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Investment, Growth and Economic Fluctuations

A self-organised criticality approach

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We study the behaviour of the average aggregate growth rate of an economy, driven by only small idiosyncratic total factor productivity shocks and where investment behaviour at the single plant level follows an (S,s) policy. We assume that in the case of investment (disinvestment) there are positive (negative) localised spill-over effects because of factor demand linkages. We show that if these spill-over effects are strong, the economy converges naturally towards a "critical" state where there exists a positive, highly volatile aggregate investment rate which depends on the size of the microeconomic non-convexities. We find so a positive, highly volatile growth rate of aggregate output.

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“...the development of the organism, whether social or physical, involves an increasing subdivision of functions between its separate parts on the one hand, and on the other a more intimate connection between them. Each part gets to be less and less self-sufficient, to depend for its wellbeing more and more on other parts, so that any disorder in any part of a highly-developed organism will affect other parts also.”

A. Marshall[1]

1. Introduction

It has been pointed out in the recent economic literature[2] that investment at the single plant level is mostly intermittent and discrete. Studying a 17-year sample of plants belonging to the U.S. manufacturing sector, Doms and Dunne (1994) evidence that plants concentrated about 50 % of their cumulative investment over the sample in the three years surrounding the year with the largest investment. Theoretical explanations for this lumpiness are increasing returns in the investment technology. They are due to sunk costs of reorganisation or indivisibilities. Given this optimal policy at the single plant level, we have to determine the aggregate behaviour of the economy.

Caballero and Engel (1998) study the aggregate behaviour of an economy where discrete and intermittent capital adjustment is optimal using the hazard function approach[3]. According to the level of capital imbalance, each firm has a probability of adjusting their capital stock, which is a function of the actual level of capital imbalance and is called hazard function. The economy is fully characterised by the probability distribution of adjustment of the single firms. In the stationary state the authors study the response of the economy in the case of aggregate shocks and show that this response is non-linear. Caballero and Engel consider these firms as independent, that is, if a firm adjusts its capital stock, this will not have any effect on other firms[4].

In this paper we consider the case where the discrete and intermittent policy for the adjustment of the capital stock leads to spill-over effects: we will assume that in the case of adjustment of the

[1] A. Marshall “Principles of Economics“, London Macmillan, Book IV, Chapter VII, 1.
[2] See for example Caballero (2000).
[3] See also Caballero and Engel (1993).
[4] Caballero and Engel (1991) study aggregate dynamics of independent firms using (S,s) policy in the case of

capital stock, “neighbouring” firms will be hit by a demand shock. These dynamic spill-over effects are because of factor demand linkages, like those pointed out in Cooper and Haltiwanger (1992). A discrete increase in the capital stock leads to an increase in the demand of those firms supplying this capital. The firm transfers in this way, at least to some extent, a demand shock to randomly matched neighbouring firms, which supply this capital. The extent to which a shock is transferred to neighbouring firms depends on the strength or degree of demand spill-over effects. The strength of these spill-over effects depends on the degree of specialisation of each firm. It has been pointed out by Alfred Marshall[5], that as the economy grows, the specialisation of each firm increases and so also the interdependence between the single firms increases. Because of these demand spill-over effects, we obtain an endogenous diffusion of the exogenous shocks. This latter is so a sort of endogenous multiplier of the exogenous shocks. The extent of this endogenous multiplier depends on the degree or strength of the spill-over effects. We find that even though the economy is driven only by small idiosyncratic technology shocks, the investment behaviour of the firms is correlated. We show that the strength of this correlation depends on the strength of the spill-over effects. In highly developed countries there is a natural tendency of the strength of these spill-over effects to become very high because of the increasing specialisation of each part of the system and because of the resulting increasing interdependence between its single parts. We will show that, in the case of strong spill-over effects, the correlation of investment activity among the single firms is high, leading to a highly volatile positive, aggregate investment rate, even though the economy is driven only by small idiosyncratic total factor productivity shocks.

Our aim is to study the growth rate of aggregate output of an economy driven by small idiosyncratic total factor productivity shocks, where investment behaviour at the single plant level is discrete and intermittent and where firms interact dynamically through spill-over effects, because of factor demand linkages. We will see that the extent to which the growth rate of aggregate output is affected by this dynamic interaction depends on the strength of the spill-over effects. If the strength of the spill-over effects is high, then the investment behaviour of the single firms will be highly correlated and the aggregate investment rate will be positive and highly volatile. Thus, the growth rate of aggregate output will depend not only on the exogenously given growth rate of total factor productivity, but also on the aggregate investment rate. Since this latter is highly volatile, this will lead to large fluctuations in aggregate output. We will see that the aggregate investment rate depends in this case on the size of the microeconomic non-convexities. In particular, we have that greater indivisibilities lead to a higher long run investment rate but also to a higher short run volatility of the same. Holding the exogenous long run growth rate of total factor productivity constant, we show that the economy will exhibit a higher long

idiosyncratic and also aggregate shocks.

[5] See footnote 2.

run growth rate of aggregate output with larger short run fluctuations[6] in the same. Thus, starting form a microeconomic framework, we obtain, in a structural way, an aggregate output with a unit root.

We are going to use self-organized criticality techniques. Self-organized criticality, which has been introduced by Bak et. al. (1987, 1988), studies the dynamics of large interacting ensembles which are governed by threshold dynamics at the microeconomic level[7]. The standard example is the sandpile model. In this model, in each point in time, a sand granule is added at a random site, and the avalanche dynamics, that is the number of sites toppling given an exogenous shock, of the sand pile is studied. These models converge naturally towards a critical state, where the system does not exhibit a characteristic event size and where a small sand granule can be propagated throughout the whole system. Even through the aggregate dynamics are complex, the aggregate behaviour of the system is describable by simple power laws[8]. In this paper we will extent the mean-field approach proposed by Vespignani and Zapperi (1997, 1998).

The paper is organised as follows. Section 2 discusses the microeconomic problem of a representative firm. The aim of this section is to develop a framework where the (S,s) policy is optimal, given an exogenous stochastic environment. In Section 3 we describe the aggregate dynamics of interacting firms using the (S,s) policy. Further, we calculate measures of investment avalanches, that is the number of firms adjusting their capital stock, in the case of a small idiosyncratic technology shock, in the stationary state. In Section 4 we consider the effect of the interaction between firms on the long run aggregate growth rate of the economy and on the variance of this growth rate. Section 5 concludes.

2. 1. The problem of the representative firm[9]

In this section we study the problem of the representative firm. As pointed out in the introduction, the representative firm applies an (S,s) rule for the adjustment of the capital stock. We are going to develop a framework where this policy is optimal given the stochastic environment the representative firms is facing. We will assume that there are sunk costs of investment and disinvestment. That is, in the case of investment (disinvestment) the firm has to pay a sunk cost which is independent of

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[6] Bak et. al. (1993) and Scheinkman and Woodford (1994) evidenced first that in order to obtain aggregate fluctuations in the case of an economy driven by small idiosyncratic shock we need strong localized interaction between the single firms and microeconomic non-convexities. The two elements taken for each own do not generate large fluctuations. See also Horvath (1998), (2000) and Cooper, Haltiwanger and Power (1999).

[7] See for example Jensen (1998) for a good introduction.

[8] A first application to economics is given in Bak et. al. (1993) where the authors study in a model of inventory dynamics, the emergence of aggregate fluctuations in an economy driven by small idiosyncratic shocks. See also Nirei (2000) for an extention of Bak et. al. (1993).

the amount of investment (disinvestment). We will assume that this cost is proportional to the optimal frictionless capital.

Each firm faces an intertemporal problem of profit maximisation. We assume that the representative firm produces according to the following production function

$$F(K,L) = AK^b$$

where K indicates actual level of capital, β < 1 indicates the capital share and $A = e^a$ indicates the total factor productivity and demand strength. We will assume that a is a random variable, following a stochastic process which will be specified further on. Profits, in each period of time, of the representative firm are given by

$$\Pi = AK^b - rK$$

Static first order condition is given by $\Pi_K = 0$ and so the optimal frictionless capital[10] is given by $K^* = \left[\frac{bA}{r}\right]^{\frac{1}{1-b}}$. We can rewrite the profit function as a function of capital imbalance σ = K/K* and as a function of the optimal frictionless capital:

$$\Pi(s, K^*) = \frac{r}{b}\left(s^b - bs\right)K^*. \tag{1}$$

Profits are homogeneous of degree one in the optimal frictionless capital K* , and are concave in σ. If there is a positive demand or total factor productivity shock, then the level of optimal frictionless capital K* increases as well while the capital imbalance σ decreases. We can write optimal frictionless capital as follows $K^*(t) = A_0 e^{x(t)}x = K^*(0)e^{x(t)}$, where $x = \frac{1}{1-b}a$ and where ξ is a constant. We assume unitary changes in the random variable x. x changes either because of a change in total factor productivity or because of a change in the demand strength through endogenous demand spill-over effects. In particular, we will see in Section 2.2. an 3.1. that in the stationary state, x increases (decreases) by unity with a certain probability and it remains constant with a certain probability. This

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[9] This section follows Caballero (2000)

[10] The frictions here are referred to the costs of adjustment. As pointed out above, in the absence of adjustment costs the dynamic optimisation problem reduces to a repetition of static problems.

stochastic process is representable, in the limit where the number of steps goes to infinity in a finite time, by an arithmetic Brownian motion (see Appendix I). Thus, A follows a geometric Brownian motion[11]:

$$dA = mAdt + sAdv \qquad (2)$$

where $dv = e^* \sqrt{dt}$ and ε* is normally distributed with mean 0 and unitary variance.

As pointed out above, we assume that in the case of investment or disinvestment the representative firm faces a sunk cost of the form

$$c(I_t, K^*) = 1(I_t \neq 0) K^* \begin{cases} c_f^+ \; if & I_t > 0 \\ c_f^- \; if & I_t < 0 \end{cases}$$

where 1(.) is an indicator function, c_f^+ is the sunk cost a firm faces in the case of investment, while c_f^- is the sunk cost in the case of disinvestment. The representative firm maximises the expected present value of profits

Max

$$E_0 \left\{ \int_0^\infty e^{-rt} \Pi(s, K^*) dt - 1(I_t > 0) K^* c_f^+ - 1(I_t < 0) K^* c_f^- \right\} \qquad (3a)$$

s.t. $$dA = mAdt + sAdv \qquad (3b)$$

and $$dK = Idt \qquad (3c)$$

where we assume for simplicity that capital does not depreciate[12]. We can rewrite (3) in a recursive way through the value function V:

$$V(s_t, K_t^*) = \Pi(s_t, K_t^*)dt + (1 - rdt)E\left[V^*(s_{t+dt}, K_{t+dt}^*)\right]$$

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[11] Here we make use of Ito's Lemma which will be explained further on. In particular we have that x follows an absolute Brownian motion of the form dx = μ"dt + σdϖ and through standard steps we obtain that $A = e^x x$ follows the geometric Brownian motion (2) where μ = μ" + ½ σ^2. See for example Dixit (1993) for a simple exposition of stochastic dynamic programming.

[12] We implicitly assume here that firms have to pay sunk costs only if they investment in "new" capital. In other words we assume that repairs or reintegration of already installed capital is done in a continuous and infinitesimal way.

$$V*\left(s_t, K_t^*\right) = MAX\left\{V\left(s_t, K_t^*\right), MAX_h\left[V\left(s_t + h, K_t^*\right) - c\left(h_t, K_t^*\right)\right]\right\}$$

subject to constraints (3b), (3c).

Since the profit and cost functions are homogeneous of degree one in K* we have that also the value function V is homogeneous of degree one in K*. Thus, the problem is representable in the space of imbalances σ. First order conditions for this problem are

$$V_\sigma(c) = 0 \qquad V_\sigma(L) = 0 \qquad V_\sigma(U) = 0 \tag{4}$$

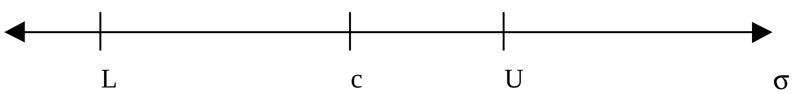
where $V_s = \frac{\partial V}{\partial s}$. Expressions (4) are smooth pasting conditions and

$$V(L, K*) = V(c, K*) - c_f^+ K* \tag{5a}$$

$$V(U, K*) = V(c, K*) - c_f^- K* \tag{5b}$$

are value matching conditions, and determine ,together with (4), the value of c, L and U (see Figure 1).

