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Abstract

Coalitional Game Contexts (CGCs) –the objects of a nonfull subcategory of the category of 'Chu spaces' or 'formal contexts' – are introduced and shown to encompass virtually all coalitional game formats as currently employed in the game- and social choice-theoretic literature. *Concept lattices* of CGCs are also discussed, and the resulting *order dimension* for CGCs is defined. Some basic spectral properties of those lattices are studied. In particular, it is shown that for any positive integer k there exists a *preconvex* –hence strongly core-stable– CGC with a concept lattice of width k.

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1 Introduction

Coalitional game contexts(CGCs) are general incidence structures arising as a common abstraction from many different coalitional game formats which have been variously used in the game- and social choice-theoretic literature. Indeed, *CGCs* can also be regarded as specialized instances of *"classifications" or "Chu spaces"* as recently introduced in logic and theoretical computer science (see e.g. Barwise and Seligman(1997), Pratt(1999)).

Concept lattices are complete lattices that arise in a 'natural' manner from the study of relational databases, enabling a detailed analysis of the 'intrinsic' hierarchical structures of the latter (see e.g. Davey and Priestley(1990), Ganter and Wille(1998)).

In this paper–which draws on, and extends, Vannucci (1999a,1999b)– *coalitional game contexts(CGCs)* are introduced and their concept lattices are defined. Properties of concept lattices arising from a few standard types of CGCs are studied, and the resulting *order dimension theory* for CGCs is outlined. A few spectral properties of CGCs arising from strongly (core-)stable effectivity functions are also considered. In particular, it is shown that there exist preconvex –hence strongly stable– CGCs of *arbitrary width.*

2 Model and results

2.1 Coalitional Game Contexts:Classifying Coalitions and Outcome-Subsets

A *coalitional game context (CGC)* is a triplet $\mathsf{G} = (\mathcal{C}, \mathcal{Z}, \Im)$ where

$\mathcal{C} = (C, \lessdot)$ and $\mathcal{Z} = (Z, \lessdot')$ are preordered sets,

i.e. $\lessdot$ and $\lessdot'$are reflexive and transitive binary relations on C and Z respectively

($\mathcal{C}$ typically denotes the coalition structure, and $\mathcal{Z}$ the outcome structure), and $\Im \subseteq C \times Z$– the coalition-outcome *incidence correspondence*– is required to satisfy a normalization condition, namely there exists a $\lessdot$ $-$*minimal* $s^\circ \in C$ with $\Im(s^\circ) = \emptyset$.

In particular, a CGC G =$(\mathcal{C}, \mathcal{Z}, \Im)$ is said to be

$\mathcal{C}$*-topped* with top element $\top_\mathsf{C}$ if there exists $\top_\mathsf{C} \in C$ such that $\top_\mathsf{C} \lessdot s$ for any $s \in C$,

$\mathcal{Z}$*-grounded* with bottom element $\bot_\mathsf{Z}$ if there exists $\bot_\mathsf{Z} \in Z$ such that $z \lessdot' \bot_\mathsf{Z}$ for any $z \in Z$, and

semi-bounded if G is both $\mathcal{C}$-topped and $\mathcal{Z}$-grounded.

The following are a few relevant properties the incidence correspondence $\Im$ of a CGC $\mathsf{G} = (\mathcal{C}, \mathcal{Z}, \Im)$ *may* satisfy

(*Normality*): $\Im(s) = \emptyset$ for any $\lessdot$ $-$*minimal* $s \in C$

(*Weak Communal Domain*): there exists a $<' -maximal$ $x^{\circ} \in Z$ such that $\Im(s) \supseteq \{x^{\circ}\}$ for any $s \in C$ such that $\Im(s) \neq \emptyset$

(*Communal Domain*): there exists a $<' -maximal$ $x^{\circ} \in Z$ such that $\Im(s) \supseteq \{x^{\circ}\}$ for any $s \neq s^{\circ}$

