QUADERNI

Università degli Studi di Siena
DIPARTIMENTO DI ECONOMIA POLITICA

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Weight and Values with Incomplete Communication

n. 314 - Gennaio 2001

Weights and values with incomplete communication

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March 23, 2001

Abstract

We focus on the space of finite coalitional games whose player set coincides with the vertex set of some exogenously given *simple graph.* We adopt the usual approach consisting in the treatment of such games in terms of the associated (projective) space of *point games*, one of whose bases is the collection of all *connected* unanimity games. In association with each element of such a basis we consider the corresponding *spanned subgraph* and use this latter to define two hierarchical structures, formalized through a vector- valued set function quantifying weights. We propose (and axiomatically characterize) two corresponding value functions that are consistent with each of such hierachical structures when the associated connected unanimity game is played. The values thus obtained result to be generalizations of weighted Shapley values. By means of a newly defined *arc game*, we also provide an alternative computation procedure for the Myerson (1977) value.

Key words: coalitional game, simple graph, connected unanimity game, spanned subgraph, weighted Shapley value, Möbius transform.

JEL classification number:

1 Intoduction

The aim of this article is to propose two generalizations of weighted Shapley (1953a) values for coalitional games with incomplete communication. Roughly speaking, coalitional games formalize situations where cooperation among individuals displays synergies quantified in terms of *transferable utility*, TU. Then the value (or solution) problem concerns the definition of a rule for sharing the fruits of such synergies between cooperating individuals. In this paper we consider finite coalitional games where, in addition, some exogenously given constraints impede complete communication among the players. Such constraints are modeled by defining a certain collection of unordered pairs of players which

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identify the only available one-to-one communication possibilities. Within the treatment of coalitional games, constrained communication had been firstly modeled by means of unordered partitions of the player set (as in Aumann and Dréze (1974) and Owen (1977)). Subsequently, the definition of ordered partitions, together with weight vectors, enabled to focus on communication structures where the constraints seemed to be due to (or to translate into) some sort of hierarchies within the player set (see Kalai and Samet (1988), and also Nowak and Radzik (1995)). Myerson (1977) was the first to use (simple) graphs for formalizing the existence of communication constraints, and solved the resulting graph-restricted games through his well known value; this latter turned out to be the standard Shapley (1953b) value of a so-called *point game* (see Borm *et al.* (1992)), which is, in fact, the projection of the original game (i.e., a generically superadditive set-function on the power set of the player set) on the collection of *connected* coalitions. Afterwards, communication constraints have been mainly modeled by means of graphs[1], and two further value functions have been proposed, that is the position (Borm *et al.* (1992)) and the Hamiache (1999) values. Owen (1986) also considered the Banzhaf (1965) value for point games.

It must be emphasized that a simple graph allows for different theoretical approaches to the solution problem of the (generic) associated graph-restricted game. In fact, each of the three values mentioned above definitely results in some hierarchical structure over the player set (which depends, in turn, on some peculiar interpretation of the information contained within the edge set of the exogenously given graph). Nevertheless, sometimes even a vague idea of such a resulting hierarchical structure is hard to be made (see Borm *et al.* (1992), final remarks p. 319).

Many of the proofs (and reasonings in general) concerning value functions for coalitional games rely on the following basic idea: any value is a mapping from the space of all games on a given player set to the euclidean space whose dimension equals the number of players, and if two mappings coincide on a basis of the mapped space, then they coincide over the whole space. Moreover, it is often useful to check how a given value behaves in association with the generic element of a basis (see Hamiache (1999), p. 64). If the purpose of any axiomatic characterization of a value is that of explaining its behavior (in general, that is for the whole space of coalitional games), then probably its precise (i.e., quantitatively defined) behavior for all the elements of a basis is very informative. Thus, we proceed as follows: we first impose a certain payoff vector, identifying a peculiar hierarchical structure, for each game that is an element of a basis, and then determine the unique value function which is consistent with each hierarchical structure when the associated game is played[2]. The

[1] Alternatively, a broad class of constraints (that may be seen as comprehending those due to lack of communication capabilities) has been succesfully modeled by means of *combinatorial structures* called *convex geometries*. See Bilbao (1998) and Bilbao, Jiménez and Lébron (1998).

[2] A vague similarity with our procedure can be found in Hamiache (1999), who also makes use of two axioms (i.e., *positivity* and *independence of irrelevant players*) that can only be

hierarchical structure we impose in association with each element of a basis is derived by focusing on the corresponding spanned subgraph. More precisely, we make use of the information this latter contains by freely borrowing from *social network analysis* (Wasserman and Faust (1994)). In particular, we treat *blocks* (of the above mentioned spanned subgraph) as unions (or cohesive subgroups) and use their *connectivity* features for measuring cohesion. In fact, we call the values we propose *block-connectivity-degree, BCd*, and *block-connectivity-connectivity, BCc, values.*

2 Formalization

Let $N = \{1, \ldots, n\} \subset \mathbb{N} \backslash \{\infty\}$ be a finite[3] *player set*. A coalitional game on N consists of a *characteristic set function* $v : 2^N \rightarrow \mathbb{R}_+$ satisfying $v(\emptyset) = 0$. A graph-restricted coalitional game requires, in addition, some exogenously given *edge set* $E(N) \subset N^{(2)} := \{S \subset N \mid \#S = 2\}$, so that $g(N) = \{N; E(N)\}$ is a *simple graph*, that is an *ordered*[4] pair of a vertex set N and an edge set $E(N)$, respectively. The edge set formalizes all the information concerning communication constraints. In fact, *"the graph $g(N)$* [...] *represents the communication channels available: $i \in N$ can communicate directly with $j \in N$ if and only if $\{i,j\} \in E(N)$. Of course, even if $\{i,j\} \notin E(N)$, it may still be possible for i to communicate with j. This will, however, require the cooperation of some intermediaries who can relay a message, i.e., players who define a path in $g(N)$, from i to j."* (Owen (1986), p. 210).

A *subgraph* is any pair $\{S \subset N; K \subset E(N)\} \subset g(N)$. Given any graph $g(N)$, let $E : 2^N \rightarrow 2^{E(N)}$ be given by $E(S) := \{\{i,j\} \in E(N) \mid \{i,j\} \subset S\}$ for all $S \subset N$, and define $g(S) := \{S; E(S)\}$ to be the subgraph *spanned* by S. For all singletons $\{i\} \in 2^N$, let $E(i) := \{\{i,j\} \subset N \mid j \in N \backslash i, \{i,j\} \in E(N)\}$, and $d_T(i) := \#\{E(i) \cap E(T)\}$ for all $T \subsetneqq N$, since $d_N(i) = d(i) = \#E(i)$ is the *degree* vertex $i \in N$ displays in $g(N)$; if $d(i) = 0$, then i is said to be an *isolated vertex*, or, equivalently, a *trivial (sub)graph*. Furthermore, if $d_T(i) = 1$, then $i \in T$ is said to be *pendant* in $g(T)$. For any pair of distinct players $i, j \in N$, an $i - j$ *path* is a peculiar subgraph $P_{ij} = \left\{V_{P_{ij}}; E_{P_{ij}}\right\} \subset g(N)$ of the form

$$
\begin{aligned}
V_{P_{ij}} &= \{i = i_0, i_1, \ldots, i_{m-1}, i_m = j\} \subset N \\
E_{P_{ij}} &= \{\{i_0, i_1\}, \{i_1, i_2\}, ..., \{i_{m-1}, i_m\}\} \subset E(N)
\end{aligned}
$$

P_{ij} is said to *connect* i and j in $g(N)$; in fact, any graph $g(N)$ is defined to be *connected* if for every pair $i, j \in N$ of distinct vertices there is an $i - j$ path

defined in terms of all games that are elements of a basis of the space of point games.

[3] Many studies focus on infinite games whose player set $U \mid \#U = \infty$ has a finite *carrier* $N \subset U \mid \#N = n \in \mathbb{N} \backslash \{\infty\}$, where $v(S) = v(S \cap N)$ for all $S \subset U$. Our treatment, which concerns finite games only, does not lead to a relevant loss of generality. Furthermore, only finite graphs have been so far applied to coalitional games with incomplete communication.

[4] We use semicolumns $\{\ldots; \ldots\}$ to denote ordered collection of subsets and/or of elements. Furthermore, we denote by $\#A$ the cardinality of any set A.

$P_{ij} \subset g(N)$, otherwise the graph is *disconnected*. A *maximal connected subgraph* $g(S) \subset g(N)$ (i.e., such that there does not exist any larger subgraph $g(T)$, such that $g(N) \supset g(T) \supsetneqq g(S)$, which is connected as well) is a *component* of $g(N)$. We denote by $S/g := \{g(T_1), \ldots, g(T_{m_S})\}$ the collection of all the components of any spanned subgraph $g(S) \subset g(N)$. We also need to consider non-spanned subgraphs: for any edge subset $K \subset E(N)$ let S/g_K denote the collection of all the components of the subgraph $\{S; (E(S) \cap K)\} \subset g(S)$. If $g(S)$ is connected, while $g(S \backslash H) = \{S \backslash H; E(S \backslash H)\}$ is disconnected, then H *separates* $g(S)$; the maximal value $h = \#H \leq \#S - 1$ such that no set of $h-1$ vertices separates $g(S)$ (or makes it trivial) is the *vertex connectivity*, or simply the connectivity, $\kappa(g(S))$, of the subgraph spanned by S; analogously, its edge connectivity $\kappa^{\mathrm{e}} g(S)$ is the maximal value $k = \#K$ such that no edge subset of $k-1$ edges separates $g(S)$ (i.e., such that $\{S; (E(S) \backslash K)\}$ is disconnected). Thus

$$\kappa(g(S)) := \min\{\#S-1, \#H \mid H \subset S \text{ separates } g(S)\} \qquad \forall S \subset N$$

Clearly, every *k-connected* graph (i.e., such that $k = \kappa(g(S))$) has at least $k+1$ vertices and the unique k-connected graph of order $k+1$ is the *complete graph of order*[5] $k+1$, that is $\{M; M^{(2)}\}$, with $\#M = k+1$. A *cutvertex* (a *bridge*) of any subgraph $g(S) \subset g(N)$ is a vertex $i \in S$ (an edge $\{i,j\} \in E(S)$) such that its deletion increases the number of components, that is $\#S/g > \#(S \backslash i)/g$ (or $\#S/g_{E(S)} > \#S/g_{(E(S) \backslash \{i,j\})}$). A subgraph $g(B) \subset g(S) \subset g(N)$ is a *block* of $g(S)$ if either $g(B)$ consists of a bridge together with its endvertices, or else it is a *maximal 2-connected subgraph* of $g(S)$. Any two blocks have at most one vertex in common. Every vertex belonging to two blocks is a cutvertx, and, conversely, every cutvertex belongs to at least two blocks; furthermore, $g(S)$ decomposes into its blocks $g(B_1^S), \ldots, g(B_{m_S}^S)$ in the following sense: $E(S) = \bigcup_{j=1}^{m_S} E(B_j^S)$ and $E(B_j^S) \cap E(B_i^S) = \emptyset$ for all $i, j \in \{1, \ldots, m_S\}$ such that $i \neq j$. In words, the edge set of any (sub)graph can be partitioned into the edge sets of its blocks.