Figure 1



Optimal policy is inaction as long as the capital imbalance σ is between the lower bound L and the upper bound U. Once σ is lower than L, it is optimal to invest in order to obtain σ = c, and once σ > U to disinvest such that σ = c. The optimal amount of investment will be I, where $\frac{K+I}{K*} = c$ and $\frac{K}{K*} < L$, while the amount of disinvestment will be I, where $\frac{K-I}{K*} = c$ and $\frac{K}{K*} > U$. Notice that, in general, c is different from 1, that is, the dynamic optimal target is different from the static one. If c_f^+ = c_f^- then we have symmetric adjustment, that is, the optimal amount of investment is the same as the optimal amount of disinvestment, while if the two costs are different we have asymmetric adjustment.

Now we are going to characterise the behaviour of the value function within the barriers of control. For this we are going to apply Ito's Lemma. We can rewrite the problem of the firm using (1) and through standard steps one can reduce the problem of the representative firm within the barriers of control as follows

$$rV(K,A) = \Pi(K,A) + \frac{1}{dt} E\left[dV(K,A)\right] \tag{6}$$

s.t. $\quad dA = \mu A dt + \sigma A dv$

$\quad\quad dK = I dt$

$dV(K,A)$ is unknown, so we can calculate it taking a Taylor expansion; we have that $dV(K,A) = \frac{\partial V}{\partial K} dK + \frac{\partial V}{\partial A} dA + \frac{1}{2}\frac{\partial^2 V}{\partial K^2}(dK)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial A^2}(dA)^2 + \frac{\partial^2 V}{\partial A \partial K}(dA)(dK) + \ldots$

Since we are interested in values for which dt tends towards zero, all values multiplied by $(dt)^y$, where y > 1 are negligible. Further have that I = 0 by assumption. Calculating the single terms we have that $dV(K,A) = \frac{\partial V}{\partial A}\left(\mu A dt + \sigma \varepsilon^* A \sqrt{dt}\right) + \frac{1}{2}\frac{\partial^2 V}{\partial A^2}\sigma^2 (\varepsilon^*)^2 A^2 dt$, where we used the fact that $dv = \varepsilon^* \sqrt{dt}$. Since $E[\varepsilon^*] = 0$ and $E[(\varepsilon^*)^2] = 1$ by definition, we can rewrite (6) as follows

$$rV(K,A) = \Pi(K,A) + \frac{\partial V}{\partial A}\mu A + \frac{1}{2}\frac{\partial^2 V}{\partial A^2}\sigma^2 A^2 \tag{7}$$

This last expression is a second order partial differential equation which solution, together with the boundaries condition (4) and (5) solves the problem of the representative firm. Next we are going to study the stochastic environment of the representative firm.

2.2. Notes on the optimality of (S,s) rules

The stochastic environment each firm is facing is partly exogenous and partly endogenous. The former is given by the exogenous driving force of the economy, represented by small idiosyncratic technology shocks. The latter is given by endogenous spill-over dynamics, represented by factor demand linkages between the single firms. Expectations at the firm level lead to a certain microeconomic policy rule. Aggregating this policy rule over all firms we obtain a macrobehaviour of the economy, and this should be consistent with the expectations at the single firm level. In other words, we need consistency between expectations at the microeconomic level and the resulting aggregate dynamics at the macroeconomic level. That is, we are looking for a fixed point. We are going to develop briefly this argument.

In the next Section we are going to study the aggregate dynamics resulting form the localised interaction between the single firms. We are going to describe these dynamics using the concept of avalanches. An investment (disinvestment) avalanche indicates the number of firms investing (disinvesting), given that an exogenous positive (negative) shock hit a single firm. In the stationary state we can calculate the average size of an investment avalanche, that is the average number of firms adjusting in the case of an exogenous technology shock and we can calculate the stationary average fraction of firms in the critical investment (disinvestments) state, that is the state where it needs just one more positive (negative) shock such that it is optimal to invest (disinvest) capital. Since avalanches are unpredictable for the single firm, each firm will calculate the probability that it receives a positive (negative) demand shock from a "neighbouring" firm, which increase (decreases) its capital stock. In particular, we are going to assume that an investing (disinvesting) firm increases (decreases), because of factor demand linkages, the demand of $Z^{+(-)}$ firms[13]. Further, we assume that these $Z^{+(-)}$ firms receiving an endogenous demand shock are chosen at random. We call r_{c_+} the stationary average density (fraction) of firms in the critical investment state, that is, the average fraction of firms, in the stationary state, which are in a state where it needs just one more positive shock, either exogenous total factor productivity or endogenous demand shock, such that this firm becomes active, and invests a discrete amount of capital[14]. Thus, in the stationary state, the probability of receiving a positive demand shock because of factor demand linkages will be $h(1-\lambda)\, r_{c_+} Z^+ + h(1-\lambda)\left(r_{c_+} Z^+\right)^2 + h(1-\lambda)\left(r_{c_+} Z^+\right)^3 + \ldots$ The first term is the probability that a neighbouring firm is in the critical investment state and that this firm receives a positive productivity shock; the second term is the probability that one of the neighbour of one of our neighbours and the neighbour itself are in the critical investment state and that the former one receives a positive productivity shock and so on. Further, we assume that with probability $h(1-\lambda)$ each firm receives a positive, exogenous, total factor productivity shock, so the probability of receiving a positive shock, either productivity shock or demand shock, is $h(1-l)\frac{1}{1-r_{c_+}Z^+}$. In the same way we can calculate the probability of receiving a negative shock. Following the arguments outlined above we have that the probability of receiving a negative shock, either total factor productivity or demand shock, is $lh\frac{1}{1-r_{c_+}Z^-}$, where r_{c_-} is the stationary average density of firms in the critical disinvestments state. As the number of steps within a given time interval goes to infinity this stochastic process is

[13] We will specify in the next Section the value of $Z^{+(-)}$.

[14] We will define this state more precisely in the next section.

representable by a Brownian motion (see Appendix I). This fact, together with the arguments outlined in section 2.1 is sufficient for the optimality of the (S,s) policy.

As pointed out at the beginning of this section, we need consistency between the optimal microeconomic policy rule and the resulting macroeconomic behaviour. Given the expectations about receiving a demand shock from a neighbouring firm, and the probability of receiving a technology shock, we have an optimal (S,s) policy with a certain bandwidth. We will see that this bandwidth is crucial in determining the stationary behaviour of the model, and in particular in determining the stationary average density of firms in the critical investment (disinvestment) state. The bandwidth influences the stationary average densities and influences the probability of receiving a demand shock from neighbouring firms. So we have to find a fixed point, where the probabilities of receiving demand shocks from neighbouring firms generates bandwidths and stationary average densities of firms in the critical investment and disinvestments state which in turn are consistent with the initial probabilities of receiving demand shocks from randomly matched neighbouring firms.

In the next Section we are going to study the aggregate dynamics of this economy. We will calculate stationary average densities of firms in active and critical investment and disinvestment state, the average investment and disinvestment avalanche size and its average change .

3.1. Aggregate Dynamics

In this Section we are going to outline the aggregate dynamics resulting from the interaction of firms which apply an (S,s) policy as shown in Section 2. The interaction between single firms is given by spill-over effects through factor demand linkages. We assume that if a firm adjusts its capital stock, it increases demand, at least to some extent, of $Z^{+(-)}$ randomly matched neighbouring firms. That is, we consider a mean-field interaction between the firms.

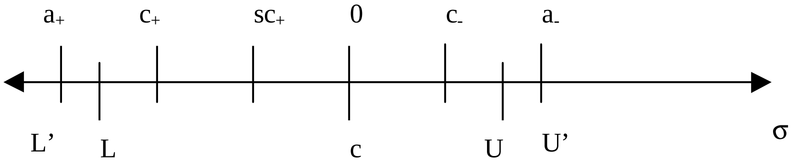
Since *x* changes in a discrete way, we have that the state variable $\sigma = K/K^*$ changes in a discrete way. In particular we have that σ assumes values belonging to a discrete state space Ω, where this state space is given by all values of σ, for each possible value of *x*. In particular, we have that $K^*(t) = K^*(0)e^{x(t)}$, where the dynamics of *x(t)* are described as follows

$$x(t+1)=\begin{cases} x(t)+1 & with \quad probability \quad h(1-l)\dfrac{1}{1-r_{c_+}Z^+} \\ x(t)-1 & with \quad probability \quad hl\dfrac{1}{1-r_{c_-}Z^-} \\ x(t) & with \quad probability \quad 1-h(1-l)\dfrac{1}{1-r_{c_+}Z^+}-hl\dfrac{1}{1-r_{c_-}Z^-} \end{cases}$$

For each value of *x* we have a value of K* and so a value of σ. Capital imbalance level for firm i, σ_i, assumes one of the values of the discrete state space Ω, that is $\sigma_i \in \Omega$, for i = {1, ..., n}. We label the elements of the state space as follows: Ω = {a_-, c_-, sc_-, sc_- - 1, ..., 0, ..., sc_+ - 1, sc_+, c_+, a_+}. a_+ indicates the active investment state, that is the state where the firms invests the optimal amount I and switches to state 0. c_+ is the critical investment state, that is, the state just before the active state. sc_+ is the subcritical investment state and so on. The same applies for the disinvestment states. All states but $a_{+(-)}$ are stable states in the sense that inaction is an optimal policy for the firm.

The economy is represented by σ* = $\{s_i\}_{i=1}^n$, where n is the number of firms in our economy. The cardinality of the state space depends on g^+ and g^-, the number of steps necessary to reach the barriers of control starting from the optimal level, which in turn depends on the size of the sunk costs and indivisibilities. In particular, g^+ and g^- are non-decreasing functions of the size of the sunk costs or indivisibilities. We can calculate the number of positive (negative) shocks necessary to reach the trigger point L' (U') (see Figure 3), once we are at the target point c. We have that the following has to be satisfied: $L'e^{g^+}=c$ $\left(U'e^{g^-}=c\right)$. Thus, rearranging terms we have that $g^+=\log\left(\dfrac{c}{L'}\right)$ $\left(g^-=\log\left(\dfrac{U'}{c}\right)\right)$[15]. For example, if g^+ = 3 and g^- = 2 then the state space will be $\Omega=\{a_-,c_-,0,sc_+,c_+a_+\}$(see Figure 3).

Figure 3



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15 g^+ and g^- are by definition of L' and U' integers.

We can express the optimal investment (disinvestment) in terms of g^+ (g^-). We have that by definition of investment (disinvestment) $\frac{I}{K*} = c - L'$ $\left(\frac{I}{K*} = c - U'\right)$. Using the condition $L' e^{g^+} = c$ $\left(U' e^{g^-} = c\right)$ we have that $\frac{I}{K*} = L'\left(e^{g^+} - 1\right)$ $\left(\frac{I}{K*} = U'\left(1 - e^{g^-}\right)\right)$. Once a firm is hit by a shock, either technology shock or demand shock, we have that it changes state. If a firm is for example in the critical investment state, and if it receives a negative shock, then it switches to the subcritical investment state, while if it receives a positive shock, then it switches to the active investment state. In this latter case the firm adjusts fully its capital stock and in this way increases demand, at least to some extent, of Z^+ randomly matched neighbouring sites because of factor demand linkages.

We assume that in the case of investment (disinvestment), firm increases (decreases), at least to some extent, the demand of $Z^{+(-)} = g^{+(-)}$ randomly matched neighbouring firms because of factor demand linkages. Thus, the number of neighbouring firms which receive a positive (negative) demand shock depends on the size of the adjustment. This because we normalise the size of the endogenous demand shocks to be equal to the size of the exogenous total factor productivity shocks. As the amount of adjustment increases, while holding the size of the demand shock constant, the number of firms from whom the firm acquires capital has to increase. Further, we will introduce the degree or strength of spill-over effects introducing a variable indicating the probability that some of the $g^{+(-)}$ randomly matched neighbouring firms do not receive a demand shock, i.e. that some of the demand shocks will be dissipated because of insufficient factor demand linkages. Neighbours receiving a positive demand shock switch state toward state a_+. If one of the neighbours receiving a positive demand shock is in the critical investment state, it switches to the active investment state, and invests an amount of $K * L'\left(e^{g^+} - 1\right)$, and increases so demand, at least to some extent, of g^+ randomly matched neighbouring firms, and so on. As long as there are firms in an active investment (disinvestment) state, this process goes on. Once there are no firms in the active investment or disinvestment state, the economy falls in an absorbing state[16].