(*$\mathcal{C}$-Monotonicity*): for any $z \in Z$, and any $s, t \in C$, if $t < s$ and $z \in \Im(s)$ then $z \in \Im(t)$ (i.e $\Im^{-1}(z)$ is an order filter of $\mathcal{C}$)

(*$\mathcal{Z}$-Monotonicity*): for any $s \in C$, and any $z, u \in Z$, if $u <' z$ and $z \in \Im(s)$ then $u \in \Im(s)$ (i.e. $\Im(s)$ is an order filter of $\mathcal{Z}$)

(*Ferrers Condition*): for any $s, t \in C$, and any $z, u \in Z$, if $z \in \Im(s) \setminus \Im(t)$ and $u \in \Im(t)$ then $u \in \Im(s)$.

Moreover, if G is $\mathcal{C}$-topped with top element $\top_{\mathsf{C}}$ then it may satisfy

(*Local $\mathcal{C}-$Monotonicity*): $\Im(\top_{\mathsf{C}}) \supseteq \Im(s)$ for any $s \in C$

(*Nonimposition*): $\Im(\top_{\mathsf{C}}) \supseteq P(Z) \setminus \{\emptyset\}$

and if G is $\mathcal{Z}$-grounded with bottom element $\bot_{\mathsf{Z}}$ then it may satisfy

(*Nonbottom-Valuedness*): $\Im(s) \cap \{\bot_{\mathsf{Z}}\} = \emptyset$ for any $s \in C$.

Other properties of interest can also be defined in a natural way when a CGC is endowed with a suitably richer latticial structure. Hence, a few basic lattice-theoretic notions are to be recalled here for the sake of completeness: a *lattice* $\mathcal{L} = (L, \geq)$ may be regarded as an *antisymmetric* preordered set – or *poset*– that is both a *join-semilattice* –i.e. for any $a, b \in L$ there exists a $\geq -$*least upper bound* $a \vee b \in L$– and a *meet-semilattice* i.e. for any $a, b \in L$ there exists a $\geq -$*greatest lower bound* $a \wedge b \in L$. A lattice is *complete* if a $\geq -$*least upper bound* and a $\geq -$*greatest upper bound* exist in L for any $A \subseteq L$, *pseudocomplemented* if L has a $\geq -$bottom element $\bot_{\mathsf{L}}$ and for any $a \in L$ there exists $a^{*} \in L$ such that $a^{*} = \max\{b \in L : b \wedge a = \bot_{\mathsf{L}}\}$, *bounded* if L has both a $\geq -$top element $\top_{\mathsf{L}}$ and a $\geq -$bottom element $\bot_{\mathsf{L}}$, *complemented* if L is bounded and for any $a \in L$ there exists $a^{\mathsf{c}} \in L$ such that $a \vee a^{\mathsf{c}} = \top_{\mathsf{L}}$ and $a \wedge a^{\mathsf{c}} = \bot_{\mathsf{L}}$, *distributive* if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for any $a, b, c \in L$, and *Boolean* if it is both distributive and complemented. Furthermore, an *atom* of a lattice $(L, \geq)$ is a $\geq -$*minimal* nonbottom element of L, and –dually– a *coatom* is a $\geq -$maximal nontop element of L. A lattice is said to be *dense* if it has a unique atom, and *codense* if it has a unique coatom). The following properties are of special interest:

(*Superadditivity*): $\mathcal{C}$ is a bounded lattice, $\mathcal{Z}$ is a lattice, and for any $s, t \in C$, and any $u, z \in Z$, if $s \wedge t = \bot_{\mathsf{C}}$, $u \in \Im(s)$ and $z \in \Im(t)$ then $u \wedge z \in \Im(s \vee t)$

($Preconvexity$): $\mathcal{C}$ and $\mathcal{Z}$ are lattices, and for any $s, t \in C$ and any $u, z \in Z$, if $u \in \Im(s)$ and $z \in \Im(t)$ then either $u \wedge z \in \Im(s \vee t)$ or $u \vee z \in \Im(s \wedge t)$

