

# **Anomalies and monotonicity in net present value calculations**

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**Abstract.** In recent issues of *Economics Letters* (**72** (2001) 127-129 and **67** (2000) 349-351), the authors find conditions on internal rates of return so that the net present value function doesn't present more than one change in sign. We show in this note a condition for the strictly monotonicity of net present value function.

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## **(Article Outline)**

The internal rate of return and the net present value are two of the most used criteria to accept or reject projects; for the internal rate of return (IRR) criterion, a project is acceptable if its IRR is major than its risk-adjusted discount rate, while for the net present value (NPV) criterion, we

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have acceptance if NPV is positive. Both procedures have been criticized from many authors; in particular, if the project has a stream of net benefits with more than one change in sign, the internal rate of return may not exist or it is possible to have two or more IRR.

The Oehmke (2000) and Domingo (2001) letters explain that when the stream of net benefits has more than one change in sign, isn't assured that the relation between NPV and the discount rate is monotonic, thus the net present value criterion, as the IRR, can generate ambiguous results. The non monotonicity of NPV can cause situations where an increment in the discount rate yields a NPV that grow from negative to positive quantities.

Before to explain our considerations on the letters above quoted, we remember that for every discount rate  $r \in \mathbb{R}^+$  and for every stream of net benefits  $(B_0, B_1, \dots, B_n) \in \mathbb{R}^{n+1}$ , ( $B_0 < 0$ ,  $B_n \neq 0$  and  $B_t > 0$  for at least one  $t$ ) we define NPV at rate  $r$  as the quantity

$$N(r) = \sum_{t=0}^{t=n} \frac{B_t}{(1+r)^t}$$

while a rate  $i$  is an IRR for the stream  $(B_0, B_1, \dots, B_n)$  if  $N(i) = 0$  ( $i$  is a positive root for the function  $N$ ).

The Oehmke paper focuses attention on the relation between function's sign of NPV and its roots; in particular is noted that "*a necessary condition for anomalous behavior*" of NPV criterion "*is that  $N(r)$  have at least two real roots. A sufficient condition is that  $N(r)$  have (at least) two distinct real roots*", and concludes "*the cases in which NPV may exhibit anomalous behavior are exactly those cases in which there are multiple*" IRRs. Domingo note instead achieves the following result: "*Let  $(B_0, B_1, \dots, B_n) \in \mathbb{R}^{n+1}$ , and  $x, y$  be positive real numbers with  $x < y$* " such that  $N(x) > 0$  and  $N(y) > 0$ . "*If the interval  $(x, y)$  contains at most one internal rate of return corresponding to  $(B_0, B_1, \dots, B_n) \in \mathbb{R}^{n+1}$ , then the NPV is positive for all  $r \in [x, y]$ .*"

If we consider the stream  $(-4, 7, -4)$  with

$$N(r) = -4 + \frac{7}{1+r} - \frac{4}{(1+r)^2} = - \left( 4 \left( 1 - \frac{1}{1+r} \right)^2 + \frac{1}{1+r} \right),$$

we can observ that even if  $N(r)$  have'nt roots, the NPV may exhibit anomalous behavior, infact in this case the function  $N$  is strictly increasing in  $(0, 1/7)$  while for  $r > 1/7$  is strictly decreasing, thus we do not believe correct the condition that if  $N(r)$  have at least two real roots is necessary for anomalous behavior of NPV. But moreover the results displayed by Ohmke and Domingo consider only the sign of  $N(r)$  and not its monotonicity; to go at one condition for it, take  $\forall k \leq n$  the quantity

$$N_k(r) = \sum_{t=0}^{t=k} \frac{B_t}{(1+r)^t} \quad (1)$$

obviously  $N_n(r) = N(r)$ . Every  $N_k(r)$  may be explained as the net present value at rate  $r$  of the truncated stream  $(B_0, B_1, \dots, B_k)$  and following the idea of maximization of the truncated net present value, see Arrow and Levhary (1969), for the monotonicity be worth the following

**Proposition 1.** *Let  $(B_0, B_1, \dots, B_n) \in \mathbb{R}^{n+1}$  (with the conditions before expressed) and  $x, y$  two positive real values with  $x < y$ . If for every  $k \leq n$  and for every  $r \in [x, y]$  is  $N_k(r) \leq N(r)$ , then  $N(r)$  is strictly decreasing in  $[x, y]$ .*

**Proof.** For every  $r \in \mathbb{R}^+$  calculate the derivative of  $N(r)$ :

$$\begin{aligned} N'(r) &= \sum_{t=1}^{t=n} \frac{-t \cdot B_t}{(1+r)^{t+1}} \\ &= \frac{1}{1+r} \sum_{t=1}^{t=n} \sum_{h=t}^{h=n} \frac{-B_h}{(1+r)^h} \\ &= \frac{1}{1+r} \sum_{t=0}^{t=n} \left( N_t(r) - N(r) \right). \end{aligned}$$