We are going to study the evolution of the configuration of the system and the associated investment dynamics. We are going to cluster firms according to their state, and study the time evolution of these clusters. In particular, we are going to define a clusters for each possible state, indicating the average density of firms being actually in that state.

We start with a local description of the dynamics. We assume that in each point of time with probability $h(1 - \lambda)$ $(h\lambda)$ each firm receives a positive (negative) total factor productivity shock respectively. Since we are interested in the time evolution of our cluster, we are going to specify transition probabilities for the single states. Once solved the model, we recover aggregate dynamics for h equal to zero[17].

In order to study the evolution of the system we have to specify the transition probability of the economy to switch from state σ*' to state σ*, which we write as W(σ*|σ*`). We assume that the transition probability is symmetric and homogeneous and such that W(σ*|σ*`) = $\prod_{i=1}^{n} w\left(\sigma_i^* \middle| \sigma^{*\prime}\right)$. $w\left(\sigma_i^* \middle| \sigma^{*\prime}\right)$ is the single site transition probability, which satisfies the normalisation property $\sum_{\sigma_i^*} w\left(\sigma_i^* \middle| \sigma^{*\prime}\right) = 1$. The probability that the system is in state σ* at time t is given by *P(σ*, t)*. We can write the master equation which describes the evolution of the probability distribution as follows

$$\frac{\partial}{\partial t} P\left(\sigma^*, t\right) = \sum_{\sigma^{*\prime}} W\left(\sigma^* \middle| \sigma^{*\prime}\right) P\left(\sigma^{*\prime}, t\right) - P\left(\sigma^*, t\right) \sum_{\sigma^{*\prime}} W\left(\sigma^{*\prime} \middle| \sigma^*\right)$$

Following Vespignani, Zapperi (1998) we take a single site mean-field approximation[18]. We denote $\rho_{a_+}, \rho_{c_+}, \rho_{sc_+}, \ldots, \rho_{sc_-}, \rho_{c_-}, \rho_{a_-}$ the average density[19] of firms, in each state. Using an indicator function we can write these densities as

$$\rho_k(t) = \sum_{\{\sigma_j^*\}} 1\left(\sigma_j^* - k\right) P\left(\sigma^*, t\right)$$

where $1\left(\sigma_j^* - k\right)$ = 1 if $\sigma_j^* = k$. The transition probabilities can be written as follows

$$\sum_{\{\sigma_j^*\}} 1\left(\sigma_j^* - k\right) \prod_i w\left(\sigma_i^* \middle| \sigma'\right) = w\left(\sigma_j^* = k \middle| \sigma^{*\prime}\right)$$

Using the master equation and the definitions above, together with the normalisation condition for the transition probabilities we can write the evolution of the average density of firms in state k as follows

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16 As we will see, these models have infinitely many absorbing states, that is a state where the system in trapped in without possibility of escape. See Dickman, Vespignani and Zapperi (1998).

17 See Vespignani and Zapperi (1997), (1998) for details.

18 Mean field theory is concerned with the study of the time evolution of clusters.

19 In our case it is the average fraction of firms in a given state.

$$\frac{\partial}{\partial t} r_k = \sum_{\{s^{*}{}'\}} w\left(k \middle| s^{*}{}'\right) \prod_i r_{s_i{}'} - r_k \qquad (8)$$

where the evolution equation has been truncated using a dynamical mean field approximation at the single firm level[20], that is $P\left(s^{*}\right) = \prod_i P\left(s^{*}_i\right) = \prod_i r_{s^{*}_i}$.

The dynamics of the system are so completely specified once transition rates at the single firm level are determined.

Given that a firm is in the critical disinvestment state, and given that no neighbouring firm is in the active disinvestment state, the single site transition probability to shift to the active disinvestment state is given by

$$w\left(a_{-} \middle| c_{-}, s'_{NN} \neq a_{-}\right) r_{c_{-}} \prod_{i \in s_{NN}} r_{s_i{}'} = l h r_{c_{-}} \left(1 - r_{a_{-}}\right)^{Z^{-}}$$

where $Z^- = g^-$ is the number of neighbouring firms. As pointed out previously, in the case of investment (disinvestment), the adjusting firm increases (decreases) demand of potentially $Z^{+(-)} = g^{+(-)}$ randomly matched neighbouring firms because of factor demand linkages. There are so potentially $g^{+(-)}$ sites involved in the dynamic relaxation process of investment (disinvestment). The degree of spill-over effects between firms is represented by the probability that a firm receives a demand shock once one of its neighbouring firms is in an active state. Given that investment-disinvestment occurs, the shock will be transferred to each neighbouring firm with probability $\frac{g}{Z}(1-e')$ where $g = g^+$ and $Z = Z^+$ in the case of investment and $g = g^-$ and $Z = Z^-$ in the case of disinvestment and where ε` is the probability that a neighbouring firm does not receive a negative shock because of insufficient spill-over effects among firms, i.e. insufficient factor demand strength. Thus, given that a firm is in the critical disinvestment state and given that one neighbouring firm is active, the single site transition probability is given by

$$w\left(a_{-} \middle| c_{-}, s^{*}_i{}' = a_{-}, s^{*}_{j \neq i} \neq a_{-}\right) = (1 - l h) \frac{g^{-}}{Z} (1 - e')$$

The probability that a shock will not be transferred depends on the strength of the factor demand linkages between firms. In the limit where $\varepsilon' \to 1$ we have that, given that a firm is in an active state, no neighbouring firm will receive a demand shock because the factor demand linkages are negligible. In this limit each firm is self-sufficient. On the other side, in the limit of ε' → 0 we have perfect spill-over effects, that is, each of the randomly matched neighbouring firms receive a demand shock of the same size of an exogenous one. This is equivalent to say that these neighbouring firms receive the exogenous shocks accumulated at the adjusting firm. In this limit, each firm is fully specialised, and the single firms are interconnected through strong localised input-output relations. Given the multiplicity of active neighbouring sites we have that

$$\sum_{\left\{s^{*}_{NN}'\right\}} w\left(a_- \middle| c_-, s_i^{*}' = a_-, s^{*}_{j\neq i} \neq a_-\right) r_{c_-} r_{a_-} \prod_{j\neq i\in s^{*}_{NN}} r_{s_j^{*}'} = \left(g^- - e^-\right)(1 - l\,h)\, r_{c_-} r_{a_-} \left(1 - r_{a_-}\right)^{Z-1} \qquad (9)$$

where we set $g^-\varepsilon' = \varepsilon^- \in [0, g^-]$, which is the average dissipation of shocks due to insufficient spill-over effects. Substituting these expressions in (8) and neglecting higher order terms in λh and $r_{a\_}$ we get the following law of motion of the average density of firms in the active disinvestment state

$$\frac{\P}{\P t} r_{a_-} = -r_{a-} + h\,l\,r_{c_-} + \left(g^- - e^-\right) r_{c_-} r_{a_-} \qquad (10a)$$

The derivation of the evolution of the average density of firms in the active investment state is symmetric and is so given by the following

$$\frac{\P}{\P t} r_{a_+} = -r_{a+} + h(1 - l)\, r_{c_+} + \left(g^+ - e^+\right) r_{c_+} r_{a_+} \qquad (10b)$$

Using the same reasoning as above we can derive the law of motions for the average density of firms in the critical investment and disinvestment states. The average density of firms in the critical investment state decreases if the firms being actually in the critical investment state are hit by an exogenous technology shock or if they receive an endogenous demand shock due to interaction through factor demand spill-over effects. On the other side this density increases if firms which are actually in the subcritical investment state receive an exogenous or an endogenous positive shock, and so switch to the critical investment state. The evolution of the average density critical investing firms is as follows

$$\frac{\P}{\P t} r_{c_+} = -\left[h + \left(g^- - e^-\right) r_{a_-} + \left(g^+ - e^+\right) r_{a_+}\right] r_{c_+} + \left[h(1 - l) + \left(g^+ - e^+\right) r_{a_+}\right] r_{sc_+} \qquad (11a)$$

[20] See Vespignani and Zapperi (1998) for details.

where ρ_{sc_+} is the average density of sites in the subcritical investment state. In the same way, we have that the time evolution of the average density of firms in the critical disinvestment state is described by the following law of motion

$$\frac{\partial}{\partial t}\rho_{c_-} = -\left[\eta + \left(\gamma^- - \varepsilon^-\right)\rho_{a_-} + \left(\gamma^+ - \varepsilon^+\right)\rho_{a_+}\right]\rho_{c_-} + \left[\eta\lambda + \left(\gamma^- - \varepsilon^-\right)\rho_{a_-}\right]\rho_{sc_-} \qquad (11b)$$

For the cluster of firms being actually in the subcritical investment (disinvestment) state we have to take into account also the event that firm which are with a higher (lower) capital imbalance could receive a negative (positive) shock, and so switch to the subcritical investment (disinvestment) state increasing its cluster. Thus, the time evolution of the average density of firms being actually in the subcritical investment state is given by the following law of motion:

$$\frac{\partial}{\partial t}\rho_{sc_+} = -\left[\eta + \left(\gamma^- - \varepsilon^-\right)\rho_{a_-} + \left(\gamma^+ - \varepsilon^+\right)\rho_{a_+}\right]\rho_{sc_+} + \left[\eta(1-\lambda) + \left(\gamma^+ - \varepsilon^+\right)\rho_{a_+}\right]\rho_{sc_+-1} + $$
$$+\left[\eta\lambda + \left(\gamma^- - \varepsilon^-\right)\rho_{a_-}\right]\rho_{c_+} \qquad (12a)$$

In the same way we have that the time evolution of the average density of firms being actually in the subcritical disinvestment state is described by the following law of motion

$$\frac{\partial}{\partial t}\rho_{sc_-} = -\left[\eta + \left(\gamma^- - \varepsilon^-\right)\rho_{a_-} + \left(\gamma^+ - \varepsilon^+\right)\rho_{a_+}\right]\rho_{sc_-} + \left[\eta(1-\lambda) + \left(\gamma^+ - \varepsilon^+\right)\rho_{a_+}\right]\rho_{c_-} + $$
$$+\left[\eta\lambda + \left(\gamma^- - \varepsilon^-\right)\rho_{a_-}\right]\rho_{c_--1} \qquad (12b)$$

In order to simplify notation we define $\alpha = \eta(1-\lambda) + \left(\gamma^+ - \varepsilon^+\right)\rho_{a_+}$ and $\beta = \eta\lambda + \left(\gamma^- - \varepsilon^-\right)\rho_{a_-}$, being the probability, for the single firm, of receiving a positive and negative shock, either exogenous total factor productivity or endogenous demand shock, respectively. Following the arguments outlined above, we have that the time evolution of the average density of firms being actually in a generic state m, where $c_+ > m > 0$ is given by the following law of motion

$$\frac{\partial}{\partial t}\rho_m = -(\alpha + \beta)\rho_m + \alpha\rho_{m-1} + \beta\rho_{m+1} \qquad (13a)$$

The time evolution of the average density of firms being actually in a generic state m*, *where 0 > m* > c_-* is given by the following law of motion

$$\frac{\P}{\P t} r_{m^*} = -(a+b)\,r_{m^*} + b r_{m^*-1} + a r_{m^*+1} \tag{13b}$$

The time evolution of the average density of firms which are actually in state 0, that is they have an optimal level of capital, is given by the following law of motion:

$$\frac{\P}{\P t} r_0 = -(a+b)\,r_0 + a r_{-1} + b r_1 + r_{a_+} + r_{a_-} \tag{13c}$$

where we have to take into account also the fact that firms in the active investment and disinvestment state switch immediately to state 0.

The dynamical system (10) – (13) describes so completely the time evolution of the average densities. In the mathematical description above we implicitly assume an infinite time separation between the two different dynamics (Vespignani and Zapperi (1998)): the exogenous driving force, given by the exogenous total factor productivity shocks, and the endogenous demand spill-over dynamics, given by factor demand linkages. In particular, we assume that no further exogenous total factor productivity shock occurs as long as endogenous dynamics through demand linkages go on, or in other words, that investment and/or disinvestment avalanches do not overlap, or at least that the event of avalanche overlapping is of measure zero. This is not a strong assumption as long as we assume that the exogenous driving force h is very small and that the number of firms living in the economy (n) is very large.