($Regularity$): $\mathcal{C}$ is a pseudocomplemented lattice, $\mathcal{Z}$ is a lattice, and for any $s \in C$, and any $u, z \in Z$, if $u \in \Im(s)$ and $z \in \Im(s^*)$ then $u \wedge z \neq \perp_{\mathsf{Z}}$

($Maximality$): $\mathcal{C}$ is a pseudocomplemented lattice, $\mathcal{Z}$ is a lattice, and for any $s \in C$ and any $u \in Z$, if $u \notin \Im(s)$ then there exists $z \in Z$ such that $z \in \Im(s^*)$ and $u \wedge z = \perp_{\mathsf{Z}}$.

As mentioned above, CGCs are a common abstraction of several coalitional game formats which have been proposed and used both in the game-theoretic and in the social-choice-theoretic literature. In order to substantiate that claim, let us consider a few salient examples.

Example 1 Effectivity Functions and Related Formats *(see e.g. Rosenthal (1972), Moulin and Peleg(1982), Moulin(1983), Peleg(1984), Abdou and Keiding(1991)). Given a player set N and an outcome set X, an* effectivity function (EF) *is usually defined as a function $E : P(N) \rightarrow P(P(X))$ such that EF i) $E(\emptyset) = \emptyset$; EF ii) $\emptyset \notin E(S)$ for any $S \subseteq N$; EF iii) $X \in E(S)$ for any $S \subseteq N, S \neq \emptyset$; EF iv) $E(N) = P(X) \setminus \{\emptyset\}$.*

A conditional EF *can also be defined by considering a function $E' : P(N) \rightarrow \prod_{\mathsf{x} \in \mathsf{X}} [\{x\} \times P(P(X))]$ such that for any $S \subseteq N$, $E'(S) = \prod_{\mathsf{x} \in \mathsf{X}} \{x\} \times E_{\mathsf{x}}(S)$ where –for each $x \in X$– E_{x} defines an EF (see e.g. Rosenthal (1972) for an early proposal of a special case of that construct).*

(Since both $\mathcal{C}-$monotonicity and $\mathcal{Z}-$monotonicity are necessary conditions –along with either Regularity or Maximality– for an EF to arise in a standard way from a strategic game form, in early treatments monotonicity requirements used to be embodied into the definition of an EF: see Moulin and Peleg(1982), Moulin(1983). This usage is now over).

Clearly enough, (conditional) characteristic function coalitional game forms *may be regarded as corestrictions of (conditional) EFs to singleton-set-values. Moreover, it should be recalled here that* simple games *may be regarded as equivalence classes of* simple EFs *namely EFs with a set $W \subseteq P(N)$ such that for any $S \in P(N) \setminus \{\emptyset\} : E(S) = P(X) \setminus \{\emptyset\}$ if $S \in W$ and $E(S) = \{X\}$ otherwise.*

Several weakenings and extensions of EFs have also been proposed in the literature under various labels: e.g. well-behaved functions *satisfying EF i),EF ii), EFiii),* coalitional game forms *(or* pseudo-EFs *or* semi-well-behaved functions *(see e.g. Ichiishi(1989)) satisfying EF i) and EF ii), or even an intermediate notion of* weak EFs *satisfying EF i), EF ii) plus EF iii')$X \in E(S)$ for any $S \subseteq N$ such that $E(S) \neq \emptyset$ and EF iv')$E(N) \supseteq E(S)$ for any $S \subseteq N$. The last additions to that list are two mutually related notions, namely* domination patterns *and* domination structures *as recently introduced by Fristrup and Keiding(2001). A* domination pattern *on (N, X) denotes an agreement on simultaneous vetoing of several outcomes on the part of certain coalitions,* *and may be represented as a set* $D_{\mathsf{B}} = \{(S(x), x) : \emptyset \neq S(x) \subseteq N, x \in B \subseteq X\}$*.* *A* domination structure $\mathcal{D}$ *is a set of domination patterns such that for any* $B \subseteq N$, $D_{\mathsf{B}}^{\mathsf{N}} = \{(N, x) : x \in X\} \in \mathcal{D}$ *, and* $D' \in \mathcal{D}$ *for any* $D' \subseteq D \in \mathcal{D}$*.*

