\ldots, x_\tau,\ldots\}$ of a point x_0 into a string of symbols from the sub-alphabet. An infinite string is the same as a coded (regime-) trajectory, or an orbit when an initial regime is specified as a symbol. An infinite string displaying one and the same symbol will represent an equilibrium in the regime sense; likewise, an infinite sequence of the type $(AB)^{\infty}$ is a period two cycle. The presence of an initial set of As (or Bs) says that there is a transient dynamics within the corresponding regime, but

[40] Hence we have to be careful if we have to recover from a coded sequence a model that generates it. This problem shows up when we come to define partitions from a mathematical viewpoint.

[41] Compare with the n-dimensional case above, where you can never be in either one alone except for mathematical fluke, or by construction.

[42] That's the purpose of the lengthy discussion on an elementary model. Moreover, the list of models that could generate it is not finished there.

eventually dynamics settles down to the cycle, which is therefore a kind of limit cycle. More complicated strings ("random strings") will have a different dynamical name, but more of this later.

We now look for models that generate a dynamic variety in the regime sense that is richer than an equilibrium string, e.g. a regular cycle to begin with. It is clear from the above that dynamics across regimes can only be generated if one or more of them are unstable without being at the same time null regimes (whose dynamics gets lost forever). Then, after transients, we may have some predictable long run in the regime sense[43]. For our exercise it is interesting to consider a hyper-model (M, Σ) inducing three regimes (M_j, R_j), j = 1,..., 3. This will also show the role of the number of regimes in determining regime dynamics. It is clear that with 2 regimes (hence, two symbols) one can generate 4 strings or typical trajectories of one or two periods, so that the portfolio of possibilities is richer than what we have seen so far. We will return to it after the three regime detour, which is also an interesting piece of history of economic analysis.

It is also clear that these same trajectories can also be generated with any k symbols (that is, k regimes), for any $k \geq 2$. One says that a model with regimes is a theory of a particular <u>regime trajectory</u> (a period-2 regime cycle) if it predicts an infinite string like $(AB)^{\infty}$ under minimal requirements (as defined later!).

7: Three regimes are sufficient for cycles.

For any given finite partition of state space in *regime domains* R_j, j=1,2…,n, richer dynamics can be constructed by concatenation of local models (or phase structures, in the language of Day). Comparing with the previous case, this is the new idea we are going to develop in the sequel. It was Hicks who introduced explicitly the notion of a concatenated three regime dynamics, with a piecewise linear model of the business cycle, actually the first nonlinear model in the history of the subject. The three regimes correspond to the behaviour of the accelerator principle in three regions of the state space, which is there taken to be the set of values of GDP. Detrending the time series, we obtain a model in deviations, whereby the long run equilibrium is an exogenously driven growth path, set at the origin of the y-line that now stands for deviations from equilibrium[44]. Locally, the trend is unstable and thus any small deviation gets amplified over time, via the working of the interaction between multiplier and a standard, linear accelerator mechanisms. It works for values of the deviations that are not too large, either way, above or below equilibrium. This gives the middle regime domain, with its own (linear) model. Dynamic behaviour however is checked above and below by the presence of a ceiling and a floor, corresponding to the maximal attainable level of output (the full employment level) and the minimal level (that corresponds to the level where worn out machinery is not being replaced). At these values of y, the investment function will switch. Hence we have a three regime structure, whereby each pair (M_j, I_j) is made up of the model with appropriate investment function restricted to the interval I_j, j ranging over the three regimes, the middle one and the two outward ones. To construct this model from the previous one, investment I_t is made again a non-linear function of the rate of change of income Δy_t. However, in Hicks a second order hypothesis is introduced, i.e. $\Delta y_{t-1} = (y_{t-1} - y_{t-2})$, but we prefer to stick to Δy_t, as the fact that the global reduced form model is nonlinear replaces the arbitrary hypothesis on the order in generating the sought oscillations. Hence, let us have

$$I_t = F(\Delta y_t) = \begin{cases} k_1 \Delta y_t + a_1, & \text{if} \quad \Delta y_t \leq a_1 / k_2 - k_1 \\ k_2 \Delta y_t, \ \text{if} & a_1 / k_2 - k_1 \leq \Delta y_t \leq a_3 / k_2 - k_3 \\ k_3 \Delta y_t + a_3, & \text{if} \quad a_3 / k_2 - k_3 \leq \Delta y_t \end{cases}$$

where we have the following conditions: $0 < k_1 < 1$, $k_2 > 1$, $0 < k_3 < 1$, $a_1 < 0$ and $a_3 > 0$.