Let $\Gamma(N) := \{g = \{N; E(N)\} \mid E(N) \subset N^{(2)}\}$ denote the set of all simple graphs on N. For N is understood to be the whole player set throughout the remainder of the paper, in order to simplify notations let $g \in \Gamma(N)$ denote the generic graph on N; thus we use parentheses $g(S)$ only for spanned subgraphs, that is when $S \subsetneqq N$. Also let $\mathcal{G}(N) := \{v : 2^N \to \mathbb{R} \mid v(\emptyset) = 0\}$ denote the vector space of all (unrestricted) coalitional games on N. Eventually, we shall denote by $\mathcal{G}_\Gamma(N) := \{(v, g) \mid v \in \mathcal{G}(N), g \in \Gamma(N)\}$ the space of all graph-restricted coalitional games on N. A value $\phi : \mathcal{G}(N), \mathcal{G}_\Gamma(N) \to \mathbb{R}^n$ is a real, vector-valued function which assigns to each player $i \in N$ his prospect $\phi_i(v)$ or $\phi_i(v, g)$ from playing a given unrestricted or graph-restricted coalitional game. For any $T \subset N$, let $u^T \in \mathcal{G}(N)$ identify the associated unanimity game, that is

$$u^T(S) = \begin{cases} 1 \text{ if } T \subset S \\ 0 \text{ if } T \backslash S \neq \emptyset \end{cases} \qquad \forall S \subset N$$

[5] The *order* of a graph $g(N) = \{N; E(N)\}$ is the cardinality $n = \#N$ of its vertex set; its *size* is the cardinality $\#E(N)$ of its edge set.

Also let $\Pi(A)$ denote the set of all permutations of any set A. In general, we shall consider games satisfying $S \subset T \subset N \Rightarrow v(S) \leq v(T)$; such games are said to be (weakly) *monotone increasing.*

Given any game $v : 2^N \rightarrow \mathbb{R}$, its *Mobiüs tranform*[6] $a^v : 2^N \rightarrow \mathbb{R}$ is a one-to-one and invertible mapping given by $a^v(T) := \sum_{S \subset T} (-1)^{\#T - \#S} v(S)$ for all $T \subset N$, the recovering of v given a^v being defined by the so-called *Zeta transform* $v(T) = \sum_{S \subset T} a^v(S)$ for all $T \subset N$. Therefore, for all $S \subset N$ we can write $v(S) = \sum_{T \subset S} a^v(T) u^T(S)$ (see Shapley (1953b), lemma 3), so to observe that the collection $\{u^T\}_{\emptyset \neq T \subset N}$ constitutes a basis of $\mathcal{G}(N)$, with finite dimension $2^n - 1$. Weber (1988) was the first to define *probabilistic* and *random-order values* as, respectively, those given by $\phi_i(v) = \sum_{T \subset N, T \ni i} p_i^T [v(T) - v(T \backslash i)]$ for all $i \in N, v \in \mathcal{G}(N)$, where $\{p_T^i\}_{T \subset N, T \ni i}$ is a probability distribution for all $i \in N$, and those given by $\phi_i(v) = \sum_{\pi \in \Pi(N)} P(\pi) [v(PR^\pi(i) \cup i) - v(PR^\pi(i))]$ for all $i \in N, v \in \mathcal{G}(N)$, where $\{P(\pi)\}_{\pi \in \Pi(N)}$ is a unique probability (i.e., constant across players) on the set $\Pi(N)$ of all orderings of the players, and where, for any permutation $\{i_{(1)}, \ldots, i_{(n)}\} = \{\pi(1), \ldots, \pi(n)\} = \pi \in \Pi(N)$ of the player set, the associated set $PR^\pi(i) := \{j \in N \backslash i \mid \pi(j) < \pi(i)\}$ denotes the collection of players that preceed i according to π.

For all $i \in N$, the Shapley (1953b) value $\phi^{Sh} : \mathcal{G}(N) \rightarrow \mathbb{R}^n$ is given by $\phi_i^{Sh}(v) = \sum_{\substack{S \subset N \\ S \ni i}} \frac{(n - \#S)!(\#S - 1)!}{n!} [v(S) - v(S \backslash i)] = \sum_{\substack{S \subset N \\ S \ni i}} \frac{a^v(S)}{\#S} =$
$= \sum_{\pi \in \Pi(N)} \frac{1}{\#\Pi(N)} [v(PR^\pi(i) \cup i) - v(PR^\pi(i))]$, where $\#\Pi(N) = n!$. Thus, $\phi_i^{Sh}(u^S) = (\#S)^{-1}$ if $i \in S$, and 0 if $i \in N \backslash S$, for each unanimity game u^S, $S \subset N$. For any $\pi \in \Pi(N)$ and $S \subset N$, define $\pi S = \{i_{(j)} \mid i_{(j)} \in S\}$. Then, for any $v \in \mathcal{G}(N)$, the game πv is defined by $\pi v(\pi S) = v(S)$ for all $S \subset N$; thus πv is simply obtained by relabeling the players. Furthermore, define the game v_S as $v_S(T) = v(T \cap S)$ for all $S, T \subset N$. Now consider the following axioms for value functions: • *efficiency (E):* $\sum_{i \in N} \phi_i(v) = v(N)$ for all $v \in \mathcal{G}(N)$

- *linearity (L):* $\phi(\alpha v + \beta z) = \alpha\phi(v) + \beta\phi(z)$ for all $\alpha, \beta \in \mathbb{R}, v, z \in \mathcal{G}(N)$
- *symmetry (S):* $\phi_{\pi(i)}(\pi v) = \phi_i(v)$ for all $v \in \mathcal{G}(N), \pi \in \Pi(N)$
- *dummy (D):* if $v(S \cup i) - v(S) = v(i)$ for all $S \subset N \backslash i$, then $\phi_i(v) = v(i)$
- *weighted balanced contributions (WBC):* for all $v \in \mathcal{G}(N), i, j \in N, \lambda \in \mathbb{R}_+^n$

$$\frac{\phi_i(v) - \phi_i(v_{N \backslash j})}{\phi_j(v) - \phi_j(v_{N \backslash i})} = \frac{\lambda_i}{\lambda_j}$$

It can be proved that ϕ^{Sh} is the unique value function satisfying *E, L, D* and *S* (see Shapley (1953b) and also Grabisch and Roubens (1999), Theorem 1); alternatively, it can also be proved that ϕ^{Sh} is univocally characterized by means of *E* and W*BC*, and by imposing $\lambda_i = 1$ for all $i \in N$ (this was firstly proved by Myerson (1980)).

[6] The ratio $a^v(T)/\#T$ is the Harsany (1963) dividend any player $i \in T$ receives from joining any coalition $T \subset N$.

3 The Myerson and position values

We already mentioned that the Myerson (1977) value is the Shapley value of a so-called point game. This latter is a peculiar interpretation of the generic graph-restricted game $(v,g) \in G_\Gamma(N)$ based upon the following approach: the exogenously given graph $g \in \Gamma(N)$ is used to determine the (unique) associated collection $\mathcal{F}_g := \{S \subset N \mid \#S/g = 1\} \subset 2^N$ of feasible (i.e., connected) coalitions; then a new game v/g is obtained as follows: v/g coincides with v on $\mathcal{F}_g$, while it is additive on S/g for each $S \in 2^N \backslash \mathcal{F}_g$. Thus, v/g can be seen as the projection of v on $\mathcal{F}_g$, and it has been firstly named the *point game* associated with (v,g) in Borm *et al.* (1992). As we shall see, v/g plays a key role for the treatment of graph-restricted games. Its definition relies upon the idea that when we consider any two coalitions $S,T \subset N$ such that $S,T \in \mathcal{F}_g$ while $(S \cup T) \notin \mathcal{F}_g$, no communication (nor, *a fortiori*, cooperation) between S and T can occur unless (i) there exists at least one path $P_{ij} \subset g$ such that $i \in S$ and $j \in T$, and (ii) all players $h \in V_{P_{ij}} \backslash (S \cup T)$ cooperate as well. Since the worth of coalition $S \cup T$ may be thought as the amount of TU that all players $j \in S \cup T$ can produce by themselves through cooperation, then the communication constraints are brought into the picture by means of a set function which is additive over maximal connected coalitions, that is $v/g(S \cup T) = v(S) + v(T)$.

It must be emphasized that given a graph g and once the set function v/g is defined, this latter may be treated as any unrestricted game. In particular, the associated Shapley value may be computed; in fact, this is exactly the way the Myerson (1977) value $\phi^{My} : \mathcal{G}_\Gamma(N) \to \mathbb{R}^n$ is computed. Formally

$$\phi_i^{My}(v,g) = \sum_{\pi \in \Pi(N)} \frac{v/g(PR^\pi(i) \cup i) - v/g(PR^\pi(i))}{\#\Pi(N)} = \phi_i^{Sh}(v/g) \qquad (1)$$

for all $i \in N$, the game $v/g : 2^N \to \mathbb{R}$ being defined as

$$v/g(S) := \sum_{g(T) \in S/g} v(T) \qquad \forall S \subset N \qquad (2)$$

Thus, any of the axiomatizations so far proposed for the Shapley value[7] may be applied to the generic point game $v/g \in \mathcal{G}(N)$ and then used for characterizing the Myerson value.

Given $g \in \Gamma(N)$, let $\mathcal{G}_g(N)$ denote the space of all the associated point games. The problem of determining a base for the space $\mathcal{G}_g(N)$, given g, was first attacked by Owen (1986, theorem 3), and, more recently, solved by Hamiache (1999, lemma 2, p. 74) as follows

$$v/g(S) = \sum_{\mathcal{F}_g \ni T \subset S} a^{v/g}(T)\, u^T(S) \qquad \forall S \subset N, g \in \Gamma(N) \qquad (3)$$

[7] Concerning the original axiomatization of ϕ^{My}, we gratefully refer to Myerson (1977).

$$a^{v/g}(T) := \begin{cases} \sum_{\substack{\mathcal{F}_g \ni R \subset T \\ T \setminus R^* = \emptyset}} (-1)^{\#T - \#R} v(R) \text{ if } T \in \mathcal{F}_g \\ 0 \text{ if } T \in 2^N \setminus \mathcal{F}_g \end{cases} \quad \forall T \subset N \tag{4}$$

where $R^* := \{i \in N \mid \exists j \in R \text{ such that } \{i,j\} \in E(N)\}$. In other terms, using the notation of the previous section, for all $R \subset N$ we can write

$$R^* \setminus R = \left\{ i \in N \setminus R \mid \emptyset \neq E(i) \cap \left(\left(\bigcup_{j \in R} E(j) \right) \setminus E(R) \right) \right\}$$

Thus, for the computation of $a^{v/g}(T)$ we must consider only those sub-coalitions $R \subset T$ which are (i) connected (i.e., $R \in \mathcal{F}_g$), and (ii) (collectively) directly linked to all remaining vertices in T (i.e., $T \setminus R^* = \emptyset$). Furthermore, given g, the collection $\{u^T \mid T \in \mathcal{F}_g\}$ of *connected unanimity games* is a basis of the space $\mathcal{G}_g(N)$ of associated point games, since there exist real coefficients $\{a^{v/g}(T)\}_{T \in \mathcal{F}_g}$ such that any point game $v/g : 2^N \to \mathbb{R}$ can be expressed as a linear combination of connected unanimity games as in (3).