Claim 2 *i) A (standard) EF* $E : P(N) \rightarrow P(P(X))$ *can be represented as a semi-bounded CGC* $\mathsf{G}(E) = (\mathcal{C}, \mathcal{Z}, \Im_{\mathsf{E}})$ *where* $\mathcal{C} = (P(N), \supseteq)$, $\mathcal{Z} = (P(X), \supseteq)$, *and* $\Im_{\mathsf{E}} = \{(S, A) : A \in E(S)\}$ *.Hence,* $\mathcal{C}$ *and* $\mathcal{Z}$ *are* Boolean lattices, *and* $\mathsf{G}(E)$ *satisfies Normality, Communal Domain, Nonbottom-Valuedness, and Nonimposition.*

ii) A domination pattern D_{B} *on* (N, X) *is representable as a semi-bounded CGC* $\mathsf{G}(D) = (\mathcal{C}, \mathcal{Z}, \Im_{\mathsf{D_B}})$ *such that* $\mathcal{C} = (P(N), \supseteq)$, $\mathcal{Z} = (P(X), \supseteq)$ *and* $\Im_{\mathsf{D_B}} = \{(S, X \setminus \{x\}) : (S, x) \in D_{\mathsf{B}}\}$ *. Thus,* $\mathsf{G}(D)$ *satisfies Normality and Communal Domain.*

Remark 3 *The foregoing definitions, however, should make it clear that the basic EF notion can be easily lifted to more general CGCs. (Incidentally, it should be noticed that such a lifting makes it quite easy to embed into an EF-format other proposed formats such as* set-theoretic forms *arising from extensive forms: see Bonanno(1991)). Similar observations hold for Pseudo-EFs, which amount to retaining Normality and Nonbottom-Valuedness and substituting Weak Common Domain for Common Domain, while dropping Nonimposition. Well-behaved functions and weak EFs are similarly represented in a CGC setting.*

Example 4 'Constitutional' Game Forms and Related Formats *(see e.g. Ferejohn and Fishburn(1979), Andjiga,Moulen(1989)). Given a player set* N *and an outcome set* X*, consider*

a nonempty *set* $\mathsf{C} \subseteq \{(S, T) : S \subseteq N, T \subseteq N \text{ and } S \cap T = \emptyset\}$ *, and*

a nonempty *set* $\mathsf{Y} \subseteq \{(A, B) : A \subseteq X, B \subseteq X \text{ and } A \cap B = \emptyset\}$*.*

Then, a 'Constitutional' Game Form(COGF) *is a tuple* $\Gamma = (N, X, \mathsf{C}, \mathsf{Y}, K)$ *where* $K : \mathsf{Y} \rightarrow P(\mathsf{C}) \setminus \{\emptyset\}$

is such that for any $(S, T) \in C$ *with* $S \neq \emptyset$ *there exists* $(A, B) \in \mathsf{Y}$ *:* $(S, T) \in K((A, B))$.

By positing $\mathsf{C} \subseteq \{(S, N \setminus S) : S \subseteq N\}$ *the subclass of* Constitution Forms *as presented in Andjiga,Moulen(1989) is defined.*

By positing $\mathsf{Y} = \{(\{a\}, \{b\}) : a, b \in X, a \neq b\}$*, and imposing the requirement that*

for any $\{\{a\}, \{b\}\} \in \mathsf{Y}$*, and any* $(S, T) \in \mathsf{C}$

if $(S, T) \in K((\{a\}, \{b\}))$ *then* $(T, S) \notin K((\{b\}, \{a\}))$

the subclass of Binary Constitutions *as defined in Ferejohn and Fishburn (1979) is immediately obtained.*