The consumption function at time t, C_t is part of income cy_{t-1} at time $t-1$:

---

43 It may not be so, as is explained in the following section.

44 This makes it formally similar to the Harrod model before, though obviously trends are defined differently.

$$C_t = cy_{t-1}, \text{ where } 0< c <1.$$

Replacing C_t and I_t in the equilibrium equation, and after some rearrangement we have the equation of the modified Hicks model

$$y_t = \begin{cases} \dfrac{c-k_1}{1-k_1} y_{t-1} + \dfrac{a_1}{1-k_1}, & if \quad y_{t-1} \le \dfrac{a_1(1-k_2)}{(k_1-k_2)(1-c)} \\ \dfrac{c-k_2}{1-k_2} y_{t-1}, \ if \quad \dfrac{a_1(1-k_1)}{(k_1-k_2)(1-c)} \le y_{t-1} \le \dfrac{a_3(1-k_2)}{(k_3-k_2)(1-c)} & \\ \dfrac{c-k_3}{1-k_3} y_{t-1} + \dfrac{a_3}{1-k_3}, & if \quad \dfrac{a_3(1-k_2)}{(k_3-k_2)(1-c)} \le y_{t-1} \end{cases}$$

If we denote : $m_1 = \dfrac{c-k_1}{1-k_1}, m_2 = \dfrac{c-k_2}{1-k_2}, m_3 = \dfrac{c-k_3}{1-k_3}, \text{-} \ n_1 = \dfrac{a_1}{1-k_1}, n_3 = \dfrac{a_3}{1-k_3}$, then the conditions :
$0 < k_1 < 1,\ k_2 > 1,\ 0 < k_3 < 1,\ a_1 < 0$ and $a_3 > 0$ imply that : $m_1 < c, m_2 > 1, m_3 < \mathrm{c}, n_1 > 0, n_3 > 0.$

If we define the function f by:

$$f(y) = \begin{cases} m_1 y - n_1, \ \text{if} \quad y \le x_1 = {}^{-n_1}\!\big/_{m_2 - n_1} \\ m_2 y, \quad \text{if} \quad x_1 \le y \le x_2 = {}^{n_3}\!\big/_{m_2 - n_3} \\ m_3 y + n_3, \qquad \text{if} \quad x_2 \le y \end{cases}$$

thus, the modified Hicks model, too, is represented by a first-order non-linear difference equation:

$$y_t = f(y_{t-1}).$$

The map f is now piecewise linear with two critical points x_1 and x_2. Dynamics, of course, depends upon the values of the five parameters: m_1, m_2, m_3, n_1, and n_3.

First a little bit of algebra. Just like in Hicks' original model, map f has a fixed point at $y = 0$ which is repelling (unstable) as $m_2 > 1$. Compared with the modified Harrod's model of section 3 above, now f has two more fixed points $p_1 > 0$ and $p_3 < 0$, (being $m_1 < 1$ and $m_3 < 1$), instead of one. Of these, we have that the fixed point p_i is stable if and only if $m_i > -1$ (that is, if and only if $k_i < ½(c+1)$) for i = 1, 3. [45]

Thus, if both m_1, $m_3 > -1$, dynamics is very simple: if $y_t \neq 0$ for all t, it goes to p_1 or to p_3 as t goes to infinity and if there is some q such that $y_q = 0$, then $y_t = 0$ for all $t \geq q$. Harrod's situation is now reproduced, and in a sense doubled up, as both non zero fixed points are attractors within their own regimes. Regime dynamics is simple in that the overall dynamics decomposes into two local dynamics, with their own attractors, while the middle domain splits up in two: the left hand (the right hand) side becoming transients with respect to the corresponding stable regimes. We are back to regime dynamics without concatenation. To generate Hicks' result, which is the novelty in such model as compared with Harrod's, we need to look for unstable regimes. We can think of the result as being obtained by smoothly changing slopes m_i, hence decreasing them to less than -1 we come to the stability bifurcation values**.** This is called a flip or period-doubling bifurcation.

In fact, when m_i = -1, p_i becomes unstable and there is a neighborhood of p_i of periodic points of period 2 ($i = 1,3$). Instead of making a detailed analysis of all other cases, we take some examples of instability, with given values for $m_1 < -1$ and/or $m_3 < -1$.

[45] The single non-zero fixed point of the Harrod's map is stable for m: $0<m<1$.