In the following we are going to solve the model in the stationary state, that is the state where the average densities of firms being in a particular state remains constant. Further, we restrict our analysis to the case where the probability of a decrease in the total factor productivity is small compared to the probability of an increase. That is, we assume that there is a positive drift in the exogenous driving force of the economy, i.e. $\lambda \leq ½$.

For the system to exhibit stationary average densities the following conservation law must be satisfied

$$(1-a)h\left(1 - r_{a_+} - r_{a_-}\right)n = \left(e^+ r_{a_+} - e^- r_{a_-}\right)n \tag{14}$$

where $\alpha = 2\lambda$. The left-hand-side of (14) represents the net - influx of small idiosyncratic total factor productivity shocks, where we have taken into account the fact that firms which are in the active investment and disinvestment state do not receive any exogenous shock. The right-hand-side represents the dissipation of shocks because of insufficient spill-over effects. Condition (14) states that every non compensated shock must be dissipated in order to maintain stationarity of the average densities. Further, from (14) we have that the stationary average density of firms in the active investment state is higher than the stationary average density of firms in the active disinvestment state, since we have that in the stationary state there must be a higher dissipation of positive shocks than negative ones.

In order to calculate the stationary average densities of firms being in each state we have to calculate the stationary state relationship between the stationary average density of firms in the active investment and disinvestment state. The stationary average densities are related in a recursive way as we will see in the proof of Proposition 1. We are now going to state our first result:

Proposition 1:

The relation between the stationary average density of firms in the active investment and disinvestment state, for $\gamma^+, \gamma^- \geq 2$, is given by the following

$$\rho_{\alpha_+}\left[1-\left(\frac{\beta}{\alpha}\right)^{\gamma^+}\right]=\left[\left(\frac{\alpha}{\beta}\right)^{\gamma^-}-1\right]\rho_{\alpha_-} \qquad (15)$$

Proof :

See Appendix II. ■

Equation (14) together with (15) define the stationary average density of active investing and disinvesting firms. We can state our second Proposition, which solves the system for the stationary average density of firms in the active investment and disinvestment state in the limit of perfect spill-over effects:

Proposition 2

Suppose there exists a Q which defines the stationary state relationship $r_{a_-} = \Theta r_{a_+}$. *In state where* $e^{+(-)}$ *® 0 and h ® 0, we have that Q is given by the following*

a) if l < ½ then " g^-, g^+ we have that Q ® 0

b) if l = ½ then " g^-, g^+ we have that either Q ® g^+/ g^- or Q ® 0

Proof:

See Appendix III. ■

Proposition 2 states that as long as there is a strictly positive drift in the exogenous driving force, in the limit of $\varepsilon^{+(-)} \to 0$ and $h \to 0$, we have that $\Theta = \frac{r_{a_-}}{r_{a_+}}$ tends towards zero. Notice that if we tune the exogenous driving force down to zero, the system will fall, sooner or later, in one of its absorbing states (Dickman, Vespignani and Zapperi (1998)), in which the average density of firms in the active investment and disinvestment state is equal to zero[21]. Thus, in this limit, where both stationary average densities are equal to zero, the average density of firms in the active disinvestment state tends towards zero faster then the average density of firms in the active investment state. The intuition for this result is that the endogenous propagation of positive shocks is much stronger than the endogenous propagation of negative shocks. On the other side, if the probability of receiving a positive exogenous shock is the same as the probability of receiving a negative one, then we have, in the limit of $\varepsilon^{+(-)} \to 0$ and $h \to 0$, two equilibria: in the first the average density of firms in the active disinvestment state and the average density of firms in the active investment state are equal to zero, while in the second one we have that $\Theta \to g^+/ g^-$. In this latter case the behaviour of the stationary average density of firms in the active disinvestment state is proportional to the stationary average density of firms in the active investment state. In this latter case, the endogenous propagation of positive shocks is proportional to the endogenous propagation of negative shocks.

We have in this model one order parameter r_{a_+} and three control parameters: ε', h and α. h is the exogenous driving rate, which in our case, together with α, determine the exogenous path of technological progress, and ε' represents the degree of complementarity among the single firms, indicating the degree or strength of factor demand linkages among (neighbouring) firms.

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[21] This will be shown in more detail later on.

The configuration of the system depends on the past dynamics, in particular on past avalanches. Past avalanches predetermine current and future avalanches in the sense that they change the density of firms in critical and lower states. This implies that the system has a memory effect. This memory effect depends on the magnitude of the exogenous driving forces: higher h decreases the memory of the system. The transmission of shocks occurs through firms in the critical state. As we will see in the next section, their density determines the possibility of avalanches and so the propagation of shocks throughout the system. The stationary average density of firms in the critical investment state determines the degree of correlation between firms. If this density is low then the correlation of investment between firms is low and the endogenous propagation mechanism is weak. A law of large numbers applies and the aggregate investment rate converges towards zero. The economy is in this case in a subcritical state. We will see that in the case of perfect spill-over effects, the stationary average density of firms the critical investment state self-organizes towards a critical value. The investment behaviour will be so highly correlated among firms. In this case investment fluctuations do not cancel out and the aggregate investment rate fluctuates around a constant, positive value. The economy is so in a critical state, where small idiosyncratic total factor productivity shocks can drive large investment fluctuations.

In the following Section we are going to study the stationary state behaviour of the model, in particular how perturbations of the system evolve over time and space. For this purpose we are going to introduce measures of average sizes of investment and disinvestment avalanches and of the volatility of these avalanches. After this we are going to study the effect of the investment behaviour on the long run growth rate and the short run fluctuations of aggregate output.

3.2. Average Avalanche size and Volatility of Avalanches

We are going to introduce in this Section measures of avalanches dynamics in the stationary state. Since we are interested in the effects of the investment activity on the long run growth and on the volatility around this average rate we have to calculate the first two moments of the distribution of avalanches. In other words, we are looking for the average number of firms adjusting their capital stock in the case of an exogenous shock, and the average change of this average number.

The average avalanche size in the stationary state has been introduced in Vespignani and Zapperi (1998). They define $c_{h,e}(\Delta x, \Delta t)$ as the response function of the system and $c_{h,e}$ as the total susceptibility, which is given by the integral over space and time of the response function. Vespignani and Zapperi (1998) show that in the stationary state

$$c_e = \lim_{h \to 0} \frac{\partial}{\partial h} r_a \qquad (16)$$

and that the relation between average avalanche size $\langle s \rangle$ and the zero field susceptibility c_e is $\langle s \rangle = c_e$. Here the driving rate (h) of the system is tuned down to zero. This tuning down to zero corresponds to an infinite time separation between two different time scales: the timing of the exogenous technology shocks and the timing of the endogenous propagation the shocks through demand linkages. Taking the limit h $\rightarrow$ 0 corresponds so to an aggregation procedure, i.e. the switching from a local analysis to an aggregate point of view. The average avalanche size indicates the average density of firms investing or disinvesting after a small idiosyncratic technology shock has hit the system. It is a stationary state average reaction function, indicating the number of firms investing or disinvesting in the case of an exogenous technology shock. It also indicates the magnitude of propagation of the shock in the system through factor demand linkages. In our case we have two different zero field susceptibilities according to the type of shock: $\langle s \rangle^{+} = c_e^{+}$ and $\langle s \rangle^{-} = c_e^{-}$ are the zero field susceptibility and average avalanche size of investment and disinvestment respectively.

In order to study the volatility of the avalanche size we define the generalized variability function f_{he} (Δx, Δt) and the total variability of the system f_{he}, which is the integral in time and space of the generalised variability function. We can state our third result.

Proposition 3:

In the stationary state we have that the following relation between second moment of the distribution of avalanches and total variability of the system holds

(17a) $$\langle s^2 \rangle = \left| f_e \right| = \left| \lim_{h \to 0} f_{he} \right| \qquad (17a)$$

and further, we have that

$$\lim_{h \to 0} f_{h,e} = \lim_{h \to 0} \frac{\partial^2 r_a}{\partial h^2} \qquad (17b)$$

Proof:

See Appendix IV. ■

Taking the limit h → 0 corresponds also in this case to an aggregation of the single firms. $\left\langle s^2 \right\rangle$ is an index of the variability of the number of firms adjusting in the case of an exogenous technology shock. It indicates the average change in the average number of firms adjusting in the case of an exogenous total factor productivity shock.

In the next two Propositions we state results about the average number of firms adjusting their capital stock in the case of an exogenous total factor productivity shock, and the average change in this average number. In particular, we have to distinguish between two different cases: a) there is a positive drift in the exogenous driving force of the economy, i.e. λ < ½ and b) there is no drift in this exogenous driving force, i.e. λ = ½.

Proposition 4:

Consider the case where l < ½. The first and the second moment of the distribution of investment and disinvestment avalanches is given by the following expressions:

$$\langle s \rangle^+ = \frac{1-a}{e^+ - e^-\Theta'} \tag{18a}$$

$$\langle s \rangle^- = \frac{(1-a)\Theta'}{e^+ - e^-\Theta'} \tag{18b}$$

$$\left\langle s^2 \right\rangle^+ = 2\frac{1-a}{\left(e^+ - e^-\Theta'\right)^2}\left|(1-a)(1+\Theta') - e^- \lim_{h\to 0}\frac{\P}{\P h}\Theta''\right| \tag{18c}$$

$$\left\langle s^2 \right\rangle^- = 2\left|\frac{1-a}{e^+ - e^-\Theta'}\lim_{h\to 0}\frac{\P}{\P h}\Theta'' - \frac{1-a}{\left(e^+ - e^-\Theta'\right)^2}\Theta'\left[(1-a)(1+\Theta') - e^- \lim_{h\to 0}\frac{\P}{\P h}\Theta''\right]\right| \tag{18d}$$

Further, we have that, in the limit of perfect spill-over effects, $e^{+(-)}$ ® 0 and h ® 0 the economy converges towards a critical state: the average number of firm investing in the case of a positive exogenous shock diverges towards infinity, while the average number of firms disinvesting capital in the case of a negative shock converges towards zero; further, the second moment of the distribution of investment avalanches diverges towards infinity, while the second moment of the distribution of disinvestment avalanches converges towards a finite value if g^- = 2 and towards zero if g^- > 2.

Proof:

A sketch of the proof will be given in Appendix V. ■

These results show that as long as the spill-over effects are less than perfect, there exists a finite average avalanche size of firms investing and disinvesting. Even though there are infinite many firms in the economy, only a finite average number of them will invest. Further, since a characteristic scale exists, a law of large numbers applies. Thus, as we will see more formally in the next section, we have that as long as the spill-over effects are less than perfect, the investment rate will be negligible since the number of firms adjusting are negligible compared to the number of firms working in the economy, and fluctuations average out in the process of aggregation since a law of large number applies.

On the other side in the case of perfect spill-over effects and in the case of a positive drift in the exogenous driving force, the economy converges towards a critical state where no finite first and second moment of the distribution of investment avalanches exists. Thus, the average number of firms investing in the case of an exogenous shock in no longer negligible compared to the total number of firms existing in the economy even though we assume that there are infinite many firms. There will be a finite aggregate investment rate as long as the first moment diverges toward infinity at the same rate as the number of firms does. Further, no characteristic scale exists and so fluctuations do not average out. Thus we will observe large fluctuations in the aggregate investment rate. This state is a critical state, since the zero field susceptibility diverges and small idiosyncratic shocks can have long ranged effects. In Appendix VI we show that in the case of perfect spill-over effects the economy converges naturally towards this critical state. In other words this state is asymptotically stable. In Appendix VI we show also that, in the critical state, the distribution of the stationary average densities is given by

$$r_{a_+} = 0 \quad r_{c_+} = \frac{1}{g^+} \quad \ldots \quad r_0 = \frac{1}{g^+} \quad r_{1_-} = 0 \quad \ldots \quad r_{a_-} = 0 \tag{19}$$

As we can see from (19) the stationary average density of firms in the critical investment state is $1/g^+$. Thus, with probability $1/g^+$ an exogenous total factor productivity shock starts an investment avalanche. Further, since in the case of investment, the firm increases demand of g^+ randomly matched neighbouring firms, and each of these firm is in a critical investment state with probability $1/g^+$, on average, the shock will be propagated further on, leading to a long-ranged investment avalanche. Investment behaviour will be so highly correlated among the single firms.