Claim 5 *A COGF* $\Gamma = (N, X, \mathsf{C}, \mathsf{Y}, K)$ *can be represented (modulo 'natural' order-isomorphisms) by a CGC* $\mathsf{G}(\Gamma) = (\mathcal{C}, \mathcal{Z}, \Im)$ *where* $\mathcal{C} = (\mathsf{C}, \sqsupseteq)$, $\mathcal{Z} = (\mathsf{Y}, \sqsupseteq)$ *and in both cases '*$\sqsupseteq$*' is defined by the rule* $(x, y) \sqsupseteq (u, z)$ *iff* $x \supseteq u$ *and* $x \cup y \supseteq u \cup z$ *(whence* $\mathcal{C}$ *and* $\mathcal{Z}$ *are both* bounded posets, *indeed bounded* $\cap$*-(meet)semilattices with* $\top_{\mathsf{C}} = (N, \emptyset), \perp_{\mathsf{C}} = (\emptyset, \emptyset), \perp_{\mathsf{Y}} = (\emptyset, \emptyset)$ *and* $\top_{\mathsf{Y}} = (X, \emptyset)$*), and* $\Im = \{(S, (A, B)) : (A, B) \in K(S)\}$ *(notice that in fact* $\Im(\emptyset) = \emptyset$ *as required). In particular, it follows that* $\Im$ *also satisfies Normality.*

Example 6 Semisimple Games *(see Blau and Brown(1989), Kalai, Pazner and Schmeidler(1976), Packel(1981)). Given a player set* N *and an outcome set* X*, a* (Monotonic) Semisimple Game (SSG) *is a* $P(N)-$*parameterized family* $\mathcal{S} = \{\mathsf{S}_{\mathsf{T}} \subseteq P(N) : T \subseteq N\}$ *such that*

SSG i): for any $S, T, U \in P(N)$*, if* $S \in \mathsf{S}_{\mathsf{T}}$ *and* $S \subseteq U \subseteq T$ *then* $S \in \mathsf{S}_{\mathsf{U}}$*, and*

SSG ii): for any $S, T, U \in P(N)$*, if* $S \in \mathsf{S}_{\mathsf{U}}$ *and* $S \subseteq U \subseteq T$ *then* $S \cup (T \backslash U) \in \mathsf{S}_{\mathsf{T}}$.

Semisimple games –which are meant to model the sets of globally 'winning' coalitions in voting-like processes with variable sets of passive players (the 'abstainers')– where first proposed in the late '70s by Blau and Brown under the label of 'neutral monotonic social functions' (see Blau and Brown(1989), a paper originally written in 1978), and subsequently dubbed 'semisimple games' in Packel(1981) (see also Kalai,Pazner and Schmeidler(1976) for a strictly related construct). Note that the original definition of SSGs did not make reference to the empty coalition, which we consider here for the sake of convenience.

Claim 7 *A SSG* $\mathcal{S} = \{\mathsf{S}_{\mathsf{T}} : \emptyset \neq T \subseteq N\}$ *can be represented–modulo 'natural' order-isomorphisms– as a CGC* $\mathsf{G}(\mathcal{S}) = (\mathcal{C}, \mathcal{Z}, \Im)$ *where*

$\mathcal{C} = (C = \{(S, T) : S \subseteq N, T \subseteq N,\ S \cap T = \emptyset\}, \sqsupseteq)$

with $(S, T) \sqsupseteq (U, V)$ *iff* $S \supseteq U$ *and* $S \cup T \supseteq U \cup V$ *(therefore* $\top_{\mathsf{C}} = (N, \emptyset)$ *, and* $\perp_{\mathsf{C}} = (\emptyset, \emptyset)$ *i.e.* $\mathcal{C}$ *is a bounded poset, indeed a bounded* $\cap$*-(meet)semilattice),* $\mathcal{Z} = (P(\{0, 1\}), \supseteq)$ *(hence* $\mathcal{Z}$ *is a Boolean lattice), and for any* $(S, T) \in C$,

$$\Im((S, T)) = \left\{ \begin{array}{c} \emptyset \text{ if } S = \emptyset \\ \{0, 1\} \text{ if } \emptyset \neq S \notin \mathsf{S}_{\mathsf{N} \backslash \mathsf{T}} \\ P(\{0, 1\}) \backslash \{\emptyset\} \text{ if } \emptyset \neq S \in \mathsf{S}_{\mathsf{N} \backslash \mathsf{T}} \end{array} \right\}.$$