We could continue using standard bifurcation techniques to see the changes in behaviour of the one-parameter family of functions *{f(y); m_1}*, while the free parameter m_1 is allowed to change, now to the left of the critical value -1. Likewise, an analogous exercise could be performed with the other parameters. Combining we would obtain a picture for a family of functions with a vector of parameters. Instead, we will limit ourselves to investigating what regime dynamics looks like for a sample of values of the chosen parameter, in the unstable interval. They show that the regime dynamics obtained via concatenation via instability is quite rich, in fact richer than Hicks' himself believed.

Let fix m_3 = -4, m_2 = 2, x_2 =1, x_1 = -1, n_3 = 6. Then, as we have n_1 = 2- m_1, the function f depends on the parameter m_1 and is given by the formula:

$$f(y) = \begin{cases} m_1 y + m_1 - 2, & if \quad y \le -1 \\ 2y, \quad if \quad -1 \le y \le 1 & where \quad m_1 < -1 \\ -4y + 6, & if \quad 1 \le y \end{cases}$$

Thus, we take three particular cases: m_1 = -2, m_1 = -3 and m_1 = -4.

If m_1 = -2, we have that $f(y) = \begin{cases} -2y - 4, & if \quad y \le -1 \\ 2y, & if \quad -1 \le y \le 1 \\ -4y + 6, & if \quad 1 \le y \end{cases}$, and the graph is given in figure.5.

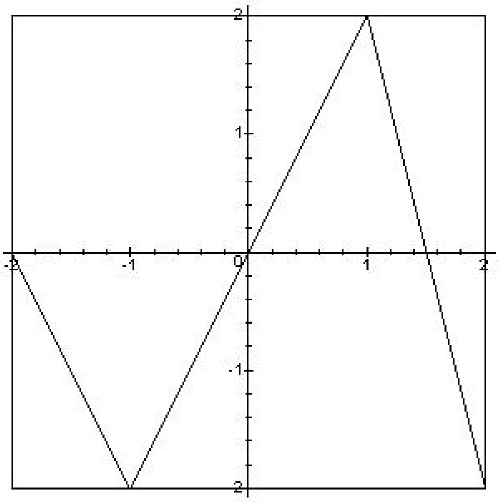


Figure 5: Hicks' model. Graph of map f for the parameter values m_1 = -2, m_2 = 2, m_3 = -4, n_1 = 4 and n_3 = 6. The interval I = [-2, 2] is invariant under the map f and all interesting dynamics lies in I. This interval was divided in four regime domains labelled by A, B, C and D: A = [-2, -1], B = [-1, 0], C = [0, 1] and D = [1, 2]. This partition verifies the relations: A∪B = f(A), A∪B = f(B), C∪D= f(C) and A∪B = f(D)

If m_1 = -3, we have that $f(y) = \begin{cases} -3y - 5, & if \quad y \le -1 \\ 2y, & if \quad -1 \le y \le 1 \\ -4y + 6, & if \quad 1 \le y \end{cases}$, and the graph is given in figure 6.

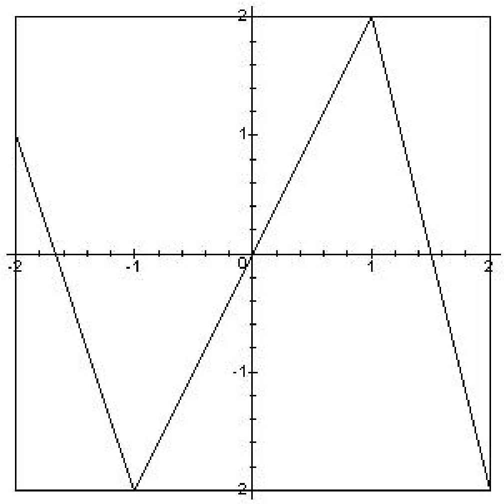


Figure 6: Hicks' model. Graph of map f for the parameter values m_1 = -3, m_2 = 2, m_3 = -4, n_1 = 4 and n_3 = 6. The interval I = [-2, 2] is invariant under the map f and all interesting dynamics lies in I. This interval was divided in four regime domains labelled by A, B, C and D: A = [-2, -1], B = [-1, 0], C = [0, 1] and D = [1, 2]. This partition verifies the following relations: A∪B∪C = f(A), A∪B = f(B), C∪D= f(C) and A∪B = f(D)

If m_1 = -4, we have that $f(y) = \begin{cases} -4y-6, & if \quad y \le -1 \\ 2y, & if \quad -1 \le y \le 1 \\ -4y+6, & if \quad 1 \le y \end{cases}$, and the graph is given in figure 7.