Conditions in proposition imply that  $N'(r) \leq 0$  for every  $r \in [x, y]$  and by  $B_n \neq 0$  follow  $N_{n-1}(r) < N(r)$  for every  $r \in \mathbb{R}^+$ , thus at least one addendum in last summation is negative and so the strict negativity of  $N'(r)$  in  $[x, y]$ . ■

If we consider the sign of  $N(r)$ , by the Proposition 1 turn out immediatly the

**Corollary 2.** *Under the hypotheses of Proposition 1, the condition  $N(y) > 0$  is sufficient to assure positivity of  $N(r)$  in the interval  $[x, y]$ .*

Unfortunately Proposition 1 and Corollary 2 provide conditions on strict monotonicity that have to be valid for every  $r$  that belong in  $[x, y]$ ; surely conditions only on one point are easier to be cheked, for this goal we have

**Proposition 3.** *Let  $(B_0, B_1, \dots, B_n) \in \mathbb{R}^{n+1}$  (with the conditions before expressed). If  $\exists x \in \mathbb{R}^+ : \forall k \leq n, N_k(x) \leq N(x)$ , then  $N(r)$  is strictly decreasing in  $[0, x]$ .*

Before to start the proof note that as before if we derive  $N(r)$  we get

$$N'(r) = \frac{1}{1+r} \sum_{t=0}^{t=n} \left( N_t(r) - N(r) \right); \quad (2)$$

rearranging the above summation follow that for every  $k \leq n - 1$

$$\begin{aligned} N'(r) &= \frac{1}{1+r} \sum_{t=0}^{t=k} \left( N_t(r) - N_k(r) \right) + \frac{1}{1+r} \sum_{t=0}^{t=k} \left( N_k(r) - N(r) \right) \\ &\quad + \frac{1}{1+r} \sum_{t=k+1}^{t=n} \left( N_t(r) - N(r) \right) \end{aligned}$$

by (1) the first addendum is the derivative of  $N_k(r)$ , thus

$$N'(r) - N'_k(r) = \frac{1}{1+r} \sum_{t=0}^{t=n} \left( N_{t \vee k}(r) - N(r) \right) \quad (3)$$

where  $t \vee k = \sup\{t, k\}$ .

To prove the proposition we need of the following

**Lemma 4.** *Under the hypotesies of Proposition 3,  $N_k(r) \leq N(r)$  for every  $k$  and for every  $r \in [0, x]$ .*

**Proof.** By the hypotesies if we calculate in  $x$  the summation in (3) every addendum is not positive, besides from condition  $B_n \neq 0$  at least one is negative and this imply  $N'(x) < N'_k(x), \forall k \leq n - 1$ . From the previous inequality follow that exist a positive  $w < x$  such that

$$N_k(r) < N(r), \forall k \leq n - 1, \forall r \in [w, x]. \quad (4)$$

Now go ad absurd and suppose that exist  $j \leq n - 1$  and  $v < x$  such that  $N_j(v) > N(v)$ ;  $v$  is less than  $w$  and by the continuity of every function  $N_k(r)$ , we may suppose without lose of generality that exist  $u$  ( $v < u < w$ ), such that:

$$\begin{aligned} N_j(r) &> N(r), \forall r \in [v, u[; \\ N_j(u) &= N(u); \end{aligned} \quad (5)$$

$$N_j(r) \leq N(r), \forall r \in ]u, w]. \quad (6)$$

Integrating the difference  $N'(r) - N'_k(r)$  we obtain

$$\begin{aligned}
N(u) - N_j(u) &= \int_x^u \left( N'(r) - N'_j(r) \right) dr + N(x) - N_j(x) \\
&\geq \int_u^w \left( N'_j(r) - N'(r) \right) dr + \int_w^x \left( N'_j(r) - N'(r) \right) dr
\end{aligned}$$

where the inequality follow from hypotesies in Lemma; by (3) the addition above may be rewritten as

$$\int_u^w \frac{1}{1+r} \sum_{t=0}^{t=n} \left( N(r) - N_{t \vee j}(r) \right) dr + \int_w^x \frac{1}{1+r} \sum_{t=0}^{t=n} \left( N(r) - N_{t \vee j}(r) \right) dr$$

and by (6) the first integral is not negative while the second is positive from (4). Thus  $N(u) - N_j(u) > 0$  in contradiction with (5) and the proof is completed. ■

**Proof of Proposition 3.** From Lemma 4 every addendum in (2) is not positive  $\forall r \in [0, x]$  and by condition  $B_n \neq 0$  at least one is negative  $\forall r \in [0, x[$ ; this imply  $N'(r) < 0$  in  $[0, x[$  and so the strictly decreasing of  $N$  in  $[0, x]$ . ■

## References

- 1) Arrow K.J. and Levhary D., 1969, Uniqueness of the internal rate of return with variable life of investment, *The Economic Journal* **79**, pp. 560-566.
- 2) Domingo C. J., 2001, Anomalies in net present value calculations?, *Economics Letters* **72**, pp. 127-129.
- 3) Oehmke J. F., 2000, Anomalies in net present value calculations, *Economics Letters* **67**, pp. 349-351.