We have seen in Proposition 2 that if there is no drift in the exogenous driving force, i.e. $\lambda = ½$, then there are two possible equilibria: $\Theta = 0$ and $\Theta = g^+/g^-$. Consider first the case where $\Theta = 0$. We can see from conservation law (14) that the stationary average density of firms in the active investment and disinvestment state is equal to zero. Thus, in this case we have that $r_{a_+} = r_{a_-} = 0$.

Consider now the equilibrium where $\Theta'' \xrightarrow[h \to 0]{} \Theta' \xrightarrow[e^{+(-)} \to 0]{} \Theta = \frac{g^+}{g^-}$. By definition, we have that $e^+ = e^- \frac{g^+}{g^-}$. In our following Proposition we state that this equilibrium is not a critical state.

Proposition 5

Consider the case where the probability of receiving a positive exogenous shock is the same as receiving a negative one, i.e. l = *½, and further where* $\Theta'' \xrightarrow[h \to 0]{} \Theta' \xrightarrow[e^{+(-)} \to 0]{} \Theta = \frac{g^+}{g^-}$ *(Proposition 2b). This equilibrium is not a critical state. In particular, a finite first moment of the distribution of avalanches exists.*

Proof:

See Appendix VII. ■

As long as there is no drift in the exogenous driving force of the economy, we have that the average number of firms investing or disinvesting in the case of an exogenous total factor productivity shock is given by a finite value. In other words, the zero field susceptibility does not diverge towards infinity. Thus, as the number of firms existing in the economy diverges towards infinity, the investment-disinvestment activity has a negligible aggregate effect. This shows that a critical state exists only if there is a positive drift[22] in the exogenous driving force. In the next Section we are going to study the average aggregate growth rate and aggregate fluctuations in this economy assuming that there is a positive drift in the exogenous driving force, i.e. $\lambda < ½$.

[22] Symmetrically, there exists also a critical state if there is a negative drift in the exogenous driving force (Andergassen (2000b))

4.1. Average Aggregate Growth Rate

In this Section we are going to study the behaviour of the average aggregate growth rate, focusing on the long run growth rate and the variance of the growth rate of our economy if there is a positive drift in the exogenous driving force of the economy, i.e. $\lambda < ½$.

There are two different dynamics: the exogenous growth of the total factor productivity and an endogenous multiplier through factor demand spill-over effects. This latter one is an endogenous multiplier of the exogenous shocks. Intuitively, as long as the number of firms adjusting is small compared to the number of firms in the economy, the effect of the endogenous propagation mechanism on the aggregate growth rate should be negligible, while in the case of large investment avalanches we have a non-negligible effect. That is, as long as we are in the subcritical state ($\varepsilon' > 0$) the average aggregate growth rate should be unaffected by the avalanche dynamics of investment. In the critical state ($\varepsilon' \to 0$) we have that the average size of the investment avalanches is non-negligible compared to the total number of firms existing in the economy, even though we consider the case where there are infinite many firms in the economy, i.e. $n \to \infty$. In this latter case the investment activity will have a non-negligible aggregate effect.

The aggregate growth rate of the economy is given by the average growth rate of the single firms in the economy. As we have seen in section 2.1 each firm produces according to the production function $Y_t^i = A_t^i \left(K_t^i\right)^b$. We have that with probability $(1 - \lambda)h$ (λh) total factor productivity increases (decreases) while with probability $1 - h$ the total factor productivity remains constant. This process is at the limit representable by a geometric Brownian motion[23]

$$dA_t^i = cA_t^i dt + s' A_t^i dv_t^i \qquad (20)$$

where $dv_t^i = e_t^i * \sqrt{dt}$, where $e_t^i *$ are independently identically normally distributed across firms and time with mean zero and unitary variance. Taking total differential of the production function we have that

$$\frac{dY_t^i}{Y_t^i} = \frac{dA_t^i}{A_t^i} + b\,\frac{dK_t^i}{K_t^i} \qquad (21)$$

[23]Notice that A is defined as $A = e^a$. In Appendix I we show that a the stochastic environment each firm is facing is representable by a an absolute Brownian motion of the form $da = \mu'' dt + \sigma d\varpi$. Using Ito's Lemma we have that A follows a geometric Brownian motion like (20) where $\chi = \mu'' + ½\,\sigma^2$.

Averaging over all firms in our economy we obtain

$$\frac{1}{n}\sum_{i=1}^{n}\frac{dY_t^i}{Y_t^i}=\frac{1}{n}\sum_{i=1}^{n}\frac{dA_t^i}{A_t^i}+b\,\frac{1}{n}\sum_{i=1}^{n}\frac{dK_t^i}{K_t^i} \tag{22}$$

Next we are going to discuss in turn the single components of (22) for large values of n.

Since e_t^i* are identically independently normally distributed across firms and time we have that for large n, $\frac{1}{n}\sum_{i=1}^{n}\frac{dA_t^i}{A_t^i}=cdt$. The dynamic capital accumulation constraint for each firm is given by $dK_t^i=I_t^i dt$. We know that the amount of investment and disinvestment is $\frac{I}{K*}=L'\left(e^{g^+}-1\right)$ if $s_t^i=L'$ and $\frac{I}{K*}=U'\left(1-e^{g^-}\right)$ if $s_t^i=U'$ respectively. The average aggregate change in capital can be written as follows $\frac{1}{n}\sum_{i=1}^{n}\frac{dK_t^i}{K_t^i}=\frac{1}{n}\sum_{i=1}^{n}\frac{dK_t^i}{K_t^{i*}}\frac{1}{s_t^i}$ = $\frac{1}{n}\sum_{i=1}^{n}\left[1\left(s_t^i=L'\right)\left(e^{g^+}-1\right)L'-1\left(s_t^i=O'\right)\left(e^{g^-}-1\right)U'\right]\frac{1}{s_t^i}dt$. This last term is a random variable. The average aggregate growth rate of output[24] $g_y=\frac{1}{n}\sum_{i=1}^{n}\frac{dY_t^i}{dt}\frac{1}{Y_t^i}$ will be so

$$g_y=c+b\,\frac{1}{n}\sum_{i=1}^{n}\left[1\left(s_t^i=L'\right)\left(e^{g^+}-1\right)L'-1\left(s_t^i=O'\right)\left(e^{g^-}-1\right)U'\right]\frac{1}{s_t^i} \tag{23}$$

We have calculated in the previous section the first two moments of the distribution of the avalanche size, and so we are able to calculate the expected value, i.e. the long run value defined as $E\left[g_y\right]$, and the variance of the average aggregate growth rate defined as $Var\left(g_y\right)=E\left[\left(\frac{1}{n}\sum_{i=1}^{n}\frac{dY_t^i}{Y_t^i}\frac{1}{dt}\right)^2\right]-E\left[\frac{1}{n}\sum_{i=1}^{n}\frac{dY_t^i}{Y_t^i}\frac{1}{dt}\right]^2$. We can state so the following proposition about the long run growth rate of aggregate output and its short run variance.

[24] Since we assume that the exogenous shocks hit each firm with the same probability and further that interaction occurs in a random way, we have that each firm will produce, in the long run, an equal share of total output.

Proposition 6:

Consider an economy with infinite many firms, i.e. n ® ¥. The long run growth rate and the short run variance of this economy depends on the strength of spill-over effects. In particular, we can distinguish two cases:

a) *ne ® ¥*

$$E\left[g_y\right] = c$$

$$Var\left(g_y\right) = 0$$

b) *ne ® 1*

$$E\left[g_y\right] = c + b\,\frac{1-a}{g^{+}}\left(e^{g^{+}} - 1\right)$$

$$Var\left(g_y\right) = b^{2}\left[\frac{1-a}{g^{+}}\left(e^{g^{+}} - 1\right)\right]^{2}$$

Proof:

See Appendix VIII. ■

Proposition 6 implies that the limiting behaviour of the long run growth rate and it short run fluctuations depend on the limit of n*e.* In case a) that is in the case of less than perfect spill-over effects, i.e. *e* > 0, there are still insufficient factor demand linkages. As we have seen in Section 3.2. there exists a finite average investment and disinvestment avalanche size and also a finite second moment of the distribution of investment and disinvestment avalanches. Thus, as the number of firms in the economy increases, the investment rate will be negligible. Further, since a characteristic scale exists, a law of large numbers applies and so there will be no aggregate fluctuations. The aggregate growth rate depends so only on the exogenously given average growth path of total factor productivity, while there will be no short run volatility of the aggregate output.

In the case of n*e* → 1, which is the case of perfect spill-over effects, we have that the long run growth rate of aggregate output and its short run fluctuations depends also on the optimal size of investment. Through factor demand linkages we have an endogenous propagation mechanism which, in the case of perfect spill-over effects, and in particular if n*e* → 1, affects in a non-negligible way the long run average aggregate growth rate of output and its short run volatility. In this limit the size of the average investment avalanches increases at the same rate as the number of firms does. Thus, there exists a finite long run aggregate investment rate which leads to a higher long run growth rate of aggregate output. Further, this aggregate investment rate will be highly volatile, leading to large fluctuations in the aggregate output.

4.2. Discussion

In Section 4.1 we have calculated the long run growth rate and the variance of the average aggregate growth rate of output of this economy. We have seen that if the rate of convergence towards perfect spill-over effects is 1/n, then small idiosyncratic total factor productivity shocks lead to a highly volatile, positive aggregate investment rate which increases the long run growth rate of aggregate output and further lead to large short run fluctuations in the same. The intuition for this particular limit is the following. As the number of firms existing in the economy increases also the degree of specialisation of each firm or plant increases, and so also the degree of interdependence between the single firms or plants increases. Thus, in growing economies, where the specialisation of each firm/plant is increasing and so also the number of firms/plants is increasing the strength of the spill-over effects is increasing, i.e. $e \xrightarrow[n\to\infty]{} 0$, because of the increasing interdependence of each part of the system. In other words, we have that e = 1/n, that is the strength of the spill-over effects depends on the number of firms existing in the economy. In highly developed countries there will be perfect spill-over effects. These countries will so grow along a critical state.

In the case of perfect spill-over effects, i.e. result b of Proposition 6, we have that the long run growth rate of aggregate output of the economy and its short run fluctuations are a function of the size of the optimal investment and so also a function of the size of sunk costs of investment and of the importance of indivisibilities. We are able to make some comparative statics, changing the size of g^+ while holding the average growth rate of total factor productivity χ constant. From Proposition 6 b), we have that higher g^+ yields a higher long run growth rate and also a higher short run variance. Thus, given the exogenous growth path of total factor productivity, the long run growth rate and its short run variance are non-decreasing functions of sunk costs of investment. Higher sunk costs lead to a larger aggregate investment rate but also to a higher volatility of the same. In particular we can distinguish two different effects as g^+ increases: a) an increase in g^+ implies by definition that, given that we are in the target point, it needs more positive shocks in order to reach the trigger point. At the same time, in the case of investment, the shocks will be transferred to more firms. As a result, the endogenous diffusion of the shocks decreases by a power law with coefficient – 1. b) as g^+ increases, the amount of capital invested by each firm increases in an exponential way. We can summarise these results as follows: given a certain exogenous growth of total factor productivity, and in the case of perfect spill-over effects, higher sunk costs of investment or disinvestment lead to a higher long run growth rate of the economy and a higher variance of the average aggregate growth rate. In other words, the economy grows at a higher rate but in the short run there are larger fluctuations. The model predicts a trade off between a higher long run growth rate but larger short run fluctuations, or lower long run growth rate but also smaller short run fluctuations. This trade off is not a positive one in the sense that an increase in the sunk costs does not necessarily increase the average aggregate growth rate of the economy. In the comparative statics outlined above, we assumed that the exogenous average total factor productivity growth rate was fixed, while an increase in the sunk costs could decrease the average aggregate path of technology growth[25].

From Proposition 6 we can see that there exists a simple power law between the long run growth rate of the economy and the its short run volatility. The model predicts the following power law:

$$\log\left(\left|E\left[g_y\right]-c\right|\right)=\frac{1}{2}\log\left[Var\left(g_y\right)\right] \tag{24}$$

(24) states a simple relationship between the logarithm of the long run growth rate of the economy, net of the average growth of total factor productivity, and the logarithm of the variance of the growth rate. This relation is somehow biased by the fact that labour is a fixed factor[26]. Proposition 4 together with Proposition 5 imply, in the case of perfect spill-over effects, large aggregate fluctuations are obtained only if there is a positive drift in the exogenous driving force of the economy, i.e. if there is a positive exogenous growth rate due to a positive technological progress. Further, through large aggregate fluctuations in investment the economy will grow at a higher rate. On the other side, in the case of perfect spill-over effects but if there is no drift in the exogenous driving force, i.e. in the absence of technological progress, there will not be large fluctuations in aggregate investment. In this case the growth rate of the economy will be zero.