In particular, $\Im$ *thus defined also satisfies Normality, Communal Domain, Nonbottom-Valuedness, Nonimposition,* $\mathcal{C}$*-Monotonicity,* $\mathcal{Z}$*-Monotonicity, and the Ferrers Condition.*

Example 8 Partition Game Forms *(see e.g. Shubik(1982), Gilboa and Lehrer(1991)). Let* N *be a player set and* X *an outcome set. Let us denote by* Π_{N} *the set of* *all partitions of N (i.e. sets of nonempty pairwise disjoint subsets of N -or 'blocks'–that cover N), and by $\succeq$ the 'coarser than' partial order on Π_{N} (i.e. $\pi_0 \succeq \pi_1$ iff for any block $x \in \pi_1$ there exists a block $y \in \pi_0$ such that $y \supseteq x$). The poset $(\Pi_{\mathsf{N}}, \succeq)$ is a* bounded lattice, *the* lattice of partitions of N, *with top element $\{\{N\}\}$ and bottom element $\{\{i\} : i \in N\}$. A* partition characteristic function *for (N, X) is a function $f : \Pi_{\mathsf{N}} \to P(X) : f$ is said to be* normalized *if $f(\{\{i\} : i \in N\}) = \emptyset$ (*partition effectivity functions *might also be defined in a similar way). A* partition game form (PGF) *is a triplet $P = (N, X, f)$; moreover, P is said to be a* normalized PGF *if f is normalized.*

Claim 9 *A normalized partition game form $P = (N, X, f)$ can be represented–modulo order-isomorphisms– by a CGC $\mathsf{G}(P) = (\mathcal{C}, \mathcal{Z}, \Im)$ where $\mathcal{C} = (\Pi_{\mathsf{N}}, \succeq)$ is the lattice of partitions of N (hence $\top_{\Pi_{\mathsf{N}}} = \{\{N\}\}$), $\mathcal{Z} = (P(X), \supseteq)$, and $\Im = \{(\pi, A) : A \in f(\pi)\}$. Moreover, $\Im$ also satisfies Normality.*

Example 10 Social Situation-Forms *(see Greenberg(1990)). Let N be a player set, and X an outcome set. A* position-form *of (N, X) is a pair $(S, A) \in P(N) \times P(X)$, hence $\mathsf{P}(N, X) = P(N) \times P(X)$ is the set of position-forms of (N, X).*

The inducement correspondence *is a correspondence*
$I \subseteq \mathsf{P}(N, X) \times P(N) \times X \times \mathsf{P}(N, X)$
such that
$I \subseteq \{((S, A), U, x, (T, B)) : x \in A, U \subseteq S, U \subseteq T\}$.

A Social Situation-Form(SSF) *for (N, X) is a pair $\mathsf{S} = (\mathsf{P}(N, X), I)$ as defined above.*

Claim 11 *A SSF $\mathsf{S} = (\mathsf{P}(N, X), I)$ can be represented –modulo order-isomorphisms– by a CGC $\mathsf{G}(\mathsf{S}) = (\mathcal{C}, \mathcal{Z}, \Im)$ where*

$$\mathcal{C} = \mathcal{Z} = (P(N) \times P(X), \mathsf{c})$$

with $(S, A)\ \mathsf{c}\ (T, B)$ iff $S \supseteq T$ and $A \supseteq B$, and

$$\Im = \left\{ \begin{array}{c} ((S, A), (T, B)) : \text{there exist } x \in X, U \subseteq S \\ \text{such that } ((S, A), x, U, (T, B)) \in I \end{array} \right\}.$$

Thus, CGCs provide a theoretical setting which is broad enough to encompass virtually all the coalitional formats which have been proposed in the extant game-theoretic literature. It should be noticed that CGCs extend the standard notions in *two* respects, namely by weakening the required properties of the 'incidence' relation *and* by generalizing the underlying coalitional and outcome structures. It should also be stressed again that the language of CGCs also provides a bridge to similar constructs which have been quite widely used in mathematical logic and in theoretical computer science in the last two decades.