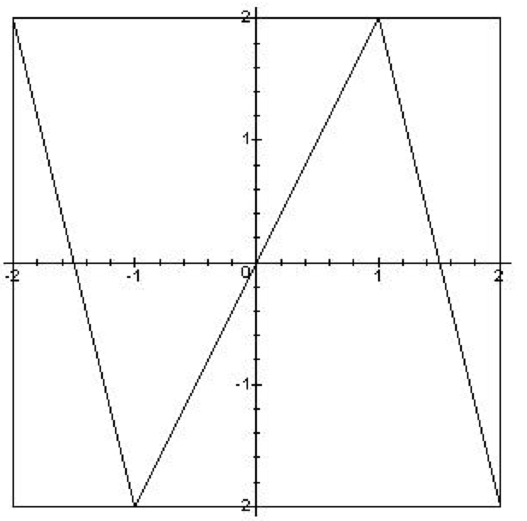


Figure 7: Hicks' model. Graph of map f for the parameter values m_1 = -4, m_2 = 2, m_3 = -4, n_1 = 4 and n_3 = 6. The interval I = [-2, 2] is invariant under the map f and all interesting dynamics lies in I. This interval was divided in four regime domains labelled by A, B, C and D: A = [-2, -1], B = [-1, 0], C = [0, 1] and D = [1, 2]. This partition verifies the relations: I=A∪B∪C∪D = f(A), A∪B = f(B), C∪D= f(C) and A∪B = f(D).

Observe that in all these cases the interval I = [-2,2] is invariant under the map f and therefore all interesting dynamics lies in that interval. We divide interval I in four pieces labeled by A, B, C and D: A = [-2, -1], B = [-1, 0], C = [0, 1] and D = [1, 2]. This partition of the phase space reflects the classification in regimes implied by economic model and we want to describe the regime switching phenomena. The above partition, however, follows the criterion of mathematical convenience (as it will be shown immediately) so that while not violating the distinction in regimes, it is in this case "finer" (and it need to be so). This appears from the observation that, while C and D do correspond to the outer regimes (as defined in Hicks' model), the middle regime corresponds to the union of two pieces, i.e. B ∪ C.[46]

All this shows that, while regimes imply a partition of state space, viceversa is not true. In general, as a rule of thumb, one can think that a regime classification can be a good starting point to obtain a state space partition that retains its fundamental dynamical features. (Thus, while we code with three letters, we need four symbols.) Thus, what we called coded dynamics, though a conceptually independent construction, is a way leading to symbolic dynamics, the latter being a mathematical branch, the former an economist's

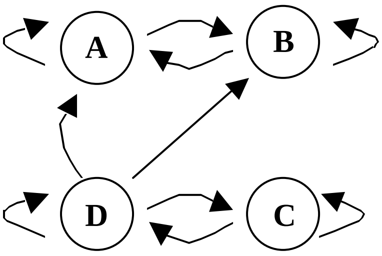
[46] In fact, distinguishing the middle region into B and C is useful also from the point of view of theory when fixed points in A and/or D contains attractors, as is in the previous case (and in our nonlinear version of Harrod).

exercise. When we are lucky, coded dynamics can also be handled by symbolic methods, as is the examples purposely chosen in this paper.

For the three cases associated with the different values of the chosen control parameter m_1, the partition into four pieces of state space I is such that the following relations hold under map f:

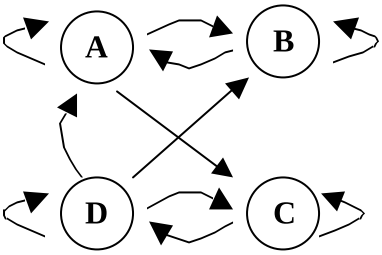
$$A \subseteq f(A),\ B \subseteq f(A),$$
$$A \cup B = f(B),$$
$$C \cup D = f(C),$$
$$A \cup B = f(D)$$

Moreover, it is $A \cup B = f(A)$ for $m_1 = -2$, $A \cup B \cup C = f(A)$ for $m_1 = -3$, and $A \cup B \cup C \cup D = f(A)$ for $m_1 = -4$. For one such partition and the related map f, therefore, the corresponding coded dynamics is represented by directed graphs and transition matrices as shown in the following figures. (The possibility of using such devices dictates the criterion of mathematical convenience.) Then, for anyone of the three values of the chosen parameter m_1, two cycles arise: one in the positive piece of I, i.e. $C \cup D$, and the other in the negative, $A \cup B$. We can also go from the positive piece to the negative piece but the converse is not valid for $m_1 = -2$. For $m_1 = -3$, we can go from A to C and then a cycle emerges involving a coded orbit with one positive piece and the other negative. For the parameter value $m_1 = -4$ this can also be obtained from A to D and there is a cycle A-D.