The model predicts also that the variance of the aggregate investment rate is higher than the variance of the growth rate of aggregate output. In other words, the model predicts that aggregate investment is more volatile than aggregate output. In particular, we have that Var (γ_y) = β^2 Var(I/K). This result is somehow biased by the fact that we assumed labour to be fixed. On the other side, if labour was fully flexible, the variance of the growth rate of output would be the same as the variance of investment since every fluctuation in capital would lead to fluctuations in the employment level. Thus,

[25] See Gilles Staint-Paul (1997) for an example related to sunk costs in the labour turnover and the rate of innovation.
[26] In Andergassen (2000b) we assume that labour, as well as capital, are quasi fixed factors.

assuming that also labour is quasi-fixed, we have that this relationship depends on the relative importance of the adjustment costs[27].

5. Conclusions

We studied a model where firms adjust their capital stock in an intermittent and discrete way. We argued that in the case of adjustment the firm increases demand of randomly matched neighbouring firms. These spill-over effects are because of factor demand linkages. So we have an endogenous propagation mechanism and a multiplier of small idiosyncratic exogenous shocks. The investment behaviour will so be correlated among the single firms. We saw that in the case of perfect spill-over effects this correlation is so high, leading to a highly volatile, positive, aggregate investment rate even though the economy is driven by only small idiosyncratic total factor productivity shocks. We argued that this is the case of highly developed countries, where each firm is highly specialised and so the single firms are highly interconnected. The aggregate investment rate has in this case a non-negligible effect on the long run growth rate of the economy and it leads to large short run fluctuations in the same. We saw that if there is a positive drift in the exogenous driving force of the economy, i.e. $\lambda < ½$, and given an exogenous average growth path of total factor productivity, greater discontinuities lead to a higher long run growth rate of aggregate output but also greater short run volatility of the same, while smaller discontinuities lead to a smaller long run growth rate of aggregate output but also to a smaller short run volatility of the same. Higher long run growth is reached through larger fluctuations in the investment rate. From this does not follow the policy advise to increase sunk cost of capital turn-over since we have not taken into account the effects of a change in the sunk costs on the average growth path of total factor productivity. Intuitively an increase in the discontinuities could also decrease the average growth path of total factor productivity. The effect of an increase in the discontinuities on the long run growth rate is so a priori ambiguous, while the short run volatility increases unambiguously.

We have seen that the model predicts a simple relationship between the logarithm of the long run growth rate, net of the average growth rate of total factor productivity, and the logarithm of the variance of the growth rate of the economy. Further, the model predicts that the variance of the investment rate is much higher than the variance of the growth rate of aggregate output. These two results are somehow biased by the assumption that labour was a fixed factor[28].

[27] See the previous footnote.

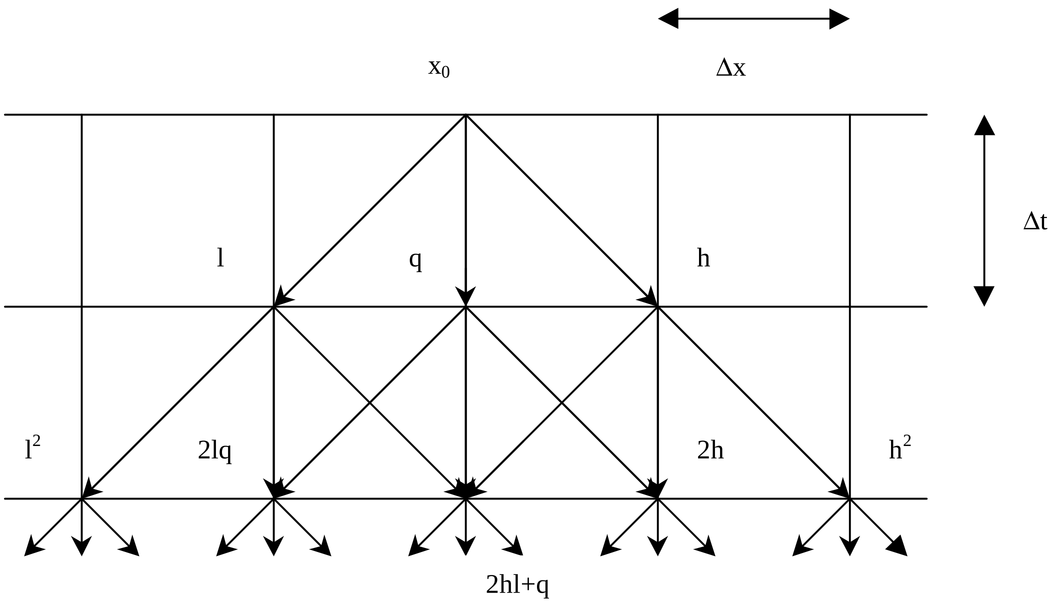
Appendix I

In this section we study the stochastic environment of the representative firm. The aim is to show that at the limit it is representable by a Brownian motion. This fact, together with the problem of the firm outlined in the previous section gives the optimality of the (S,s) policy. We follow Dixit (1993) and enlarge his analysis taking into account that with a positive probability a firm receives no shock at all in each period of time. The stochastic environment can be described as follows.

We assume that in each period of time a firm can either be hit by a positive, negative or no shock at all. The shock can either be an exogenous total factor productivity shock or an endogenous demand shock.

We start describing the distribution of the changes in x. The probability of receiving a positive shock, that is $\Delta x > 0$ is given by h, the probability of receiving a negative shock, that is $\Delta x < 0$ is given by l, and so the probability of receiving no shock, that is $\Delta x = 0$, is given by $q = 1 - h - l$. We consider a time interval Δt. This problem is similar to the problem of the position of a particle which is immersed in a liquid and can move with probability h to the right, with probability l to the left and with probability q remains in the same position. The probability of change in the relative position or cumulative change after a given time interval Δt, starting from position x_0 is described in the following figure (Figure AI).

Figure AI



[28] See footnote 29.

We can calculate the average change in position of a particle given an arbitrary initial position. If x indicates the position of the particle, than the average change in position is given by $E[\Delta x]$ = hΔx + l$(-\Delta x)$ = (h - l) Δx. In order to calculate the variance we have to calculate the second moment that is $E\left[(\Delta x)^2\right]$ which is equal to h$(\Delta x)^2$ + l$(-\Delta x)^2$ = (h + l) $(\Delta x)^2$. The variance of the change in x will be in this case $VAR[\Delta x]$ = $E\left[(\Delta x)^2\right]$- $E[\Delta x]^2$ = (h + l)$(\Delta x)^2$ - (h - l)$^2\Delta$x = [(1 - q)q + 4hl]$(\Delta x)^2$. In order to simplify notation we call 1 - q = q' and set Δx = $\frac{s}{\sqrt{q'}}\sqrt{\Delta t}$. Using this notation we have that $(\Delta x)^2 = \frac{s^2}{q'}\Delta t$. A time interval of length t has n = t/(Δt) steps, so we can calculate the mean of the cumulative change in x after n steps which is n(h - l) Δx = t(h - l) $\frac{\Delta x}{\Delta t}$ and the variance after n steps will be n[(1 - q)q + 4hl]$(\Delta x)^2$ = t(q'q + 4hl)$\frac{(\Delta x)^2}{\Delta t}$. Define now h = $\frac{1}{2}\left(q' + \frac{m\sqrt{q'}}{s}\sqrt{\Delta t}\right)$ and l = $\frac{1}{2}\left(q' - \frac{m\sqrt{q'}}{s}\sqrt{\Delta t}\right)$. Using these definitions we have that h + l = q', as required. Further we have that h - l = $\frac{m\sqrt{q'}}{s}\sqrt{\Delta t}$. Evaluating this last expression at the limit Δt $\rightarrow$ 0, we have that t(h - l) $\frac{\Delta x}{\Delta t}$ = tμ. The cumulative average change in position after n steps is tμ. Now lets calculate the variance of the cumulative change in position after n steps. Using the above definitions we can write the variance as follows n$\left[q'q + \left(q'^2 - \frac{m^2}{s^2}q'\Delta t\right)\right](\Delta x)^2$. Since q = 1 - q' we can simplify this expression and write the variance as $t\left(q' - \frac{m^2}{s^2}q'\Delta t\right)\frac{1}{\Delta t}(\Delta x)^2$, which is equal to $t\left(q' - \frac{m^2 q'}{s^2}\Delta t\right)\frac{s^2}{q'}\frac{\Delta t}{\Delta t}$. Evaluating this expression at the limit Δt $\rightarrow$ 0 we have that the variance of the change in position is equal to tσ^2.

The cumulative process for Δt $\rightarrow$ 0, that is at the limit in which the number of steps goes to infinity for a fixed time interval of length t, has the characteristics of a Brownian motion of the form

$$dx_t = mdt + s\sqrt{q'}dv_t \quad \text{(AI)}$$

where $dv_t = \frac{e_t}{\sqrt{q'}}\sqrt{dt}$ and e_t are normally distributed increments with mean zero and unitary variance. From (AI) we can calculate the average change and variance of the change in position which is given respectively by $E[\Delta x]$ = μdt and $VAR[\Delta x]$ = σ^2dt. This implies that the stochastic process the firms are facing is in the limit of $\Delta t \rightarrow 0$ representable by a Brownian motion. As pointed out in a previous footnote, ϑ ,defined as $\vartheta = e^x\xi$, where x follows a Brownian motion (AI), follows a geometric Brownian motion (2), with mean and variance appropriately redefined.

Appendix II: Proof of Proposition 1

The stationary average densities for *m* > 0 are given by the following system

$$r_{c_+} = \frac{1}{a} r_{a_+}$$

$$r_{sc_+} = \frac{a+b}{a} r_{c_+}$$

$$r_{sc_+-1} = \frac{a+b}{a} r_{sc_+} - \frac{b}{a} r_{c_+}$$

$$r_{sc_+-2} = \frac{a+b}{a} r_{sc_+-1} - \frac{b}{a} r_{sc_+}$$

...

$$r_0 = \frac{a+b}{a} r_{1_+} - \frac{b}{a} r_{2_+}$$

For $g^+ = 2$ we have that $r_0 = r_{sc_+}$ while for $g^+ = 2 + i$ we have that $r_0 = r_{sc_+ + i}$. The stationary average densities are related in a recursive way: $x_{t-2} = \frac{a+b}{a} x_{t-1} - \frac{b}{a} x_t$. This latter is a second order linear difference equation with roots 1 and a/b and so general solution $x_t = c_1 + \left(\frac{a}{b}\right)^t c_2$. In order to find the value of the two constants we impose the conditions that $x_0 = r_{sc_+} = \frac{a+b}{a^2} r_{a_+}$ and that

$x_{-1} = \frac{(a+b)^2 - ab}{a^3} r_{a_+}$. Solving this linear system and substituting the result into the general solution we find the following result:

$$r_0 = r_{sc_+ - g^+ + 2} = \frac{1}{a-b}\left[1 - \left(\frac{b}{a}\right)^{g^+}\right] r_{a_+} \qquad \text{(AII1)}$$

which is the relation between stationary average density of firms in state 0 and firms in active investment state a function of g^+. Since we assumed that $\lambda \leq ½$, we have that $a \geq b$. The same applies for $m < 0$ and applying the same reasoning to this part of the system we get the follow

$$r_0 = r_{sc_- - g^- + 2} = \frac{1}{a-b}\left[\left(\frac{a}{b}\right)^{g^-} - 1\right] r_{a_-} . \qquad \text{(AII2)}$$

Putting (AII1) and (AII2) together we get (15) stated in Proposition 1.