An alternative approach to the solution problem for graph restricted games, leading to the position value $\phi^{Po} : \mathcal{G}_\Gamma(N) \to \mathbb{R}^n$, has been furnished by Borm *et al.* (1992). They have defined a so-called *arc game* $w^v : 2^{E(N)} \to \mathbb{R}$, whose player set is the edge set $E(N)$ of the given graph, as follows

$$w^v(K) = v/g_K(V(K)) \quad \forall K \subset E(N) \tag{5}$$
$$V(K) := \{i \in N \mid K \cap E(i) \neq \emptyset\} \subset N \tag{6}$$
$$v/g_K(S) := \sum_{g(T) \in S/g_K} v(T) \quad \forall S \subset N \tag{7}$$

This allows to define the following edge-position value $\psi^{Po} : \mathcal{G}_\Gamma(N) \to \mathbb{R}^{\#E(N)}$

$$\psi_{ij}^{Po}(v,g) = \sum_{\pi \in \Pi(E(N))} \frac{w^v(PR^\pi(\{i,j\}) \cup \{i,j\}) - w^v(PR^\pi(\{i,j\}))}{\#\Pi(E(N))} \tag{8}$$

for all edges $\{i,j\} \in E(N)$, where $\Pi(E(N))$ and $PR^\pi(\{i,j\}) \subset E(N) \setminus \{i,j\}$ denote, respectively, the set of all orderings of $E(N)$ and the set of edges that preeced $\{i,j\}$ in $\pi \in \Pi(E(N))$. Thus, $\psi_{ij}^{Po}(v,g) = \phi_{\{i,j\}}^{Sh}(E(N), w^v)$. Then the position value is obtained by equally sharing edges' payoffs between associated endvertices, that is

$$\phi_i^{Po}(v,g) = \begin{cases} \sum_{\{i,j\} \in E(i)} \frac{1}{2} \psi_{ij}^{Po}(v,g) \text{ if } d(i) > 0 \\ v(i) \text{ if } d(i) = 0 \end{cases} \tag{9}$$

It should be noted that, given $(v,g) \in \mathcal{G}_\Gamma(N)$, the definition of w^v implies that, for the computation of ϕ^{Po}, an entire family $\{v/\widetilde{g} \mid \widetilde{g} \subset g\}$ of point games (and not only the unique v/g) is taken into account. Unfortunately, there does not exist any general axiomatic characterization of ϕ^{Po} (see Borm et al. (1992), p. 319). In particular, the payoffs $\left\{\phi^{Po}\left(u^T,g\right)\right\}_{g\in\Gamma(N),T\in\mathcal{F}_g}$ defined by the position value when (i) a connected unanimity game is played, and (ii) the graph $g\in\Gamma(N)$ is generic, do not satisfy any specific axiomatization. Nevertheless, one can say which players $i\in N$ get a strictly positive payoff $\phi_i^{Po}\left(u^T,g\right)>0$ in such a situation. In fact, for any $g\in\Gamma(N)$ and any $u^T\mid T\in\mathcal{F}_g$, we have

$$\phi_i^{My}\left(u^T,g\right)=\begin{cases}\frac{1}{\#T}\text{ if }i\in T\\ 0\text{ if }i\in N\backslash T\end{cases}\tag{10}$$

$$\phi_i^{Po}\left(u^T,g\right)=\begin{cases}>0\text{ if }i\in P_{hk}\subset g\mid h,k\in T\\ 0\text{ otherwise}\end{cases}\tag{11}$$

In words, unanimous cooperation within T may produce a unit of TU; of course, given necessity of unanimity, each player clearly has a *veto power*. By focusing on such a basic idea, the former value (i.e., ϕ^{My}, as well as ϕ^{Sh} for unrestricted games) equally shares the unit of TU among all the $\#T$ involved players; thus we might say that it attaches only a binary information to the edge set $E(T)$ of the spanned subgraph $g(T)$, that is $T\in\mathcal{F}_g$ or $T\notin\mathcal{F}_g$ (and in this latter case $u^T/g=\sum_{R\subset N\mid R\in\mathcal{F}_g}a^{u^T/g}(R)\,u^R$). On the other hand, the latter value (i.e., ϕ^{Po}) is perhaps utopian, in that it assigns a strictly positive payoff to some players, even though their non-cooperation would not prevent the unitary TU production. This is because, for any graph $g\in\Gamma(N)$ such that $g(T)\notin N/g$ (and even if $g(T)\subset g$ was a complete subgraph), there always exists some edge subset $K\subset E(N)$ satisfying

$$T\subsetneq V(K),\quad\begin{array}{c}\#V(K)/g_K>1\\ \#V(K)/g_{K\cup\{i,j\}}=1\end{array},\quad\begin{array}{c}\{i,j\}\in P_{hk}\subset g\\ i\text{ and/or }j\notin T\ni h,k\end{array}$$

$$\Rightarrow\psi_{ij}^{Po}(v,g)\geq\frac{(\#E(N)-\#K-1)!\#K!}{\#E(N)!}\Rightarrow\phi_i^{Po}(v,g)>0$$

In words, as long as we assume the uniform distribution over $\Pi(E(N))$, any edge $\{i,j\}\in E(N)$ belonging to any path connecting any two players $h,k\in T$ has a positive probability of being, in a random order π of $E(N)$, exactly the one which makes $\{T;PR^{\pi}(\{i,j\})\cup\{i,j\}\}\subset g$ a connected subgraph.

For any $g\in\Gamma(N)$, a further value $\phi^{Ha}:\mathcal{G}_g(N)\rightarrow\mathbb{R}^n$, which, if was to be expressed as a probabilistic or random-order one, is proved[8] to apply only to the space $\mathcal{G}_g(N)$ of associated point games, has also been recently proposed by Hamiache (1999). For reasons of space, we now briefly put on record only four out of the five axioms that characterize it. This is because such four axioms are also satisfied by the new values we are to define.

[8] See Hamiache (1999), Theorem 2 (p. 71), and its proof (pp. 75-77).

• *component efficiency (CE):* $\sum_{i \in S} \phi_i(v, g) = v(S)$ for all $(v, g) \in \mathcal{G}_\Gamma(N)$ and $g(S) \in N/g$

• *linearity (L):* $\phi(\alpha v + \beta w, g) = \alpha \phi(v, g) + \beta \phi(w, g)$ for all $\alpha, \beta \in \mathbb{R}$ and $v, w \in \mathcal{G}(N)$, for fixed $g \in \Gamma(N)$

• *independence of irrelevant players (IIP):* for all $(u^T, g) \in \mathcal{G}_\Gamma(N)$ and $T \in \mathcal{F}_g$, it holds $\phi_i(u^T, g) = \begin{cases} \phi_i(u_T^T, g(T)) \text{ if } i \in T \\ 0 \text{ if } i \in N \backslash T \end{cases}$

• *positivity (P):* $\phi_i(u^T, g) \geq 0$ for all $i \in T$ and $(u^T, g) \in \mathcal{G}_\Gamma(N)$ and $T \in \mathcal{F}_g$

It should be noted that while the Myerson value also satisfies axiom IIP, the position value does not. Nevertheless, ϕ^{My} also satisfies the much stronger condition (10), so that when the generic connected unanimity game u^T is played, the $\#T$-dimensional payoff vector $\left(\phi_i^{My}(u^T, g)\right)_{i \in T}$ (or, equivalently, $\phi^{My}(u_T^T, g(T))$) does not depend on the peculiar features displayed by the spanned subgraph $g(T)$. On the contrary, for any connected unanimity game u^T that may be played, the vector $\left(\phi_i^{Po}(u^T, g)\right)_{i \in N}$ not only does depend on $g(T)$, but also on the features displayed by the whole graph g. Eventually, the Hamiache value defines a payoff vector which does depend solely on $g(T)$ (and not on the whole g); nevertheless, it seems[9] to associate particularly high payoffs to those players $i \in T$ which are cutvertices within $g(T)$. In fact, when playing any connected unanimity game u^T, those players $i \in T \subset N$ who are cutvertices within $g(T)$ do not have more bargaining (i.e., veto) power than non-cutvertices $j \in T$. Thus, if we want to use $g(T)$ for defining a hierachical structure over T according to which players $i \in T$ are unequally rewarded when playing u^T, then cutvertices should not automatically occupy the higher positions of such a hierachical structure.

4 A new value for point games

In association with connected unanimity games, the Myerson and position values display the two different features shown above because in the former case only spanned subgraphs are considered, while in the latter case one is allowed to consider any subgraph, that is possibly non-spanned by any coalition $S \in 2^N$. We now propose a peculiar arc game $\widehat{w}^v$ $(\neq w^v)$ over $2^{E(N)}$ which, in our opinion, may be seen as the analogue of the point game among individuals, since it considers spanned subgraphs only. Given $(v, g) \in \mathcal{G}_\Gamma(N)$, let

$$\widehat{w}^v(K) := \max\{v/g_K(S) \mid E(S) \subset K\} \qquad \forall K \subset E(N) \tag{12}$$

Comparing (5) and (12) we see that $\widehat{w}^v(S) \leq w^v(S)$ for all $S \subset N$, and for any graph-restricted game $(v, g) \in \mathcal{G}_\Gamma(N)$. We may appreciate the difference between the two arc games by considering the generic connected unanimity game

[9] See Table A.I in Hamiache (1999), p. 72.

$u^T \mid T \in \mathcal{F}_g$. In fact, we have

$$\widehat{w}^{u^T}(K) = \begin{cases} 1 \text{ if } E(T) \subset K \\ 0 \text{ if } E(T) \backslash K \neq \emptyset \end{cases}$$

$$\phi^{Sh}_{\{i,j\}}\left(E(N), \widehat{w}^{u^T}\right) = \begin{cases} \frac{1}{\#E(T)} \text{ if } \{i,j\} \in E(T) \\ 0 \text{ if } \{i,j\} \in E(N) \backslash E(T) \end{cases}$$

One possible interpretation is that $g(S) \subset g$ not only identifies all the available communication channels among players $i \in S$, but a more complex *social network*, according to which there exists a unique set of one-to-one cooperative behaviors (or attitudes) through which all individuals $i \in S$ can globally cooperate. Therefore, $K \subsetneqq E(S)$ identifies a situation where some pairs of players $\{i,j\} \in E(S) \backslash K$ voluntarily do not cooperate with each other (thus preventing any unanimous agreement within S).

We might define a value function $\widehat{\phi} : \mathcal{G}_\Gamma(N) \to \mathbb{R}^n$ as follows

$$\widehat{\phi}_i(v,g) := \frac{1}{2} \sum_{\{i,j\} \in E(i)} \phi^{Sh}_{\{i,j\}}\left(E(N), \widehat{w}^v\right)$$

$$\Rightarrow \widehat{\phi}_i\left(u^T, g\right) = \begin{cases} \frac{d_T(i)}{2\#E(T)} \text{ if } i \in T \\ 0 \text{ if } i \in N \backslash T \end{cases} \tag{13}$$

Thus, when u^T is played, $\widehat{\phi}$ rewards players $i \in T$ according to a hierarchiacal structure defined by the degrees $(d_T(i))_{i \in T}$ such players display within the spanned subgraph $g(T)$. In other words, the basic idea (that is going to be developed and enriched) is that, when the connected unanimity game u^T is played, the fact that unanimous cooperation within T may produce a unit of TU is firstly known by a (uniformly randomly chosen) single player $i \in T$ (or, equivalently, by a proper subset $\widetilde{T} \subset T$ of players), and that such an information gets subsequently diffused, through edges and while negotiating, one player with another at a time, til all edges have been vehicles of information. By putting $\widehat{\phi}_T(v,g) := \sum_{i \in T} \widehat{\phi}_i(v,g)$ for all $T \subset N$, one gets that for any pair T_1, T_2 satisfying $T_1 \cap T_2 = \emptyset$ and $T_1 \cup T_2 = T$

$$\#E(T_1) > \#E(T_2) \Rightarrow \widehat{\phi}_{T_1}\left(u^T, g\right) > \widehat{\phi}_{T_2}\left(u^T, g\right)$$

for all $g \in \Gamma(N)$. Therefore, as well as the Myerson value, not only $\widehat{\phi}$ ignores the different features displayed by the two edge subsets $E(T_1)$ and $E(T_2)$, but also, and most importantly, it ignores the features displayed by the edge subset $E(T) \setminus (E(T_1) \cup E(T_2))$, and this latter, in our opinion, formally translates how negotiation between T_1 and T_2 does occur. Please note that it may well be $T_1 \notin \mathcal{F}_g \ni T_2$. In order to axiomatically model negotiation between subcoalitions of $T \in \mathcal{F}_g$ when u^T is played, let $\mathcal{B}(g(T)) := \left\{g\left(B^T_1\right), \ldots, g\left(B^T_{m_T}\right)\right\}$ denote the (unordered) collection of all blocks of the spanned subgraph $g(T)$. Within social network analysis blocks are typically recognized to be *cohesive* *subgroups* (Wasserman and Faust (1994)). In section 2 we explained that $\mathcal{B}(E(T)) := \left\{E\left(B_1^T\right), \ldots, E\left(B_{m_T}^T\right)\right\}$ constitutes an unordered partition of $E(T)$, and the *unions* Owen (1977) refers to may be considered to be a particular kind of cohesive subgroups. Therefore, let $\Pi^T(E(N)) \subset \Pi(E(N))$ denote the collection of permutations of the whole edge set $E(N)$ that are *admissible* with respect to $\mathcal{B}(E(T))$, that is