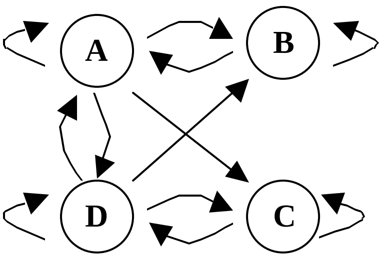
$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Figure 8: Transition graph and transition matrix M for the partition $P = \{A,B,C,D\}$ and the map f that represents Hicks' model when the parameter values are $m_1 = -2$, $m_2 = 2$, $m_3 = -4$, $n_1 = 4$ and $n_3 = 6$. All random walks on this graph are allowed symbolic sequences of the regime dynamics.



$$M = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Figure 9: Transition graph and transition matrix M for the partition $P = \{A,B,C,D\}$ and the map f that represents Hicks' model when the parameter values are $m_1 = -3$, $m_2 = 2$, $m_3 = -4$, $n_1 = 4$ and $n_3 = 6$. A new arrow from A to C emerged allowing the traverse from the negative piece to the positive.



$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Figure 10: Transition graph and transition matrix M for the partition $P = \{A,B,C,D\}$ and the map f that represents Hicks' model when the parameter values are $m_1 = -4$, $m_2 = 2$, $m_3 = -4$, $n_1 = 4$ and $n_3 = 6$. In this case the positive and negative piece are fully connected.

Thus, the structurally stable feature of the dynamics associated with the three regimes (and a geometric partition into four pieces) is the presence of two apparently distinct cycles, $(AB)^{\infty}$ and $(CD)^{\infty}$. (All other features can be more or less imputed to particular values of the control variable, as indicated above). Here, intuition should help to understand the dynamics implied by the two symbolic cycles. Note that regions A, B and C, D share a frontier state that has two names: it belongs to both pieces (and it is their closure(s)). Let us then look at the structure of the outer regions, i.e. A and D. In A (say, in D), there is a unique fixed point that is a repulsor, hence all dynamics surging out of the fixed point will point outwards. Thus, there will be a direction that will travel the representative point to loose itself (the left hand side for A**,** the right hand side for *D***,** with respect to the respective fixed point). Thus, refining partition as $A/\{p_j\} = A_+ \cup A_-$ (with A_+, A_- the right and the left hand side open intervals)[47], we see that while A_+ acts as a null phase regime for the local dynamics, A_- acts as part of a basin of attraction for now the whole of the inner regime, i.e. $B \cup C$**.** In other words, all paths starting with A**.** But with $x_0 \in A_-$ will travel towards it. The same argument applies for the outer-regime D. And for an analogous but opposite reasoning, we can say that from the inner regime, representative points are flown outwards towards A_- (from B) and D_+ (from D**,** respectively). This set of opposite forces can only settle down to a path that is both $(DC)^\infty$ and $(AB)^\infty$**,** i.e. it travels from one frontier point to the other, with cycle 2. This is Hicks' result.

8: But two regimes are already sufficient to generate a regular cycle.

This was Goodwin's original intuition i.e. that the minimal number of regimes needed to generate a regular cycle would be two, rather than the three of the Hicksian approach[48]. This can be easily seen from the following version of the above model, in a sense a simplified version for it comprises only one linearity, rather than two, but of course of a particular kind. Going over the example also shows how a particular dynamics (in this case a regular oscillation) can be built up from the careful use of a chosen control parameter. (To exhibit this point we go over the analysis with some extra care, the control parameter being chosen to be the slope of the investment function.) In fact we have chosen with the nonlinear Harrod that the latter plays a crucial role in determining the internal structure of the regime(s) and in generating the cross regime dynamics. This is a more elaborated version of the above. Hence, with the function