Appendix III: Proof of Proposition 2

Consider the case where h, $\varepsilon^{+(-)} > 0$, and so we have that $r_{a_-} = \Theta'' r_{a_+}$. Using (14) we can rewrite a and b as follows

$$a = \frac{g^+ h(1-a) + h^2(1-a)(1+\Theta'') + ahe^+ - he^-\Theta''}{e^+ - e^-\Theta'' + h(1-a)(1+\Theta'')} \qquad \text{(AIII1a)}$$

$$b = \frac{g^- h(1-a)\Theta'' + h^2(1-a)(1+\Theta'') + ahe^+ - he^-\Theta''}{e^+ - e^-\Theta'' + (1-a)h(1+\Theta'')} \qquad \text{(AIII1b)}$$

Inserting (AIII1) into (14) we obtain the following relationship which defines implicitly Θ’

$$\frac{\left[g^+(1-a) + h(1-a)(1+\Theta'') + ae^+ - e^-\Theta''\right]^{g^+} - \left[g^-(1-a)\Theta'' + h(1-a)(1+\Theta'') + ae^+ - e^-\Theta''\right]^{g^+}}{\left[g^+(1-a) + h(1-a)(1+\Theta'') + ae^+ - e^-\Theta''\right]^{g^-} - \left[g^-(1-a)\Theta'' + h(1-a)(1+\Theta'') + ae^+ - e^-\Theta''\right]^{g^-}} =$$

$$= \frac{\left[g^{+}(1-a)+h(1-a)(1+\Theta'')+ae^{+}-e^{-}\Theta''\right]^{g^{+}}}{\left[g^{-}(1-a)\Theta''+h(1-a)(1+\Theta'')+ae^{+}-e^{-}\Theta''\right]^{g^{-}}}\Theta'' \qquad \text{(AIII2)}$$

We define $\gamma = g^{-}/g^{+}$, and evaluating (AIII2) in the limit of $\varepsilon^{+(-)} \to 0$ and $h \to 0$, we obtain the following relationship which defines implicitly Θ

$$\frac{1-(g\Theta)^{g^{+}}}{1-(g\Theta)^{gg^{+}}}(g\Theta)^{gg^{+}} = \Theta \qquad \text{(AIII3)}$$

As long as $\alpha \neq 1$ we have that (AIII3) has a unique solution, given by $\Theta = 0$. On the other side, if $\alpha = 1$, then we have that (AIII3) has two solutions, $\Theta_1 = 0$ and $\Theta_2 = 1/\gamma$, where $1/\gamma < 1$, i.e. $g^{+} < g^{-}$. By definition we have that $e^{+} = e^{-}\frac{g^{+}}{g^{-}}$ or equivalently $e^{-} = e^{+}g$. Thus, we have that in the case of solution Θ_2 the conservation law (14) is satisfied only if $\lambda = ½$, i.e. $\alpha = 1$. Consider now the case where $\alpha \neq 1$ and $\frac{1}{\Theta} r_{a_-} = r_{a_+}$. We obtain in this case equation

$$\frac{1-\left(\frac{g}{\Theta}\right)^{g^{+}}}{1-\left(\frac{g}{\Theta}\right)^{gg^{+}}}\left(\frac{g}{\Theta}\right)^{gg^{+}} = \frac{1}{\Theta} \qquad \text{(AIII4)}$$

which has two solutions: $\Theta_1 = 0$ and $\Theta_2 = \gamma$, where $\gamma < 1$, i.e. $g^{+} > g^{-}$.

Appendix IV[29]: Proof of Proposition 3

We define the generalised variability function as $f_{h,e}(x - x', t - t')$ and we define

$$\Delta^2 r_a(x,t) = \int\int f_{h,e}(x-x',t-t')(\Delta h(x',t'))^2 d^d x' dt' + o\left((\Delta h)^2\right) \qquad \text{(AIV1)}$$

[29] We follow the steps of the proof in Vespignani and Zapperi (1998) for the derivation of the average

If we set the perturbation $[\Delta h(x', t')]^2 = \delta(t)\delta^d(x)$ and so solving the integral over time and space we have that $\Delta^2 r_a(x,t) = f_{h,e}(x,t)$. We define the total variability of the system as

$$f_{h,e} = \int\int f_{h,e}(x,t) d^d x dt$$

The variation of the average number of firms adjusting in the case of an exogenous technology shock is given by

$$\Delta N_a = \int\int \Delta^2 r_a(x,t) d^d x dt = f_{h,e}$$

In the absence of external driving force, that is in the limit of $h \to 0$, we have that the change in the number of firms adjusting is $|\Delta N_a| = \langle s^2 \rangle$. We have also that the following relationship is satisfied

$$|f_e| = \left|\lim_{h\to 0} f_{h,e}\right| = \langle s^2 \rangle .$$

Now we assume that the perturbation applied to the field is given by $[\Delta h(x',t')]^2 = (\Delta h)^2$ and further we define for simplicity t'' = t - t' and x'' = x - x'. We can rewrite (AIV1) as follows

$$\Delta^2 r_a = (\Delta h)^2 \int\int f_{h,e}(x'',t'') d^d x'' dt'' = (\Delta h)^2 f_{h,e}$$

We have so that $f_{h,e} = \lim_{\Delta h \to 0} \frac{\Delta^2 r_a}{(\Delta h)^2} = \frac{\partial^2 r_a}{\partial h^2}$. Using the above relationships we obtain the following

$$\langle s^2 \rangle = |f_e| = \left| \lim_{h\to 0} \frac{\partial^2 r_a(h)}{\partial h^2} \right|$$

which is relation stated in Proposition 2.

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avalanche size.

Appendix V: Proof of Proposition 4

In order to be able to calculate the first and the second moment of the distribution of avalanches we need the stationary average density of firms in the active investment and disinvestment state. Form (14), (15) and Proposition 2 we have that if λ < ½, then the limiting behaviour, of approaching the critical state, of the stationary average densities of firms in the active and critical investment and disinvestment state is described by

$$r_{a_+} = \frac{h(1-a)}{e^+ + h(1-a) - \left[e^- - h(1-a)\right]\Theta''} \qquad \text{(AV1a)}$$

$$r_{c_+} = \frac{1}{\dfrac{ae^+ + h(1-a) - \left[e^- - h(1-a)\right]\Theta''}{1-a} + g^+}$$

$$r_{a_-} = \frac{h(1-a)\Theta''}{e^+ + h(1-a) - \left[e^- - h(1-a)\right]\Theta''} \qquad \text{(AV1b)}$$

$$r_{c_-} = \frac{1}{\dfrac{ae^+ + h(1-a) - \left[e^- - h(1-a)\right]\Theta''}{(1-a)\Theta''} + g^-}$$

where we have seen, from Proposition 2, that, in the critical state, $\Theta'' \xrightarrow[h \;\to\; 0]{} \Theta' \xrightarrow[e^{+(-)} \;\to\; 0]{} \Theta = 0$. As we have already pointed previously, in the absence of an exogenous driving force, i.e. h → 0, we have that the system falls in one of its absorbing state, where the stationary average density of firms in the active investment and disinvestment state is equal to zero. Using (16), (17) and (AV1) we can calculate the average avalanche size and the second moment of the distribution of avalanches in the case of λ < ½.

In order to evaluate the limiting behaviour of Θ' in the case of perfect spill-over effects we take a first order approximation of Θ" around its limit 0. The limiting behaviour of Θ" is approximately described by $\Theta'' = \dfrac{1 - \left[\dfrac{ae^+ + (1-a)h}{ae^+ + \left(h+g^+\right)(1-a)}\right]^{g^+}}{\left[\dfrac{ae^+ + \left(h+g^+\right)(1-a)}{ae^+ + (1-a)h}\right]^{g^-} - 1}$ < 1. If we tune the exogenous driving force down to zero, i.e. h $\rightarrow$ 0, we have that $\Theta' = \dfrac{1-\left[\dfrac{ae^{+}}{ae^{+}+g^{+}(1-a)}\right]^{g^{+}}}{\left[\dfrac{ae^{+}+g^{+}(1-a)}{ae^{+}}\right]^{g^{-}}-1}$. Thus, from this last expression we see that in the limit of perfect spill-over effects we have that $\Theta \rightarrow 0$ which is a result which has already been stated in Proposition 2. Thus, we have for every $g^{+} \geq 2$ and $g^{-} \geq 2$, the average avalanche size of firms investing tends to infinity while the average avalanche size of firms disinvesting tends to zero. Thus, we have that the average investment avalanche size diverges toward infinity, while the average disinvestment avalanche size converges towards zero.

We turn now to study the behaviour of the second moment of the distribution of avalanches. First, we have to show that the limit $\lim_{h\rightarrow 0}\dfrac{\P}{\P h}\Theta''$ is a finite value. We have that $\Theta'' = \dfrac{1-\left[\dfrac{ae^{+}+(1-a)h}{ae^{+}+\left(h+g^{+}\right)(1-a)}\right]^{g^{+}}}{\left[\dfrac{ae^{+}+\left(h+g^{+}\right)(1-a)}{ae^{+}+(1-a)h}\right]^{g^{-}}-1}$. Taking the first derivative with respect to the driving rate h and afterwards evaluating the derivative in the limit of h $\rightarrow$ 0 we obtain

$$\frac{\P\Theta''}{\P h}\xrightarrow[h\rightarrow 0]{}-g^{+}\frac{\left[\dfrac{ae^{+}}{ae^{+}+g^{+}(1-a)}\right]^{g^{+}-1}}{\left[\dfrac{ae^{+}+g^{+}(1-a)}{ae^{+}}\right]^{g^{-}}-1}\frac{(1-a)^{2}g^{+}}{\left[ae^{+}+(1-a)g^{+}\right]^{2}}+$$

$$+g^{-}\frac{1-\left[\dfrac{ae^{+}}{ae^{+}+g^{+}(1-a)}\right]^{g^{+}}}{\left\{\left[\dfrac{ae^{+}+g^{+}(1-a)}{ae^{+}}\right]^{g^{-}}-1\right\}^{2}}\left[\frac{ae^{+}+g^{+}(1-a)}{ae^{+}}\right]^{g^{-}-1}\frac{(1-a)^{2}g^{2}}{\left(ae^{+}\right)^{2}} \quad \text{(AV2)}$$

which is a finite value.

We start with the study of the second moment of the distribution of investment avalanches (18c). Taking into account (AV2) we see that the first term in the absolute value of (18c) divided by $1/(\varepsilon^{+})^{2}$, evaluated at the limit of $\varepsilon^{+} \rightarrow 0$, is either zero or a finite value, depending on whether $g^{-} = 2$ or $g^{-} > 2$. The second term in the absolute value of (18c), divided by $1/(\varepsilon^{+})^{2}$ and evaluated at the limit of ε^{+}

$\rightarrow$ 0 diverges always towards infinity. So the second moment of the distribution of investment avalanches diverges always towards infinity, with the power $1/(\varepsilon^+)^2$, where $\varepsilon^+ \rightarrow 0$.

We study now the limiting behaviour of the second moment of the distribution of the disinvestment avalanches (18d). The third term in the absolute value of (18d), evaluated in the limit of $\varepsilon^- \rightarrow 0$ is equal to zero. The second term in the absolute value of (18d) in the case of $g^- = 2$ converges in the limit of $\varepsilon^+ \rightarrow 0$ towards $-\frac{a^2}{(g^+)^2}$, while in the case of $g^- > 2$ this term converges, in the limit of $\varepsilon^+ \rightarrow 0$, towards zero. The first term in the absolute value of (18d) in the case of $g^- = 2$ converges in the limit of $\varepsilon^+ \rightarrow 0$ towards $2\frac{a}{(g^+)^2}$, while in the case of $g^- > 2$ this term converges, in the limit of $\varepsilon^+ \rightarrow 0$, towards zero. Putting these results together we obtain the results stated in Proposition 4.