$$\Pi^T(E(N)) := \left\{ \pi \in \Pi(E(N)) \mid \left(\begin{array}{c} \forall \{i,j\}, \{h,k\}, \{x,y\} \in E(T) \\ \left.\begin{array}{c} \{i,j\}, \{h,k\} \in E\left(B_m^T\right) \text{ and} \\ \pi\{i,j\} < \pi\{x,y\} < \pi\{h,k\} \end{array}\right\} \Rightarrow \\ \Rightarrow \{x,y\} \in \left(B_m^T\right), 1 \leq m \leq m_T \end{array} \right) \right\}$$

so that, for all $\{i,j\} \in E(T)$

$$\sum_{\pi \in \Pi^T(E(N))} \frac{\widehat{w}^{u^T}\left(PR^\pi(\{i,j\}) \cup \{i,j\}\right) - \widehat{w}^{u^T}\left(PR^\pi(\{i,j\})\right)}{\#\Pi^T(E(N))} =$$

$$= E_{U(\Pi^T(E(N)))}\left[\widehat{w}^{u^T}\left(PR^\pi(\{i,j\}) \cup \{i,j\}\right) - \widehat{w}^{u^T}\left(PR^\pi(\{i,j\})\right)\right] =$$

$$= \left\{ \begin{array}{c} \left(\frac{1}{m_T}\right)\left(\frac{1}{\#E(B_m^T)}\right) \text{ if } \{i,j\} \in E\left(B_m^T\right), 1 \leq m \leq m_T \\ 0 \text{ if } \{i,j\} \notin E(T) \end{array} \right.$$

would constitute an arbitrary way of applying[10] the Owen value for games with coalition structure to the game $\widehat{w}^{u^T}$, with player set $E(N)$ and coalition structure $\mathcal{B}(E(T))$, where $E_P[x]$ denotes the expectation of random variable x with respect to a given probability distribution P, and $U\left(\Pi^T(E(N))\right)$ denotes the uniform distribution over $\Pi^T(E(N))$. As known, the Owen (1977) value has been derived under the assumption that unions (here $E\left(B_1^T\right), \ldots, E\left(B_{m_T}^T\right)$) bargain with each other as units (here when the connected unanimity game u^T is played). Together with the idea that blocks' edge sets represent cohesive subgroups, we also propose to use connectivity as a measure of cohesion. Thus, in our context, it is no longer true that all coalitions behave as units in the same way: those with higher connectivities (and smaller sizes) will be assumed to be the more cohesive (and thus more powerful in bargaining).

Let $\kappa\left(g\left(B_m^T\right)\right), 1 \leq m \leq m_T$ denote the vertex connectivity of subgraph $g\left(B_m^T\right)$. A *minimally k-connected graph* $g(S) = \{S; E(S)\}$ is such that, for any edge $\{i,j\} \in E(S)$, the following holds

$$\kappa(g(S)) = \kappa(\{S; E(S)\}) = k > \kappa(\{S; (E(S) \setminus \{i,j\})\})$$

[10] A fully general application of the Owen value for games with coalition structure to graph-restricted games, by means of $\widehat{w}^v$, is made in Rossi (2000b).

Thus, in a minimally k-connected graph each and every edge is strictly necessary for the given connectivity level. For $s = \#S$ and $k \leq s-1$, let $\Gamma_* (s; k)$ denote the set of minimally k-connected graphs of order s. Given two blocks

$$g\left(B_{m_1}^T\right), g\left(B_{m_2}^T\right) \mid \kappa\left(g\left(B_{m_1}\right)\right) = k = \kappa\left(g\left(B_{m_2}\right)\right)$$

we want to define a rule stating wheteher $E\left(B_{m_1}^T\right)$ or else $E\left(B_{m_2}^T\right)$ has more bargaining power than the other (that is, which one is more cohesive than the other) when playing $\left(E(N), \widehat{w}^{u^T}\right)$. One way of achieving this is by defining some sort of *distance* between $g(B_m)$ and $\Gamma_*(\#B_m; k)$, so that the larger such a distance, the weaker cohesion. In fact, we interpret such a distance (we are to formalize) as a measure of the inefficiency that characterizes the collection of (non-cooperative) negotiations that occur within the generic coalition.

Unfortunately, for $k \geq 4$ the structure of k-connected graphs is not well understood yet, and general results concerning $\Gamma_*(s; k)$ are hard to be found. A special role, within this branch of *extremal graph theory*, is played by the so-called *k-core* of a graph, that is *"the largest subgraph with minimum degree at least k"* (Pittel, Spencer and Wormald (1996)). More precisely, given $g(S)$ and denoting its k-core by $g^k(S) \subset g(S)$, we have

$$g^k(S) := \max_{k+1 \leq \#S^* \leq \#S} \left\{ g(S^*) \subset g(S) \mid d_{S^*}(i) \geq k, i \in S^* \right\}$$

In fact, if $g(S) \in \Gamma_*(s; k)$, then $\#\{i \in S \mid d_S(i) = k\} \geq \frac{(k-1)s+2}{2k-1}$ (Bollobás (1978), theorem 4.8, p. 25). Thus, given $k_m = \kappa\left(g\left(B_m^T\right)\right), 1 \leq m \leq m_T$, let

$$\Delta\left(B_m^T\right) := \left(\#\left\{i \in B_m^T \mid d_{B_m^T}(i) \geq k_m\right\} - \frac{(k_m - 1)\left(\#B_m^T\right) + 2}{2k_m - 1}\right) + 1$$

We add 1 because we do not want to deal with indeterminacies, i.e., in case we actually had some block $g\left(B_m^T\right) \in \Gamma_*\left(\#B_m^T; k_m\right)$, for some $T \in \mathcal{F}_g$ and $m \in \{1, \ldots, m_T\}$, which is minimally k_m-connected.

We want to define an edge value function $\psi^{BC} : \mathcal{G}_\Gamma(N) \to \mathbb{R}^{\#E(N)}$ such that, when the generic connected unanimity game u^T is played, the bargaining power of coalition $E\left(B_m^T\right) \subset E(N)$ (of edges; or, equivalently, that of coalition $B_m^T \subset N$ of individuals) is given by $\frac{\kappa\left(g\left(B_m^T\right)\right)}{\Delta\left(B_m^T\right)}$, so that the following holds

$$\psi_{E(B_m^T)}^{BC}\left(u^T, g\right) := \sum_{\{i,j\} \in E(B_m^T)} \psi_{ij}^{BC}\left(u^T, g\right) = \left(\gamma_{g(T)}\right) \frac{\kappa\left(g\left(B_m^T\right)\right)}{\Delta\left(B_m^T\right)}$$

and

$$\frac{\sum_{\{i,j\} \in E\left(B_{m_1}^T\right)} \psi_{ij}^{BC}\left(u^T, g\right)}{\sum_{\{h,k\} \in E\left(B_{m_2}^T\right)} \psi_{hk}^{BC}\left(u^T, g\right)} = \frac{\left[\kappa\left(g\left(B_{m_1}^T\right)\right)\right]\left[\Delta\left(B_{m_2}^T\right)\right]}{\left[\Delta\left(B_{m_1}^T\right)\right]\left[\kappa\left(g\left(B_{m_2}^T\right)\right)\right]} \qquad (14)$$

for all graphs $g \in \Gamma(N)$ and for any pair $m_1, m_2 \in \{1, \ldots, m_T\}$. In terms of the usual value functions, defining payoffs for individuals (and not for edges), we shall define a *bolck-connectivity-degree value* $\phi^{BCd} : \mathcal{G}_\Gamma(N) \to \mathbb{R}^n$ satisfying

$$\phi_i^{BCd}\left(u^T, g\right) = \begin{cases} \sum\limits_{\substack{m \in \{1,\ldots,m_T\} \\ E\left(B_m^T\right) \cap E(i) \neq \emptyset}} \dfrac{\left(\frac{d_{B_m^T}(i)}{2\#E\left(B_m^T\right)}\right)\left(\frac{\kappa\left(g\left(B_m^T\right)\right)}{\Delta\left(B_m^T\right)}\right)}{\sum\limits_{m=1}^{m_T} \frac{\kappa\left(g\left(B_m^T\right)\right)}{\Delta\left(B_m^T\right)}} \text{ if } i \in T \\ 0 \text{ if } iN \backslash T \end{cases} \tag{15}$$

for any connected unanimity game $\left(u^T, g\right) \in \mathcal{G}_\Gamma(N)$. Note that for any vertex-player $i \in T$ which is not a cutvertex in $g(T)$ we have

$$\#\left\{m \in \{1, \ldots, m_T\} \mid E\left(B_m^T\right) \cap E(i) \neq \emptyset\right\} = 1$$

If we focus (for the time being) only on some given connected unanimity game u^T, conditions (14) and (15) can be easily obtained by means of an $\#E(T)$-dimensional vector $\lambda^T \in \mathbb{R}^{\#E(T)}$ of weights $\lambda^T = \left(\lambda_1^T, \ldots, \lambda_{\#E(T)}^T\right)$, where the (unordered) edge subset $E(T)$ of the spanned subgraph $g(T)$ is here denoted by $E(T) =$

$$\left\{\{i,j\}_1, \ldots, \{i,j\}_{\#E\left(B_1^T\right)}, \ldots, \{i,j\}_{1+\sum\limits_{m=1}^{m_T-1} \#E\left(B_m^T\right)}, \ldots, \{i,j\}_{\#E(T)}\right\}$$

and the (unordered) edge set of block $\left(g\left(B_m^T\right)\right)_{1 \leq m \leq m_T}$ is here denoted by

$$E\left(B_m^T\right) = \left\{\{i,j\}_{1+\sum\limits_{k=1}^{m-1} \#E\left(B_k^T\right)}, \ldots, \{i,j\}_{\#E\left(B_m^T\right)+\sum\limits_{k=1}^{m-1} \#E\left(B_k^T\right)}\right\}$$

and by setting $\lambda_h^T := \left(\frac{\kappa\left(g\left(B_m^T\right)\right)}{\Delta\left(B_m^T\right)}\right)\left(\frac{1}{\#E\left(B_m^T\right)}\right)$ for all edges $\{i,j\}_h \in E\left(B_m^T\right)$, and for all blocks $\left(g\left(B_m^T\right)\right)_{1 \leq m \leq m_T}$. The weight vector λ^T is then used[11] to define a probability $\{P_T(\pi)\}_{\pi \in \Pi(E(N))}$ over the collection of all orderings of the whole edge set. In fact, any ordering $\pi \in \Pi(E(N))$ of the whole edge set induces a unique ordering $\pi_{E(T)} \in \Pi(E(T))$ of the edge subset $E(T)$, where the first (second,...) edge in $\pi_{E(T)}$ is the (generic) edge $\{i,j\}_h \in E(T)$ appearing first (second,...) within the unique maximal chain in $2^{E(T)}$

$$\emptyset = A_0 \subset \cdots \subset A_{\#E(T)} = E(T) \mid A_k \in 2^{E(T)}, \#A_k = k, k = 0, 1, \ldots, \#E(T)$$

defined by $\pi \in \Pi(N)$. We use the following notation: $\{i,j\}_{(h)} \in E(T)$ denotes the edge which occupies the h-th position in $\pi_{E(T)}$; thus $\{i,j\}_{(h)} \neq \{i,j\}_h$. For

[11] This approach develops from the contribution of Kalai and Samet (1988).

any $\pi_{E(T)} = \left\{ \{i,j\}_{(1)}, \ldots, \{i,j\}_{(\#E(T))} \right\}$ we associate a probability $P_T(\pi)$ to all orderings $\pi \in \Pi(E(N))$ that induce (or are consistent with) such an ordering $\pi_{E(T)} \in \Pi(E(T))$ given by

$$P_T(\pi) = \frac{\prod_{h=1}^{\#E(T)} \lambda^T_{(h)} / \sum_{k=1}^{h} \lambda^T_{(k)}}{(\#E(N))_{\#E(N)-\#E(T)}} \tag{16}$$

where, for any two integers x, r such that $x \geq r$, we denote by

$$(x)_r := x(x-1)(x-2)\cdots(x-r+1)$$

the so-called *falling factorial*. In fact, there clearly exist $\#\Pi(E(T)) = \#E(T)!$ different orderings of $E(T)$, but for any given ordering $\pi_{E(T)} \in \Pi(E(T))$ within $E(T)$ there only exist $\#E(N)!/\#E(T)! = (\#E(N))_{\#E(N)-\#E(T)}$ orderings of the whole edge set that are consistent with (or induce) $\pi_{E(T)}$. Please note that, given the definition of the arc game $\widehat{w}^v$ (i.e., in association with any connected unanimity game u^T, for any edge $\{i,j\} \in E(N) \backslash E(T)$ and for any edge subset $K \subset E(N) \setminus \{i,j\}$, we always get $\widehat{w}^{u^T}(K \cup \{i,j\}) - \widehat{w}^{u^T}(K) = 0$), when the generic connected unanimity game u^T is played, the position of edges $\{i,j\} \in E(N) \backslash E(T)$ within any ordering $\pi \in \Pi(E(N))$ of the whole edge set is irrelevant.