$$I_t = \begin{cases} k_2 \Delta Y_t, \quad k_2 > 1, \quad \Delta Y_t \le \dfrac{a_3}{k_2 - k_3} \\ k_3 \Delta Y_t + a_3, \quad 0 < k_3 < 1, \quad a_3 > 0, \quad \dfrac{a_3}{k_2 - k_3} \le \Delta Y_t \end{cases}$$

we get the reduced form equation

$$Y_t = \begin{cases} \dfrac{c - k_3}{1 - k_3} Y_{t-1} + \dfrac{a_3}{1 - k_3}, \qquad \dfrac{a_3(1 - k_2)}{(k_3 - k_2)s} \le Y_{t-1} \\ \dfrac{c - k_2}{1 - k_2} Y_{t-1}, \quad Y_{t-1} \le \dfrac{a_3(1 - k_2)}{(k_3 - k_2)s} \end{cases}$$

Let us rename the coefficients:

$$\frac{c - k_2}{1 - k_2} = A, \ \frac{c - k_3}{1 - k_3} = B \text{ and } \frac{a_3}{1 - k_3} = C. \text{ Then we have the conditions : } A > 1, \ B < c < 1 \text{ and } C > 0.$$

Then, our version of the Goodwin's nonlinear accelerator-multiplier model in discrete time can be represented by the first order difference equation $y_{t+1} = f(y_t)$, where the function f is a piecewise linear map with two, instead of three, branches of the type:

$$f(y) = \begin{cases} Ay, \quad if \ y \le y_0 = \dfrac{C}{A - B} \\ By + C, \quad if \ y_0 \le y \end{cases} \quad \text{where } A > 1, B < 1 \text{ and } C < 0.$$

47 The same can be done for D.

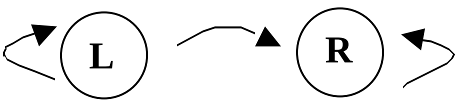
48 See Goodwin (1950); see also Punzo and Velupillai (1997) on Goodwin's use of *Occam's razor.*

The map still has two fixed points: 0 and y_1 ($y_1 > y_0 > 0$). Given the conditions that must verify A, B and C, the character of the fixed points depends only on the values of the parameter B: 0 is a repulsor for all values of B and y_1 is stable for $-1 < B < 1$ and unstable for $B \leq -1$.

We distinguish two regimes in the model: one for each monotone piece of the function f. We label L for the left and R or the right interval in the division of the phase space via the turning point y_0. For simplicity, we divide the left regime L in the three pieces $L^- = (-¥, 0)$, $L^+ = (0, y_0]$ and $\{0\}$, according to the fixed point because the dynamics there is of the following form:

a) Starting at 0, we remain there forever.
b) Starting at a point of L^-, the orbit goes to infinity and the system is self-destructing.
c) Starting at a point of L^+, the orbit goes to the stable fixed point if $-1 < B$ and when $-1 = B$ goes to the regime R and remains there in a periodic orbit.

Then, for $0 \leq B$, the symbolic sequences representing the regime dynamics are LLLL… = $(L)^\infty$, LL…LRRR… = LL…L$(R)^\infty$ and RRR… = $(R)^\infty$. Then, the regime dynamics can be represented by the transition graph and adjacency matrix of figure 11. Note that in this case we reproduce the situation seen in the Harrodian generalized model above.



$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Figure 11: Transition graph and transition matrix for the partition $P = \{L, R\}$ and the map f that represents Goodwin's model when $0 \leq B$.

For $B<0$, on the other hand, there is a positive zero of f and is the point $y_2 = -C/B$. In figure 12 we show two different representative graphs of f for $B<0$. This allows orbits starting in R to go to L^-. For this kind of sequences, the symbolic representation is $RLLL... = R(L)^¥$. Starting at a point $y \leq 0$, the orbit remains in 0 or L^- and then the symbolic representation will be $LLL... = (L)^¥$. Thus, all the interesting dynamics occurs in the interval $[0, y_2]$ where the map is like a tent map. In the interval $[0, y_2] \cap R$ there is a point p such that $f(p) = y_0$; this point bifurcates orbits starting in R in those that remains in R (when the initial condition is in $[y_0, p)$) and those that goes to L (when the initial condition is in the interval $[p, y_2]$). We will separate the case $B<0$ in two cases: $-1 \leq B < 0$ and $B < -1$ according to the different symbolic dynamics we will obtain.