Appendix VI: Distribution of the stationary average densities in the critical state and stability

We are going to derive the stationary state distribution of the average densities in the case of perfect spill-over effects. Further, we are going to show that this state is asymptotically stable. In Appendix II we have described the stationary state of the model. This is given by

$$r_{c_+} = \frac{1}{a} r_{a_+} \qquad \text{(AVI1a)}$$

$$r_{sc_+} = \frac{a+b}{a} r_{c_+} \qquad \text{(AVI1b)}$$

$$r_{sc_+-1} = \frac{a+b}{a} r_{sc_+} - \frac{b}{a} r_{c_+} \qquad \text{(AVI1c)}$$

$$r_{sc_+-2} = \frac{a+b}{a} r_{sc_+-1} - \frac{b}{a} r_{sc_+} \qquad \text{(AVI1d)}$$

...

$$r_0 = \frac{a+b}{a} r_{1_+} - \frac{b}{a} r_{2_+} \qquad \text{(AVI1e)}$$

We have seen in Appendix V that we can rewrite (AVI1a) in the following way:

$$r_{c_+} = \frac{1}{\frac{ae^+ + h(1-a) - [e^- - h(1-a)]\Theta''}{1-a} + g^+} \qquad \text{(AVI2)}$$

Thus, in the limit of perfect spill-over effects: $\varepsilon^{+(-)} \to 0$ and $h \to 0$ we have that $r_{c_+} = \frac{1}{g^+}$. In order to be able to evaluate (AVI1b) we have to calculate b/a in the limit of perfect spill-over effects. By definition, we have that

$$\frac{b}{a} = \frac{l\,h + (g^- - e^-)\Theta''\, r_{a_+}}{(1-l)h + (g^+ - e^+)\, r_{a_+}} \qquad \text{(AVI3)}$$

Using (AV1a) we can rewrite (AVI3) as follows:

$$\frac{b}{a} = \frac{\frac{l}{1-a}\left[e^+ + h(1-a) - (e^- - h)\Theta''\right] + (g^- - e^-)\Theta''}{\frac{1-l}{1-a}\left[e^+ + h(1-a) - (e^- - h)\Theta''\right] + (g^+ - e^+)} \qquad \text{(AVI4)}$$

From Proposition 2 we know if there is a positive drift in the exogenous driving force, then in the limit of perfect spill-over effects, $\Theta'' \xrightarrow[h \to 0]{} \Theta' \xrightarrow[e^{+(-)} \to 0]{} \Theta \to 0$. Thus, the RHS of (AVI4) converges in this limit to zero. (AVI1b) becomes so $r_{sc_+} = r_{c_+} = \frac{1}{g^+}$. From (AVI1c) and (AVI1d) and (AVI1e) we see that $r_0 = r_1 = \ldots = r_{sc_+} = r_{c_+} = \frac{1}{g^+}$.

We have now to consider states where the firm has more capital than optimal. We have in this case a system symmetric to (AVI1):

$$r_{c_-} = \frac{1}{b} r_{a_-} \qquad \text{(AVI5a)}$$

$$r_{sc_-} = \frac{a+b}{b} r_{c_-} \qquad \text{(AVI5b)}$$

$$r_{sc_{-}-1} = \frac{a+b}{b} r_{sc_{-}} - \frac{a}{b} r_{c_{-}} \qquad \text{(AVI5c)}$$

$$r_{sc_{-}-2} = \frac{a+b}{b} r_{sc_{-}-1} - \frac{a}{b} r_{sc_{-}} \qquad \text{(AVI5d)}$$

...

$$r_0 = \frac{a+b}{b} r_{1_{-}} - \frac{a}{b} r_{2_{-}} \qquad \text{(AVI5e)}$$

Using (AV1b) seen in Appendix V we see that in the limit of perfect spill-over effects the stationary average density of firms in the critical disinvestment state converges towards zero. We can rewrite (AVI5c) as $\frac{b}{a} r_{sc_{-}-1} = \frac{a+b}{a} r_{sc_{-}} - r_{c_{-}}$. Thus, taking into account that in the limit of perfect spill-over effects b/a converges towards zero we obtain that $r_{sc_{-}} = r_{c_{-}}$. In the same way, we can rewrite (AVI5d) as $\frac{b}{a} r_{sc_{-}-2} = \frac{a+b}{a} r_{sc_{-}-1} - r_{sc_{-}}$ and so we obtain in the limit of perfect spill-over effects $r_{sc_{-}-1} = r_{sc_{-}}$. Thus, we obtain that $r_{1_{-}} = \ldots = r_{sc_{-}-1} = r_{sc_{-}} = r_{c_{-}} = r_{a_{-}} = 0$.

We are now going to proof that this is the asymptotic distribution in the case of perfect spill-over effects. In other words, we are now going to show, that the critical state

$$r_{a_{+}} = 0 \qquad r_{c_{+}} = \frac{1}{g^{+}} \qquad r_{sc_{+}} = \frac{1}{g^{+}} \qquad \cdots \qquad r_0 = \frac{1}{g^{+}} \qquad r_{1_{-}} = 0 \qquad \cdots$$

$$r_{a_{-}} = 0$$

is asymptotically stable.

In order to study the local stability of critical state of the system we take a first order approximation of the dynamical system. Written in matrix form we obtain

$$\frac{d}{dt} z = Bz$$

where

$$z^{T} = \begin{bmatrix} r_{a_{+}} & r_{c_{+}} - \frac{1}{g^{+}} & r_{sc_{+}} - \frac{1}{g^{+}} & \ldots & r_0 - \frac{1}{g^{+}} & \ldots & r_{sc_{-}} & r_{c_{-}} & r_{a_{-}} \end{bmatrix}$$

and

$$B=\begin{bmatrix} -\frac{e^+}{g^+} & h & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & 0 & 0 \\ 0 & -(1+a)h & h & \ldots & 0 & \ldots & 0 & \ldots & 0 & 0 & 0 \\ 0 & ah & -(1+a)h & \ldots & 0 & \ldots & 0 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 0 & 0 & \ldots & -(1+a)h & \ldots & 0 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & \ldots & -(1+a)h & \ldots & 0 & 0 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & -(1+a)h & h & 0 \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & ah & -(1+a)h & 0 \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & ah & -1 \end{bmatrix}.$$

In order to study the stability of the system write down the characteristic equation of B

$$\left(-\frac{e^+}{g^+}-l\right)\left[-(1+a)h-l\right]^{g^+ + g^- -1}(-1-l)=0 \qquad \text{(AVI6)}$$

Solution of (AVI6) gives the eigenvalues of the system, which are, $l_1=-\frac{e^+}{g^+}$, $l_3=-(1-a)h$ and $l_3=-1$ which are all negative. This is sufficient to show that the system is locally asymptotically stable.

Appendix VII: Proof of Proposition 5

We are going to proof Proposition 5 by contradiction. Suppose equilibrium $\Theta'' \xrightarrow[h\ \to\ 0]{} \Theta' \xrightarrow[e^{+(-)}\ \to\ 0]{} \Theta=\frac{g^+}{g^-}$ is a critical state. Thus we should be able to get the stationary average densities of firms in the active investment/disinvestment state. We can rewrite the conservation law (14) as follows

$$\frac{1}{2}h^{+}\left(1-r_{a_+}\right)+\frac{1}{2}h^{-}\left(1-r_{a_-}\right)=e^{+}r_{a_+}-e^{-}r_{a_-} \qquad \text{(AVII1)}$$

where h^+ and h^- are positive (negative) shocks. Taking into account conservation law (AVII1) we have that the limiting behaviour, of approaching the critical state, of the stationary average density of firms in the active and critical investment and disinvestment state is described, in this state, by

$$r_{a_+}=\frac{h}{2e^{+}+h} \qquad \text{(AVII2a)}$$

$$r_{a_-}=\frac{h}{2e^{-}+h} \qquad \text{(AVII2b)}$$

Using (AVII2) we have that the average investment and disinvestment avalanche size in this case is, respectively, $\langle s\rangle^{+}=\frac{1}{2e^{+}}$ and $\langle s\rangle^{-}=\frac{1}{2e^{-}}$. Thus, in the limit of perfect spill-over effects, the average avalanche size of firms investing and disinvesting diverges towards infinity. We have also that, in the limit of perfect spill-over effect, also the second moment of the distribution of investment and disinvestment avalanches diverges towards infinity, being consistent with a critical state where no characteristic scale exists.

It is also possible to calculate the distribution of the stationary average densities in this state. Following the steps of the proof in Appendix VI and taking into account that the average densities have to sum up to one, we have that the stationary distribution of average densities will be

$$r_{a_+}=0 \quad r_{c_+}=\frac{1}{2g^{+}} \quad \ldots \quad r_{1_+}=\frac{1}{2g^{+}} \quad r_{0}=\frac{1}{2g^{+}}+\frac{1}{2g^{-}} \quad r_{1_-}=\frac{1}{2g^{-}} \quad \ldots \quad r_{c_-}=\frac{1}{2g^{-}} \quad r_{a_-}=0$$

(AVII3)

Consider now for example the average investment avalanche size, given that a firm is in an active investment state. This expectation can be written as follows:

$$E\left[s\middle|s_{i}=a^{+}\right]=1+\left(g^{+}-e^{+}\right)r_{c_+}+\left[\left(g^{+}-e^{+}\right)r_{c_+}\right]^{2}+\left[\left(g^{+}-e^{+}\right)r_{c_+}\right]^{3}+\ldots. \qquad \text{(AVII4)}$$

where s is the number of firms investing. (AVII4) takes into account that the probability that there is only one firm adjusting its capital stock is one, the probability that there are two firms adjusting their capital stock is $\left(g^{+}-e^{+}\right)r_{c_+}$ and so on.

As long as $\left(g^{+}-e^{+}\right)r_{c_+}$ < 1 we have that (AVII4) will be

$$E\left[s \middle| s_i = a^{+}\right] = \frac{1}{\left[1-\left(g^{+}-e^{+}\right)r_{c_+}\right]} \qquad \text{(AVII5)}$$

Criticality is reached if $\left(g^{+}-e^{+}\right)r_{c_+} \rightarrow 1$. But according to (AVII5) we have that in the limit of perfect spill-over effects $r_{c_+} = \frac{1}{2g^{+}}$ and so $E\left[s \middle| s_i = a^{+}\right] = 2$ and so we got a contradiction.

Appendix VIII: Proof of Proposition 6

Consider first the long run growth rate of aggregate output. Applying the expectation operator E(.) to (23) we have that the second term of (23) is given by $\frac{1}{n}\left[\langle s\rangle^{+}\left(e^{g^{+}}-1\right)-\langle s\rangle^{-}\left(e^{g^{-}}-1\right)\right]$, which is the long run aggregate net-investment rate. The long run average aggregate growth rate of output is given by

$$E\left[g_y\right] = c + b\,\frac{1}{n}\left[\langle s\rangle^{+}\left(e^{g^{+}}-1\right)-\langle s\rangle^{-}\left(e^{g^{-}}-1\right)\right] \qquad \text{(AVIII1)}$$

Inserting results stated in Proposition 4 into (AVIII1) we obtain the following expression

$$E\left[g_y\right] = c + b\left[\frac{1-a}{g^{+}ne}\,\frac{e^{g^{+}}-1}{1-\Theta} - \frac{1-a}{g^{-}ne}\,\frac{\Theta\left(e^{g^{-}}-1\right)}{1-\Theta}\right]$$

Evaluating this last expression case by case we obtain the results stated in Proposition 6.

Consider now the variance of the growth rate of aggregate output. Taking into account the fact that

$$E\left[\left(\frac{1}{n}\sum_{i=1}^{n}\left[1\left(s_t^i = L'\right)\left(e^{g^+}-1\right)L' - 1\left(s_t^i = U'\right)\left(e^{g^-}-1\right)U'\right]\frac{1}{s_t^i}\right)^2\right] =$$

$\frac{1}{n^2}\left[\langle s^2\rangle^+\left(e^{g^+}-1\right)^2 - 2\langle s^+, s^-\rangle\left(e^{g^+}-1\right)\left(e^{g^-}-1\right) + \langle s^2\rangle^-\left(e^{g^-}-1\right)^2\right]$, where $\langle s^+, s^-\rangle$ is the covariance of the investment and disinvestment avalanches, we have that the variance of the aggregate growth rate is given by

$$Var\left(g_y\right) = b^2\frac{1}{n^2}\left[\langle s^2\rangle^+\left(e^{g^+}-1\right)^2 - 2\langle s^+, s^-\rangle\left(e^{g^+}-1\right)\left(e^{g^-}-1\right) + \langle s^2\rangle^-\left(e^{g^-}-1\right)^2\right] +$$

$$-\left\{b\frac{1}{n}\left[\langle s\rangle^+\left(e^{g^+}-1\right) - \langle s\rangle^-\left(e^{g^-}-1\right)\right]\right\}^2 \qquad \text{(AVIII2)}$$

Inserting results stated in Proposition 4 into (AVIII2) we have that the variance of the average aggregate growth rate is

$$Var\left(g_y\right) = b^2\left[\frac{1-a}{g^+ n(1-d)}\left(e^{g^+}-1\right)\right]^2$$

Evaluating this last expression case by case gives the expressions stated in Proposition 6.

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