Lemma 1 *For any connected unanimity game $u^T \mid T \in \mathcal{F}_g$, condition (14) is satisfied by the edge value function*

$$\psi^{BC}_{ij}\left(u^T, g\right) = E_{P_T}\left[\widehat{w}^{u^T}\left(PR^\pi(\{i,j\}) \cup \{i,j\}\right) - \widehat{w}^{u^T}\left(PR^\pi(\{i,j\})\right)\right]$$

Proof. Given (16), it is easily checked that

$$E_{P_T}\left[\widehat{w}^{u^T}\left(PR^\pi(\{i,j\}_h) \cup \{i,j\}_h\right) - \widehat{w}^{u^T}\left(PR^\pi(\{i,j\}_h)\right)\right] =$$

$$= \sum_{\pi \in \Pi(E(N)) \mid \{i,j\}_h = \{i,j\}_{(\#E(T))} \text{ is last (swings) in } \pi_{E(T)}} P_T(\pi) =$$

$$= \frac{\lambda^T_h}{\sum_{k \in \{1,\ldots,\#E(T)\}} \lambda^T_k} \times$$

$$\times \frac{\left(\prod_{k \in \{1,\ldots,\#E(T)\} \backslash h} \lambda^T_k\right)(\#E(T)-1)!\,(\#E(N))_{\#E(N)-\#E(T)}}{\left(\prod_{k \in \{1,\ldots,\#E(T)\} \backslash h} \lambda^T_k\right)(\#E(T)-1)!\,(\#E(N))_{\#E(N)-\#E(T)}} =$$

$$= \left(\frac{\frac{\kappa(g(B_m^T))}{\Delta(B_m^T)}}{\sum_{m=1}^{m_T} \frac{\kappa(g(B_m^T))}{\Delta(B_m^T)}} \right) \left(\frac{1}{\#E\left(B_m^T\right)} \right)$$

$$\Rightarrow \sum_{\{i,j\} \in E(B_m^T)} \psi_{ij}^{BC}\left(u^T, g\right) = \frac{\frac{\kappa(g(B_m^T))}{\Delta(B_m^T)}}{\sum_{m=1}^{m_T} \frac{\kappa(g(B_m^T))}{\Delta(B_m^T)}}$$

for all $m \in \{1, \ldots, m_T\}$ as wanted. ■

Lemma 2 *For any* $g \in \Gamma(N)$ *and* $T \subset S \subset N \mid T, S \in \mathcal{F}_g$

$$a^{u^T/g}(S) = \sum_{\substack{R \subset S \\ R \supset T \\ \#R/g = 1 \\ S \backslash R^* = \emptyset}} (-1)^{\#S - \#R} = \begin{cases} 1 \text{ } if\, T = S \\ 0 \text{ } if\, T \neq S \end{cases}$$

Proof. There always exists a unique minimal (with respect to cardinality) $\widetilde{R} \subset S$ such that $S \backslash \widetilde{R}^* = \emptyset, \#\widetilde{R}/g = 1$ and $T \subset \widetilde{R}$; that is

$$\widetilde{R} := \min_{\#R} \{T \subset R \subset S \mid \#R/g = 1, S \backslash R^* = \emptyset\}$$

In fact, let $\widetilde{S} := \{i \in S \backslash T \mid d_S(i) = 1\}$ denote the collection of vertices $i \in S \backslash T$ that are *pendant*. Also let $H := \left\{ h \in S \backslash \widetilde{S} \mid \emptyset \neq E(h) \cap \left(\bigcup_{j \in \widetilde{S}} E(j) \right) \right\}$ denote the collection of vertices that are adjacent to at least one pendant vertex $\widetilde{i} \in \widetilde{S}$. Let $d(i_1) < \cdots < d(i_m)$ be the (increasing) degree sequence, in $g(S)$, of all vertices $i_j \in M := S \backslash \left(H \cup T \cup \widetilde{S} \right)$. Then the desired subset $\widetilde{R}$ clearly is the ending point of the following deletion process

$$\begin{aligned} \widetilde{R}_h &:= \begin{cases} \widetilde{R}_{h-1} \text{ if } \# \left(\widetilde{R}_{h-1} \backslash i_h \right) / g > 1 \\ \widetilde{R}_{h-1} \backslash i_h \text{ if } \# \left(\widetilde{R}_{h-1} \backslash i_h \right) / g = 1 \end{cases} \\ \widetilde{R}_0 &:= S \backslash \widetilde{S} \text{ (note that there is no } d(i_0) \text{)} \end{aligned}$$

Therefore, we obtain
$a^{u^T/g}(S) = \left((-1)^{\#S - \#\widetilde{R}} \right) \left(\sum_{R \subset S \backslash \widetilde{R}} (-1)^{-\#R} \right) = 0$ for all $\widetilde{R} \neq \emptyset$ ■

Lemma 3 *For any graph $g \in \Gamma(N)$ there exists a unique value ϕ^{BCd} satisfying (15) for all point games in $\mathcal{G}_g(N)$, which may be obtained by means of $\widehat{w}^v$ in terms of random-order values as follows*

$$\phi_i^{BCd}(v,g) = \frac{1}{2} \sum_{\{i,j\} \in E(i)} \psi_{ij}^{BC}(v,g)$$

$$\psi_{ij}^{BC}(v,g) = E_{P^{v/g}}\left[\widehat{w}^v\left(PR^\pi(\{i,j\}) \cup \{i,j\}\right) - \widehat{w}^v\left(PR^\pi(\{i,j\})\right)\right]$$

$$P^{v/g}(\pi) := \sum_{T \in \mathcal{F}_g} \frac{abs\left(a^{v/g}(T)\right)}{\sum_{T \in \mathcal{F}_g} abs\left(a^{v/g}(T)\right)} P_T(\pi) \qquad \forall \pi \in \Pi(E(N))$$

where $P^{v/g} : \mathcal{G}_\Gamma(N) \to \Delta^{\#E(N)!-1}$ maps graph-restricted coalitional games (i.e., pairs $(v,g) \in \mathcal{G}(N) \times \Gamma(N)$) into probability distributions $\left\{P^{v/g}(\pi)\right\}_{\pi \in \Pi(E(N))}$; furthermore, for any real $x \in \mathbb{R}$, define $abs(x) = x$ if $x \geq 0$, $= -x$ if $x < 0$.

Proof. Since (i) $\left\{u^T\right\}_{T \in \mathcal{F}_g}$ constitutes a basis of $\mathcal{G}_g(N)$ for all $g \in \Gamma(N)$, and (ii) ϕ^{BCd} is univocally defined for each element of such a basis according to (15), this latter is unique on $\mathcal{G}_g(N)$. Thus, we simply need to show that ϕ^{BCd} actually satisfies condition (15) for each connected unanimity game u^T. In fact, $P^{u^T/g}(\pi) = P_T(\pi)$ for all $\pi \in \Pi(E(N))$ by means of lemma 2. Furthermore, by means of lemma 1, we get $\sum_{\{i,j\} \in E(i)} \frac{\psi_{ij}^{BC}(u^T,g)}{2} =$

$$= \frac{1}{2} \sum_{\substack{\{i,j\} \in \left\{E(i) \cap E\left(B_m^T\right)\right\} \\ m \in \{1,\ldots,m_T\}}} \left(\frac{\frac{\kappa\left(g\left(B_m^T\right)\right)}{\Delta\left(B_m^T\right)}}{\sum_{m=1}^{m_T} \frac{\kappa\left(g\left(B_m^T\right)\right)}{\Delta\left(B_m^T\right)}} \right) \left(\frac{1}{\#E\left(B_m^T\right)} \right) =$$

$$= \sum_{\substack{m \in \{1,\ldots,m_T\} \\ E\left(B_m^T\right) \cap E(i) \neq \emptyset}} \frac{\left(\frac{\kappa\left(g\left(B_m^T\right)\right)}{\Delta\left(B_m^T\right)}\right)\left(\frac{d_{B_m^T}(i)}{2\#E\left(B_m^T\right)}\right)}{\sum_{m=1}^{m_T} \frac{\kappa\left(g\left(B_m^T\right)\right)}{\Delta\left(B_m^T\right)}}$$ as wanted. It remains to check whether $\left\{P^{v/g}(\pi)\right\}_{\pi \in \Pi(N)}$ constitutes a probability distribution for all pairs (v,g). This is definitely the case, because $P^{v/g}$ is a *convex* combination of probabilities $\{P_T\}_{T \subset N}$ (i.e., $0 \leq \frac{abs(a^v(T))}{\sum_{T \subset N} abs(a^v(T))} \leq 1$ and $\sum_{T \subset N} \frac{abs(a^v(T))}{\sum_{T \subset N} abs(a^v(T))} = 1$), and thus a probability as well. This completes the proof. ■

Remark: *given $g \in \Gamma(N)$, the value ϕ^{BCd} defined by (15) identifies a unique n-dimensional payoff vector for all $v \in G(N)$ if, and only if, the pair (v,g) is interpreted as v/g. Otherwise, that is if $v(S)$ remains undefined for all $S \in 2^N \backslash \mathcal{F}_g$, then there exists an entire family $\{\phi^* \mid \phi^* : \mathcal{G}_\Gamma(N) \to \mathbb{R}^n\}$ of* *(probabilistic and/or random-order) value functions such that each $\phi^*(v,g)$ satisfies (15) for all $(u^T, g) \in G_\Gamma(N)$ when $T \in \mathcal{F}_g$. In other words, any game $v_g : 2^N \to R$ satisfying $v_g = v : \mathcal{F}_g \to R$ might be used. Thus, the value ϕ^{BCd} defined by (15) is unique on the space $G_g(N)$ of point games associated with g, for all $g \in \Gamma(N)$; but it is not unique over the space $G_\Gamma(N)$ of graph-restricted games associated with $g \in \Gamma(N)$, since these latter are pairs (v,g), that can be turned into games (i.e., set functions on 2^N) only if somehow interpreted.*

The block-connectivity-degree value for point games may be derived directly; that is, without using the arc game $\widehat{w}^v$. Given any $g \in \Gamma(N)$ and the associated set $\mathcal{F}_g \subset 2^N$ of feasible coalitions, for each $T \in \mathcal{F}_g$ we define weights $\omega^T \in \mathbb{R}^n$ such that $\omega_i^T > 0$ if $i \in T$ and $\omega_i^T = 0$ if $i \notin T$. Furthermore, we impose that $\phi_i(u^T, g) = \omega_i^T / \left(\sum_{j\in N} \omega_j^T\right)$ for all $i \in N$ and for any connected unanimity game u^T. As previously explained, ω_i^T is to be interpreted as the probability that, when u^T is played, $i \in T$ is last for T. Thus, here again, we may use the family $\{\omega^T\}_{T\in\mathcal{F}_g}$ of weights to obtain a probability (given by a linear combination) over the collection $\Pi(N)$ of all permutations of the player set N. For $T \in \mathcal{F}_g$ and any $\pi \in \Pi(N)$, let $t = \#T$ and $\pi_T = \{i_{(1)}, \ldots, i_{(t)}\}$ denote the order of the t players in T induced by π. Let $P_{\omega^T}(\pi) := \frac{\prod_{j=1}^{t} \omega_{(j)}^T / \sum_{h=1}^{j} \omega_{(h)}^T}{(n)_{n-t}}$ be the probability over $\Pi(N)$ associated with the playing of unanimity game u^T, with $T \in \mathcal{F}_g$.