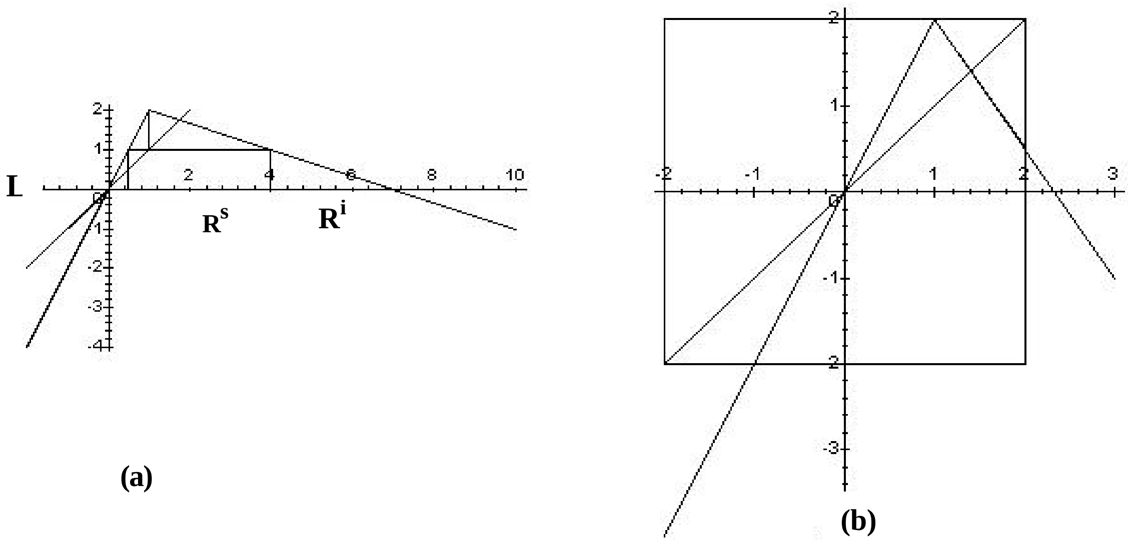
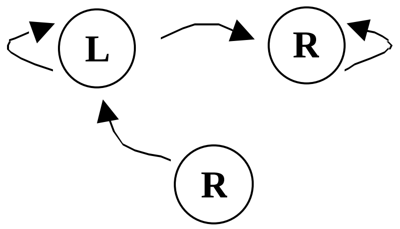


Figure 12: **(a)** The graph of the map f for $-1<B<0$. **(b)** The graph of the map f for $B < -1$. We have also drawn the graph of the identity map and the square $[0, f(y_0)]^2$. The fixed points 0 and y_1 are repulsors. All interesting dynamics is in the interval $[0, y_2]$. If $f(y_0) \leq y_2$, like the case in this figure, the interval $[0, f(y_0)]$ is invariant.

1) If $-1 < B < 0$ the fixed point is an attractor. Figure 12 (a) shows a representative graph of f in this case. Orbits starting in $[0, y_0]$, after a finite number of iterations where they stay at L, enters R and remain there; orbits starting in $[y_0, p)$ remains in R for ever, and orbits starting in $[p, y_2]$ goes to L^+ in the first iteration. Then the symbolic sequences for orbits starting in $[0, y_2]$ are of the form $S_1\, S_2 \ldots S_K\, RRR\ldots = S_1\, S_2 \ldots S_K\, (R)^{\infty}$, where the initial string $S_1\, S_2 \ldots S_K$ can be $LL\ldots L = (L)^K$, $RR\ldots R = (R)^K$ or $RL\ldots L = (L)^{K-1}$. Symbolic sequences are the same because although the fixed point is unstable, around it all orbits are periodic of period two.

To obtain a good representation of the regime dynamics via a directed graph we have to do the following partition: L, $R^{(1)} = [y_0, p)$ and $R^{(2)} = [p$, ¥); i.e. we have to divide regime R into the two pieces, the stable one $\mathbf{R}^{\mathbf{s}}$ and the transient $\mathbf{R}^{\mathbf{i}}$ to L.

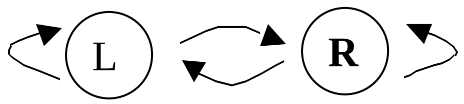


$$A=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Figure 13: Transition directed graph and transition matrix for the partition P = {L, R^s, R^i} and the map f that represents Goodwin's model when it is $-1 \le B < 0$.

2) If $B < -1$ the fixed point y_1 is unstable. Figure 12 (b) shows a representative graph of f in this case.

We have to analyze the dynamics of the function f in the interval $[0, y_2]$, where the piece $[0, y_0]$ has the label L and the piece $[y_0, y_2]$ the label R. Orbits starting in $[0, y_0]$, after a finite number of iterations where they stay at L, enters R. Orbits starting in points of $[y_0, y_2]$ can remain there for ever (like the orbit starting in the fixed point y_1) or after finite iterations, can leave R. So, in this case the regime dynamics is represented by the one-side full shift of two symbols and then the transition graph and the transition matrix for this case are the following:



$$A=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Figure 14: Transition graph and transition matrix for the partition P = {A, B} and the map f that represents Goodwin's model when $B < -1$.

Thus, two regimes may yield if properly dynamically coupled a regular cycle. In fact, they can yield much more than that, for the above tent-like map is known to be capable to a much wider variety of dynamics (this is indeed shown by the all-ones transition matrix above). This is something to investigate further, dealing once again with the three-regime structure. Nevertheless, the intuition is vindicated, even in discrete time dynamics.

9: Conclusions

This paper is written in the philosophy, so to speak, that all measurement-based approaches should accept that measurement is approximation, in other words that it is a finite precision exercise. The infinite precision requirement that goes together with the conventional modelling approach has placed demands on economic models that they simply could not satisfy. The dream of an infinite precision in describing dynamics has sclerotized, among other things, the dichotomy between stochastic and deterministic approaches, that has been the nightmare of the last years.

Chaos theory has taught us that almost always the fine structure implied by the classical state space approach is just too fine to account for all the possibilities that may arise, even in the simplest economic models. The approach proposed in this paper incorporates this basic message and tries to see a way out of what may have looked to many either as an impasse or as a sterile theoretical and mathematical exercise.

Coded dynamics and symbolic dynamics are not the same thing, nor do they not share origins. In this paper, we try to marry the two notions by making first precise the economic notion of regime that is being intuitively used in the economic literature. The definition we introduce, is such as to allow coding of economic dynamics (and economic times series): i.e. converting continuous time series into discrete sequences of finite symbols. Alternative definitions are left out, as they are fit for other uses. Coded dynamics, then, becomes not only a statistical device to handle noisy time series, but also a theoretical approach to represent the possibility of having qualitative dynamics dependent on state space in a discontinuous fashion. This is the important implication of the multi-regime model, or the hyper-model of a system dynamics. This notion is best suited to multivariable systems, though we only gave here one-variable examples. Among the phenomena that can thus be captured, one enlists chaotic and strange behaviours, but also path dependence and regular cycles, among them the most regular of all oscillations, stationary equilibria. Regimes and the technique of coding bring back into dynamics the unity between two extremes: chaos versus stationariness, quasi-stochasticity versus full predictability, the unity that recent literature has undermined. In a coded history, there is only more or less complex dynamics to talk about, and there is a natural way to measure such complexity. Hence, it is no longer a matter of "going chaotic", or else "going fully predictable", and adjust our models accordingly. We model for given or chosen levels of complexity. Thus, the need for a coding approach emerges from the economist's understanding of the issues at stake in modelling dynamics.

Regimes coincide with local models, in the sense in which economists understand them, and coded dynamics is in principle a sub-field of symbolic dynamics. This shows up in particular in the sects. 7-8, where we deal with sets of locally different models, whereby the appropriate symbolic partition is finer than the regime partition, and in particular it requires multiple criteria (in fact, the cross product of two of them). Symbolic dynamic techniques can however still be effectively used to understand the dynamics of the multi-regime model for the latter's partition becomes embedded into the one induced by the former. This appears at this stage as a reasonable criterion to justify use of that mathematical technique.

We have investigated alternative structures obtained by coupling nonlinearly local models that can be of a linear or of a non-linear type, i.e. of different degrees of mathematical "hardness". The strategy, however, is still one dictated by the principle of economy in modelling, i.e. look for the simplest (set of) models, or hypotheses that can generate/simulate the desired outcome. In this case, this has been applied by simplifying as far as possible the local structure and complexifying the interlocking. This corresponds to a connectionist philosophy that has many different versions in different sciences (i.e. neural nets and the like). Ours is a preliminary chart of what can be obtained from such an approach; more seems to lie ahead. Among other things, the use of entropy measures appears to offer a useful bridge between modelling and statistical techniques. The "empirical law" that seems to link functionally the measure of entropy to the tuning parameter, as observed earlier, non only is amenable to statistical investigation, but also, if generally true, it would place restrictions onto a meaningful modelling that start from actual data.

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