Theorem 4 *For any $(v,g) \in \mathcal{G}_\Gamma(N)$, the unique random-order value for the associated point game v/g which is consistent with (15) is given by*

$$\phi_i^{BCd}(v,g) = E_{P_\omega^{v/g}}\left[v/g\left(PR^\pi(i) \cup i\right) - v/g\left(PR^\pi(i)\right)\right]$$

for all $i \in N$, where

$$P_\omega^{v/g}(\pi) = \sum_{T\in\mathcal{F}_g} \frac{abs\left(a^{v/g}(T)\right)}{\sum_{T\in\mathcal{F}_g} abs\left(a^{v/g}(T)\right)} P_{\omega^T}(\pi) \qquad \forall \pi \in \Pi(N)$$

$$\omega_i^T = \sum_{\substack{m=1,\ldots,m_T \\ E\left(B_m^T\right)\cap E(i)\neq\emptyset}} \left(\frac{d_{B_m^T}(i)}{2\#E\left(B_m^T\right)}\right)\left(\frac{\kappa\left(g\left(B_m^T\right)\right)}{\Delta\left(B_m^T\right)}\right) \qquad \forall i \in N$$

Proof. First please note, here again, that

$$E_{P_{\omega^T}}\left[u^T/g\left(PR^\pi(i) \cup i\right) - u^T/g\left(PR^\pi(i)\right)\right] =$$

$$= \sum_{\pi\in\Pi(N)|i=i_{(t)} \text{ is last for } \pi_T} P_{\omega^T}(\pi) = \frac{\left(\omega_i^T\right)(n)_{n-t}(t-1)!}{\left(\sum_{h\in T} \omega_h^T\right)(n)_{n-t}(t-1)!} =$$

$$= \sum_{\substack{m=1,\dots,m_T \\ E\left(B_m^T\right)\cap E(i)\neq\emptyset}} \frac{\left(\frac{d_{B_m^T}(i)}{2\#E(B_m^T)}\right)\left(\frac{\kappa\left(g\left(B_m^T\right)\right)}{\Delta(B_m^T)}\right)}{\sum\limits_{m=1}^{m_T}\frac{\kappa(g(B_m^T))}{\Delta(B_m^T)}}$$

Now consider the probability P^{u^T} over $\Pi(N)$ associated with the generic unanimity game $u^T \mid T \in \mathcal{F}_g$

$$P^{u^T}(\pi) = \sum_{S\in\mathcal{F}_g} \frac{abs\left(a^{u^T/g}(S)\right)}{\sum\limits_{T\in\mathcal{F}_g} abs\left(a^{v/g}(T)\right)} P_{\omega^S}(\pi) = P_{\omega^T}(\pi) \qquad \forall\pi\in\Pi(N)$$

Therefore $\phi_i^{BC}(v,g) = E_{P_\omega^{v/g}}\left[v/g\left(PR^\pi(i)\cup i\right) - v/g\left(PR^\pi(i)\right)\right]$ for all $i \in N$, since they coincide on each element of the basis $\left\{u^T\right\}_{T\in\mathcal{F}_g}$. Here again, please note that $\{P_{\omega^T}(\pi)\}_{\pi\in\Pi(N)}$ clearly is a probability distribution on $\Pi(N)$ for all $T \subset N$; furthermore, $\left\{P_\omega^{v/g}(\pi)\right\}_{\pi\in\Pi(N)}$ is a convex combination of such probabilities, ad thus a probability as well. This completes the proof. ■

It must be emphasized that $\phi^{BCd}(v,g)$ coincides with the unrestricted Shapley value $\phi^{Sh}(v)$ whenever the exogenously given graph is the complete one, that is when $g = \left\{N; N^{(2)}\right\}$. This is definitely a positive feature ϕ^{BCd} displays, which also characterizes both ϕ^{My} and ϕ^{Ha} (and also the block-connectivity-connectivity value ϕ^{BCc} we are to develop in section 6).

5 The Myerson value through $\widehat{w}^v$

Roughly speaking, the construction of the arc game $\widehat{w}^v$ enables to get a link between point games (where, as we explained, only spanned subgraphs are considered) and arc games in general (that is, where $E(N)$ is treated as the player set). Thus, since the Myerson value is the Shapley value of the point game, we may get a computational method for obtaining ϕ^{My} by means of $\widehat{w}^v$. In doing so, we need to determine, once again, an $\#E(N)$-dimensional vector of weights for each connected unanimity game. But, in addition, we now need to determine also a (non-symmetric, that is $\neq \frac{1}{2}$) rule for distributing edges' payoffs over associated endvertices.

According to the notation used so far, let $\psi^{My} : \mathcal{G}_\Gamma(N) \to \mathbb{R}^{\#E(N)}$ denote the Myerson edge-value function we are to define. For any connected unanimity game u^T, consider the weights

$$\omega_{ij}(u^T) = \omega_{ij}^T = \begin{cases} \frac{d_T(i)+d_T(j)}{d_T(i)d_T(j)} \text{ if } \{i,j\}\in E(T) \\ 0 \text{ if } \{i,j\}\notin E(T) \end{cases}$$

$$\Rightarrow \sum_{\{i,j\}\in E(T)} \omega_{ij}^{T} = \sum_{\{i,j\}\in E(T)} \left(\frac{1}{d_T(i)} + \frac{1}{d_T(j)}\right) = \sum_{i\in T} \left(\sum_{k=1}^{d_T(i)} \frac{1}{d_T(i)}\right) = \#T$$

for all[12] $T \subset N$. Furthermore, for any order $\pi \in \Pi(E(N))$ of the whole edge set, let $\pi_{E(T)} = \left\{\{i,j\}_{(1)};\ldots;\{i,j\}_{(\#E(T))}\right\}$ denote the induced ordering of $E(T)$. Then we associate a probability $P_T(\pi)$ to such an order (and thus to all $\pi \in \Pi(N)$ that induce the same $\pi_{E(T)} \in \Pi(E(T))$) given by

$$P_T(\pi) = \frac{\prod_{j=1}^{\#E(T)} \omega_{(j)}^{T} / \sum_{h=1}^{j} \omega_{(h)}^{T}}{(\#E(N))_{\#E(N)-\#E(T)}} \tag{17}$$

We can now define the Myerson edge-value function associated with the generic connected unanimity game u^T to be given by $\psi_{ij}^{My}(u^T, g) =$

$$= E_{P_T}\left[\widehat{w}^{u^T}\left(PR^{\pi}(\{i,j\}) \cup \{i,j\}\right) - \widehat{w}^{u^T}\left(PR^{\pi}(\{i,j\})\right)\right] =$$

$$= \left(\frac{1}{\#T}\right)\left(\frac{d_T(i)+d_T(j)}{d_T(i)d_T(j)}\right)$$

Lemma 5 *For any connected unanimity game u^T, and for any graph $g \in \Gamma(N)$*

$$\frac{1}{\#T} = \phi_i^{My}(u^T, g) = \sum_{\{i,j\}\in E(i)} \frac{d_T(j)}{d_T(j)+d_T(i)} \psi_{ij}^{My}(u^T, g)$$

Proof. $\sum_{\{i,j\}\in E(i)} \frac{d_T(j)}{d_T(j)+d_T(i)} \psi_{ij}^{My}(u^T, g) =$
$= \sum_{\{i,j\}\in\{E(i)\cap E(T)\}} \frac{d_T(j)}{d_T(j)+d_T(i)} \left(\frac{1}{\#T}\right)\left(\frac{d_T(i)+d_T(j)}{d_T(i)d_T(j)}\right) =$
$= \frac{1}{\#T} \sum_{\{i,j\}\in\{E(i)\cap E(T)\}} \frac{1}{d_T(i)} = \frac{1}{\#T}$ as wanted ■

Now consider a generic game (i.e., which is not a connected unanimity game); we need to use, here again, the Mobiüs transform. More precisely, for any connected unanimity game u^T, and for any player $i \in N$ which is an endvertex of a generic edge $E(T) \ni \{i,j\} \in E(i) \subset E(N)$, let $\theta_i^T(i,j) := \frac{d_T(j)}{d_T(j)+d_T(i)}$

Theorem 6 *For any graph-restricted coalitional game $(v,g) \in \mathcal{G}_{\Gamma}(N)$*

$$\phi_i^{My}(v,g) = \sum_{\{i,j\}\in E(i)} \theta_i^{v}(i,j)\, \psi_{ij}^{My}(v,g)$$

[12] In fact, it may well be $T \notin \mathcal{F}_g$. This result is a direct consequence of the well known *handshaking lemma*, which is basic in Graph Theory. See Bollobás (1979), p. 4.

where

$$\theta_i^v(i,j) = \sum_{T\in\mathcal{F}_g} \frac{abs\left(a^{v/g}(T)\right)}{\sum\limits_{T\in\mathcal{F}_g} abs\left(a^{v/g}(T)\right)} \theta_i^T(i,j)$$

$$\psi_{ij}^{My}(v,g) = E_{P^{v/g}}\left[\widehat{w}^v\left(PR^\pi(\{i,j\}) \cup \{i,j\}\right) - \widehat{w}^v\left(PR^\pi(\{i,j\})\right)\right]$$

$$P^{v/g}(\pi) = \sum_{T\in\mathcal{F}_g} \frac{abs\left(a^{v/g}(T)\right)}{\sum\limits_{T\in\mathcal{F}_g} abs\left(a^{v/g}(T)\right)} P_T(\pi) \qquad \forall\pi\in\Pi(N)$$

Proof. As usual, consider the generic element u^T of the basis $\left\{u^S\right\}_{S\in\mathcal{F}_g}$. We have already shown that $P^{u^T} = P_T$; thus $\psi_{ij}^{My}\left(u^T, g\right)$ is definitely as above (i.e., in particular, $\psi_{ij}^{My}\left(u^T, g\right) = 0$ for all $\{i,j\} \in \{E(i) \setminus E(T)\}$. Furthermore, $\theta_i^{u^T}(i,j) = \frac{d_T(j)}{d_T(j)+d_T(i)}$, so that $\sum\limits_{\{i,j\}\in E(i)} \theta_i^v(i,j)\,\psi_{ij}^{My}(v,g)$ and $\phi_i^{My}(v,g)$ coincide for each element of a basis of the space $\mathcal{G}_g(N)$ of point games, and thus they coincide for each point game v/g, as was to be shown. ■

The above result is useful for understanding the interpretation of the edge set which is made (i.e., implicitly) by the Myerson value. More precisely, when any connected unanimity game $u^T \mid T \in \mathcal{F}_g$ is played (and if, according to $\widehat{w}^{u^T}$ and/or v/g, all existing arcs $\{i,j\} \in E(T)$ are needed for achieving unanimous agreement within T), the approach leading to ϕ^{My} implies that (1) edges one whose endvertices is pendant within $g(T)$ have higher probabilities of being last in a random order of $E(T)$, and (2) edges payoffs are shared (between associated endvertices) so that pendant vertices get higher fractions. (2) is definitely hard to justify; (1) may be accepted if we assume, as previously explained, that synergies of cooperation are initially unknown to all players apart from one, (uniformly) randomly chosen, who starts diffusing (while possibly bargaining) such an information to its adjacent vertices.

6 An alternative hierarchical structure

Let us focus once again on the generic spanned subgraph $g(T)$, with $T \in \mathcal{F}_g$, and let $g\left(B_1^T\right), \ldots, g\left(B_{m_T}^T\right)$ denote the collection of its blocks and $\#T = t$ as before. We now explore the possibility of defining an alternative hierarchical structure (or weight vector) $\mathbb{R}^t \ni \lambda^T = \left(\lambda_i^T\right)_{i\in T}$ according to which, when u^T is played, each player $i \in T$ gets a payoff given by $\lambda_i^T / \sum_{j\in T} \lambda_i^T$, while each player $h \in N \backslash T$ gets a payoff equal to zero. The hierarchical structure defined by $\left\{\omega^T\right\}_{T\in\mathcal{F}_g}$ (as in theorem 4) is based upon the assumption that, when u^T is played, the collection of vertex-players $i \in B_m^T$ belonging to the generic m-th block do not perform an overall agreement on how to share the available amount $\left(\frac{\kappa\left(g\left(B_m^T\right)\right)}{\Delta\left(B_m^T\right)}\right) / \left(\sum_{m=1}^{m_T} \frac{\kappa\left(g\left(B_m^T\right)\right)}{\Delta\left(B_m^T\right)}\right)$ of TU. In fact, the underlying assumption is that any player $i \in B_m^T$ gets rewarded in a way which is proportional to the degree $d_{B_m^T}(i)$ he displays within $g\left(B_m^T\right)$ simply because (i) all players are assumed to be equally good in one-to-one negotiations, and (ii) the larger the degree, the larger the number of such negotiations the corresponding player is assumed to make. On the opposite, we may well assume that the exogenously given graph $g \in \Gamma(N)$ (as well as the peculiar game which is played) is known to all players, so that when u^T is played all players $i \in B_m^T$, for $m = 1, \ldots, m_T$, know which players occupy more (or less) central positions within $g\left(B_m^T\right)$. According to such an assumption, we may impose that all players $i \in B_m^T \subset T$, for $m = 1, \ldots, m_T$, agree on sharing the total amount $\left(\frac{\kappa\left(g\left(B_m^T\right)\right)}{\Delta\left(B_m^T\right)}\right) / \left(\sum_{m=1}^{m_T} \frac{\kappa\left(g\left(B_m^T\right)\right)}{\Delta\left(B_m^T\right)}\right)$ of TU available to them so that any player $i \in B_m^T$ gets a share which is proportional to $\kappa\left(g\left(B_m^T\right)\right) / \kappa\left(g\left(B_m^T \backslash i\right)\right)$, where $\kappa(g(S))$ denotes the connectivity of any (spanned) subgraph $g(S)$. In fact, by definition, there is no cutvertex within any block $g\left(B_m^T\right)$, since $\kappa\left(g\left(B_m^T\right)\right) \geq 2$ for all $m = 1, \ldots, m_T$, so that such a sharing rule never displays indeterminacies. Consider a value function $\phi^{BCc} : \mathcal{G}_\Gamma(N) \rightarrow \mathbb{R}^n$ such that

$$\phi_i^{BCc}\left(u^T, g\right) = \left(\frac{\kappa\left(g\left(B_m^T\right)\right) / \kappa\left(g\left(B_m^T \backslash i\right)\right)}{\sum_{j \in B_m^T} \kappa\left(g\left(B_m^T\right)\right) / \kappa\left(g\left(B_m^T \backslash j\right)\right)}\right) \left(\frac{\frac{\kappa\left(g\left(B_m^T\right)\right)}{\Delta\left(B_m^T\right)}}{\sum_{m=1}^{m_T} \frac{\kappa\left(g\left(B_m^T\right)\right)}{\Delta\left(B_m^T\right)}}\right) \qquad (18)$$

for all players $i \in B_m^T \subset T$ and $m = 1, \ldots, m_T$, and for all connected unanimity games $u^T \mid T \in \mathcal{F}_g$. We omit the proof of the following theorem since the details are the same as those of the proofs of theorems 4 and 6.

Theorem 7 *The unique random-order value for point games consistent with (18) is given by*

$$\phi_i^{BCc}(v, g) = E_{P_\lambda^{v/g}}\left[v/g\left(PR^\pi(i) \cup i\right) - v/g\left(PR^\pi(i)\right)\right]$$

for all $i \in N$ and $(v, g) \in \mathcal{G}_\Gamma(N)$, where

$$P_\lambda^{v/g}(\pi) = \sum_{T \in \mathcal{F}_g} \left(\frac{abs\left(a^{v/g}(T)\right)}{\sum_{T \in \mathcal{F}_g} abs\left(a^{v/g}(T)\right)}\right) \left(\frac{\prod_{j=1}^{t} \lambda_{(j)}^T / \sum_{h=j}^{t} \lambda_h^T}{(n)_{n-t}}\right)$$

for all $\pi \in \Pi(N)$ that are consistent with (or induce) any given ordering $\left\{i_{(1)}, \ldots, i_{(t)}\right\} = \pi_T \in \Pi(T)$, with $T \in \mathcal{F}_g$, and where

$$\lambda_i^T = \sum_{\substack{m \in \{1, \ldots, m_T\} \\ \left\{E(i) \cap E\left(B_m^T\right)\right\} \neq \emptyset}} \left(\frac{\kappa\left(g\left(B_m^T\right)\right) / \kappa\left(g\left(B_m^T \backslash i\right)\right)}{\sum_{j \in B_m^T} \kappa\left(g\left(B_m^T\right)\right) / \kappa\left(g\left(B_m^T \backslash j\right)\right)}\right) \left(\frac{\kappa\left(g\left(B_m^T\right)\right)}{\Delta\left(B_m^T\right)}\right)$$

7 ϕ^{BCd} and ϕ^{BCc} in terms of probabilistic values for point games[13]

We have expressed both ϕ^{BCd} and ϕ^{BCc} solely as random-order values, that is through, repectively, the two probability distributions $\left(P_{\omega}^{v/g}(\pi)\right)_{\pi\in\Pi(N)}$ and $\left(P_{\lambda}^{v/g}(\pi)\right)_{\pi\in\Pi(N)}$ over the collection of permutations of the whole player set. Furthermore, since these latter are both constant across players, theorems 12 and 13 in Weber (1988) apply here as well. In particular, it is clear that

$$\phi_i^{BC(d \text{ or } c)}(v,g) = \sum_{\substack{S\subset N\\ S\ni i}} \left(\sum_{\substack{\pi\in\Pi(N)\\ PR^{\pi}(i)=S\backslash i}} P_{(\omega \text{ or } \lambda)}^{v/g}(\pi) \right) [v/g(S) - v/g(S\backslash i)] \quad (19)$$

Furthermore, as we shall see, both ϕ^{BCd} and ϕ^{BCc} constitute generalizations of *weighted Shapley values for point games*. Weighted Shapley values were first proposed by Shapley (1953a). Subsequently, such a family of value functions has been characterized by Kalai and Samet (1988), by means of a so-called *partnership* axiom, and by Hart and Mas-Colell (1989), by means of the *potential* function. More recently, Calvo and Santos (2000) have also proposed and characterized the related family of *weighted weak semivalues*. Given a game $v \in \mathcal{G}(N)$, the definition of any associated weighted Shapley value requires first the introduction of an n-dimensional weight vector $\mathbb{R}^n \ni \omega = (\omega_i)_{i\in N}$. In section 2 we have reported that the standard Shapley value $\phi^{Sh}(v)$ can be expressed as

$$\phi_i^{Sh}(v) = \sum_{\substack{S\subset N\\ S\ni i}} \frac{a^v(S)}{\#S} \qquad \forall i \in N$$

Let $\phi^{\omega-Sh}(v)$ be the weighted Shapley value of game v associated with the weight vector ω; then

$$\phi_i^{\omega-Sh}(v) := \sum_{\substack{S\subset N\\ S\ni i}} \frac{[a^v(S)]\,\omega_i}{\sum_{j\in S}\omega_j} \qquad \forall i \in N$$

so that

$$\phi_i^{\omega-Sh}\left(u^S\right) := \begin{cases} \frac{\omega_i}{\sum_{j\in S}\omega_j} \text{ if } i \in S \\ 0 \text{ if } i \in N\backslash S \end{cases} \qquad \forall S \subset N$$

for all unanimity games $u^S, S \subset T$. It is easily understood that any weight vector $\omega \in \mathbb{R}^n$ formalizes the existence of some hierarchy within the player set.

[13] This section and the following one develop from the companion paper Rossi (2000a)

Most importantly, such a hierarchy is unique; that is to say, it is maintained unchanged for all unanimity games. On the opposite, in the previous sections we have proposed the idea that any connected unanimity game have an associate peculiar and graph-induced hierarchical structure, and, given $g \in \Gamma(N)$, both ϕ^{BCd} and ϕ^{BCc} reward the players in a way which is consistent with *all* such hierarchical structures associated with connected unanimity games. This is the reason why we argue that the two values here proposed constitute generalizations of weighted Shapley values for point games , where the unique non-weighted Shapley value for point games is, of course, the Myerson value. In particular, by defining

$$\omega_i^S = \begin{cases} \sum\limits_{\substack{m\in\{1,\dots,m_S\} \\ \{E(i)\cap E(B_m^S)\}\neq\emptyset}} \left(\frac{d_{B_m^S}(i)}{2\#E(B_m^S)}\right)\left(\frac{\kappa(g(B_m^S))}{\Delta(B_m^S)}\right) & \text{if } i\in S \\ 0 \text{ if } i\in N\backslash S \end{cases}$$

$$\lambda_i^S = \begin{cases} \sum\limits_{\substack{m\in\{1,\dots,m_S\} \\ \{E(i)\cap E(B_m^S)\}\neq\emptyset}} \left(\frac{\kappa(g(B_m^S))/\kappa(g(B_m^S\backslash i))}{\sum_{j\in B_m^S}\kappa(g(B_m^S))/\kappa(g(B_m^S\backslash j))}\right)\left(\frac{\kappa(g(B_m^S))}{\Delta(B_m^S)}\right) & \text{if } i\in S \\ 0 \text{ if } i\in N\backslash S \end{cases}$$

for all $S\in\mathcal{F}_g$, it is straightforwardly verified that

$$\left.\begin{aligned} \phi_i^{BCd}(v,g) &= \sum_{\substack{S\in\mathcal{F}_g \\ S\ni i}} \frac{\left[a^{v/g}(S)\right]\omega_i^S}{\sum_{j\in S}\omega_j^S} \\ \phi_i^{BCc}(v,g) &= \sum_{\substack{S\in\mathcal{F}_g \\ S\ni i}} \frac{\left[a^{v/g}(S)\right]\lambda_i^S}{\sum_{j\in S}\lambda_j^S} \end{aligned}\right\} \quad \forall i\in N, (v,g)\in\mathcal{G}_\Gamma(N) \tag{20}$$

Furthermore, these two last expressions allow us to put both ϕ^{BCd} and ϕ^{BCd} under the probabilistic form. In fact, for any game $v\in\mathcal{G}(N)$ (recall that $\mathcal{G}_g(N)\subset\mathcal{G}(N)$ for all $g\in\Gamma(N)$) and for all $S\subset N$ such that $S\ni i$, we have (see, for example, Grabisch (1997), proof of Theorem 1, p. 174)

$$v(S)-v(S\backslash i) = a^v(i) + \sum_{j\in S\backslash i} a^v(\{i,j\}) + \sum_{\substack{T\subset S\backslash i \\ \#T=2}} a^v(i\cup T) +$$

$$+\cdots+\sum_{\substack{T\subset S\backslash i \\ \#T<\#S-2}} a^v(i\cup T) + \sum_{\substack{T\subset S\backslash i \\ \#T=\#S-2}} a^v(i\cup T) + a^v(S) \tag{21}$$

Thus, by combining (19), (20) and (21), we might temptatively approach the problem of expressing the block-connectivity-degree and the block-connectivity-connectivity values *restricted to point games* as probabilistic values in the following way. Assume, for simplifying notations, that $N\in\mathcal{F}_g$. Furthermore, $\mathcal{F}_g$

typically being a *set system* (i.e., in 2^N) for all $g \in \Gamma(N)$, we adopt the following notational conventions that are standard in Combinatorics (see Bollobás (1989)). For any integer k such that $0 \leq k \leq n$, let $\mathcal{F}_g^{(k)} := \{S \in \mathcal{F}_g \mid \#S = k\}$ denote the k-th *level set* of $\mathcal{F}_g$. Furthermore, for any set $S \in \mathcal{F}_g$, integer k satisfying $\#N = n \geq k \geq \#S$, let $\mathcal{F}_g^{(k)}(S) := \{T \in \mathcal{F}_g \mid \#T = k, T \supset S\}$. Eventually, for any $T \in \mathcal{F}_g^{(k)}(S)$ and integer h such that $\#S \leq h \leq k$, let $\partial_S^{(h)}(T) := \left\{R \in \mathcal{F}_g^{(h)}(S) \mid R \subset T\right\}$. Then, simply by substituting, we obtain

$$\phi_i^{BCd}(v,g) = \sum_{\substack{S \in \mathcal{F}_g \\ S \ni i}} \frac{\left[a^{v/g}(S)\right]\omega_i^S}{\sum_{j \in S}\omega_j^S} =$$

$$= \sum_{\substack{S \in \mathcal{F}_g \\ S \ni i}} \frac{\omega_i^S}{\sum_{j \in S}\omega_j^S} \left([v/g(S) - v/g(S \backslash i)] - \sum_{\substack{T \subsetneq S \\ T \ni i \\ T \in \mathcal{F}_g}} a^{v/g}(T) \right) =$$

$$= \sum_{\substack{S \in \mathcal{F}_g \\ S \ni i}} [v/g(S) - v/g(S \backslash i)] \times$$

$$\times \left[\sum_{k=\#S}^{n} \sum_{T \in \mathcal{F}_g^{(k)}(S)} (-1)^{k-\#S} \frac{\omega_i^T}{\sum_{j \in T}\omega_j^T} \left(\prod_{h=\#S}^{k} \#\partial_S^{(h)}(T) \right) \right]$$

$$\phi_i^{BCc}(v,g) = \sum_{\substack{S \in \mathcal{F}_g \\ S \ni i}} [v/g(S) - v/g(S \backslash i)] \times$$

$$\times \left[\sum_{k=\#S}^{n} \sum_{T \in \mathcal{F}_g^{(k)}(S)} (-1)^{k-\#S} \frac{\lambda_i^T}{\sum_{j \in T}\lambda_j^T} \left(\prod_{h=\#S}^{k} \#\partial_S^{(h)}(T) \right) \right]$$

Furthermore, simply by setting ω_i^S or, equivalently, $\lambda_i^S = (\#S)^{-1}$ for all players $i \in N$ and for all coalitions $N \supset S \ni i$, we have that one way the Myerson value may be expressed as a probabilistic value is the following

$$\phi_i^{My}(v,g) = \sum_{\substack{S \in \mathcal{F}_g \\ S \ni i}} [v/g(S) - v/g(S \backslash i)] \times$$

$$\times \left[\sum_{k=\#S}^{n} \sum_{T \in \mathcal{F}_g^{(k)}(S)} (-1)^{k-\#S} \frac{1}{\#T} \left(\prod_{h=\#S}^{k} \#\partial_S^{(h)}(T) \right) \right] \quad (22)$$

Note that if g is the complete graph so that $\mathcal{F}_g \equiv 2^N$, then $v/g(S) = v(S)$ for all $S \subset N$, and $\sum_{k=\#S}^{n} \sum_{T \in \mathcal{F}_g^{(k)}(S)} (-1)^{k-\#S} \frac{1}{\#T} \left(\prod_{h=\#S}^{k} \#\partial_S^{(h)}(T) \right) =$

$= \sum_{k=\#S}^{n} \sum_{\substack{N \supset T \supset S \\ \#T=k}} (-1)^{k-\#S} \left(\frac{1}{k}\right) \prod_{h=\#S}^{k} \binom{k-\#S}{h-\#S} = \frac{(n-\#S)!(\#S-1)!}{n!}$. As known, within the broad class of probabilistic values, the Shapley value definitely plays a special role, since by satisfying symmetry it implies $p_i^S = p^{\#S}$ for all players and coalitions $i \in S \subset N$. That is to say, the probability each individual attributes to the event of joining any coalition $S \subset N$ must depend solely on the cardinality $\#S$. And since the Myerson value is the Shapley value of v/g (and therefore satisfies symmetry with respect to v/g), it must also be consistent with such a condition. In fact, (1) can clearly be written as $\phi_i^{My}(v,g) = \sum_{\substack{S \subset N \\ S \ni i}} [v/g(S) - v/g(S \backslash i)] \frac{(n-\#S)!(\#S-1)!}{n!}$. Nevertheless, expression (22) implies something different, in that $p_i^S = 0$ for all $S \notin \mathcal{F}_g$; more precisely, we can see that coalitions $S \in \mathcal{F}_g$ are grouped in classes according to the number of connected larger coalitions, of any given cardinality, they are contained within, that is on the basis of the $n - \#S$ integers given by $\#\mathcal{F}_g^{(k)}(S)$, for $k = \#S, \ldots, n$ (recall we are assuming $\#N/g = 1$); let some integer x be the total number of such classes; also let integers $x_1, \ldots, x_x$ denote the numbers of coalitions belonging to each class. Then (22) implies that the model for choosing a random connected coalition S is the following: first, a class $j \in \{1, \ldots, x\}$ is chosen randomly (i.e., with probability $1/x$ each); second, within such a (random) class a coalition is chosen randomly (i.e., with probability $1/x_j$ each).

For all players $i \in N$, let $p_i(BCc), p_i(BCd) : \{S \subset N \mid S \ni i\} \to \mathbb{R}$ be defined as follows

$$p_i^S(BCc) = \sum_{k=\#S}^{n} \sum_{T \in \mathcal{F}_g^{(k)}(S)} (-1)^{k-\#S} \frac{\lambda_i^T}{\sum_{j \in T} \lambda_j^T} \left(\prod_{h=\#S}^{k} \#\partial_S^{(h)}(T) \right)$$

$$p_i^S(BCd) = \sum_{k=\#S}^{n} \sum_{T \in \mathcal{F}_g^{(k)}(S)} (-1)^{k-\#S} \frac{\omega_i^T}{\sum_{j \in T} \omega_j^T} \left(\prod_{h=\#S}^{k} \#\partial_S^{(h)}(T) \right)$$

for all $S \in \mathcal{F}_g \mid S \ni i$, while $p_i^S(BCc) = p_i^S(BCd) = 0$ for all $i \in S \notin \mathcal{F}_g$. Thus, by noting that each weight term $\frac{\omega_i^S}{\sum_{j \in S} \omega_j^S}$ appears only within those $p_i^T(BCd)$ such that $\mathcal{F}_g \ni S \supset T$, we obtain

$$\sum_{\substack{S \in \mathcal{F}_g \\ S \ni i}} p_i^S(BCd) = \sum_{\substack{S \in \mathcal{F}_g \\ S \ni i}} \left(\frac{\omega_i^S}{\sum_{j \in S} \omega_j^S} \right) \left(\sum_{\substack{R \subset S \\ R \in \mathcal{F}_g \\ R \ni i}} (-1)^{\#S-\#R} \prod_{h=\#R}^{\#S} \#\partial_R^{(h)}(S) \right)$$

We must emphasize that nor $\left\{p_i^S(BCd)\right\}_{S\in\mathcal{F}_g, S\ni i}$, neither $\left\{p_i^S(BCc)\right\}_{S\in\mathcal{F}_g, S\ni i}$ have to constitute probability distributions. Nevertheless, there definitely exists some (possibly stochastic) quantitative interdependence between $\frac{\omega_i^S}{\sum_{j\in S}\omega_j^S}$ and (for $i\in S\supset R\ni i$) $\prod_{h=\#R}^{\#S}\#\partial_R^{(h)}(S)$, as well as between $\frac{\omega_i^T}{\sum_{j\in T}\omega_j^T}$ and (for $i\in T\supset S\ni i$) $\prod_{h=\#S}^{k}\#\partial_S^{(h)}(T)$, and the same applies to $\left\{\lambda_i^S\right\}_{S\in\mathcal{F}_g, S\ni i}$. This is because both ω, λ and the various $\#\partial_R^{(h)}(S)$, $\#\partial_S^{(h)}(T)$ are determined on the basis of the peculiar connectivity features of the graph under concern[14].

We shall now determine under which conditions it may be possible to express ϕ^{BCd} and ϕ^{BCc} in terms of probabilistic values. Extending the treatment of Calvo and Santos 2000, we can say that a solution ϕ admits a *w-potential* (for $w=\left(\left(w_i^S\right)_{i\in S}\right)_{\emptyset\neq S\subset N}\in\mathbb{R}^{n2^{n-1}}$) if there exists a function $\mathcal{P}_w:\mathcal{G}(N)\to\mathbb{R}^{2^n-1}$ such that

$$\phi_i(v)=\sum_{\substack{S\subset N\\ S\ni i}} w_i^S\left[\mathcal{P}_w(v(S))-\mathcal{P}_w\left(v_{N\setminus i}(S)\right)\right]$$

for all $i\in N$ and $v\in\mathcal{G}(N)$. Given (20), it is clear that we must have $\mathcal{P}_w(v(S))-\mathcal{P}_w\left(v_{N\setminus i}\right)=\frac{a^v(S)}{w^S}$, where $w^S:=\sum_{j\in S}w_i^S$. Thus the above mentioned possibility exists if, and only if, $\sum_{\emptyset\neq S\subset N}\frac{1}{w^S}=1$

8 Axiomatic characterization

It is straightforward to check that the restriction of both ϕ^{BCd} and ϕ^{BCc} to point games satisfy CE, L, IIP and P described in section 3. Thus such axioms clearly are not sufficient for obtaining uniquess. In (20) the restrictions of both ϕ^{BCd} and ϕ^{BCd} to point games are expressed in terms of the (unique) associated potential function (see Hart and Mas-Colell (1989)). Accordingly, the simplest way of characterizing such values for point games is by means of a single axiom. In particular, please consider the following axioms

- *degree weighted balanced contributions (DWBC):*

$$\phi_i(v_S, g(S))-\phi_j(v_S, g(S))=\sum_{\substack{T\subset S\\ T\in\mathcal{F}_g\\ j\in T\ni i}}\frac{a^{v/g}(T)\left[\omega_i^T-\omega_j^T\right]}{\sum_{h\in T}\omega_h^T}$$

for all $S\subset N, i, j\in S, v\in\mathcal{G}(N), g\in\Gamma(N)$

[14] In order to determine such a possibly stochastic relationship, interesting results could be found in Pittel, Spencer and Wormal (1996)

- *connectivity weighted balanced contributions (CWBC):*

$$\phi_i(v_S, g(S)) - \phi_j(v_S, g(S)) = \sum_{\substack{T \subset S \\ T \in \mathcal{F}_g \\ j \in T \ni i}} \frac{a^{v/g}(T)\left[\lambda_i^T - \lambda_j^T\right]}{\sum_{h \in T} \lambda_h^T}$$

for all $S \subset N, i, j \in S, v \in \mathcal{G}(N), g \in \Gamma(N)$

It is then immediately verified that

Theorem 8 *The restriction of* ϕ^{BCd} *to point games is the unique value function on the space of such games satisfying DWBC.*

Theorem 9 *The restriction of* ϕ^{BCc} *to point games is the unique value function on the space of such games satisfying CWBC.*

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