In fact, CGCs are a special case of 'Chu spaces' or '(formal) classifications', which can be roughly described as an abstract representation of classifications of certain 'tokens' by means of certain 'types' (including the prominent case of mathematical logic where 'tokens' are structures and 'types' are sentences of a certain formal language: see e.g. Barwise and Seligman(1997)). Here, under the most natural interpretation, coalitions are the 'tokens' to be classified according to their a priori decision power, while outcome-subsets are the classifying 'types' (of course, a 'dual' perspective is also available). In that connection, a (concrete) *category of CGCs* can be defined having CGCs as *objects,* and *pairs* of order-homomorphisms –between outcome-structures and between coalition-structures, respectively– as *morphisms*: the resulting category of CGCs is clearly a subcategory of the category of all classifications (a nonfull one, because of the order-structure embodied in CGCs). The details, however, will not be pursued here.

2.2 Concept Lattices of Coalitional Game Contexts: Classifying Classifications

Let $\mathsf{G} = (\mathcal{C}, \mathcal{Z}, \Im)$ be a CGC with $\mathcal{C} = (C, <), \mathcal{Z} = (Z, <')$. The *concept lattice* of G is defined as follows:

for any $D \subseteq C$, $Y \subseteq Z$ posit

$\mathsf{h}_{\Im}(D) = \{z \in Z : (d, z) \in \Im$ for all $d \in D\}$ and

$\mathtt{i}_{\Im}(Y) = \{c \in C : (c, y) \in \Im$ for all $y \in Y\}$.

Then, consider

$$\mathsf{C}(\mathsf{G}) = \{(D, Y) \in P(C) \times P(Z) : D = \mathtt{i}_{\Im}(Y), \text{ and } Y = \mathsf{h}_{\Im}(D)\}.$$

In the language of formal concept analysis an element (D, Y) of $\mathsf{C}(\mathsf{G})$ is said to be a *concept* of the context G, with *extent* D and *intent* Y (the latter notions are amenable to straightforward dualizations).

The *concept lattice* of G (sometimes also referred to as its *Galois lattice*) is $\mathsf{L}(\mathsf{G}) = (\mathsf{C}(\mathsf{G}), \succeq)$

with $(D_1, Y_1) \succeq (D_2, Y_2)$ iff $Y_1 \supseteq Y_2$ (which is provably equivalent to $D_2 \subseteq D_1$), and

$$(D_1, Y_1) \wedge (D_2, Y_2) = (\mathtt{i}_{\Im}(\mathsf{h}_{\Im}(D_1 \cup D_2)), Y_1 \cap Y_2)$$
$$(D_1, Y_1) \vee (D_2, Y_2) = (D_1 \cap D_2, \mathsf{h}_{\Im}(\mathtt{i}_{\Im}(Y_1 \cup Y_2)).$$

It is also well-known and easily shown that both $(\mathtt{i}_{\Im} \circ \mathsf{h}_{\Im}) : P(D) \to P(D)$ and $(\mathsf{h}_{\Im} \circ \mathtt{i}_{\Im}) : P(Z) \to P(Z)$ are closure operators with respect to set-inclusion (recall that a *closure operator* K on a preordered set $(Y, \geq)$ is a function $K : Y \to Y$ such that for any $y, x \in Y$: $K(y) \geq y, K(y) \geq K(x)$ whenever $y \geq x$, and $K(y) \geq K(K(y))$), and *extents* and *intents* of concepts are precisely the *closed* elements of $(\mathtt{i}_{\Im} \circ \mathsf{h}_{\Im})$and $(\mathsf{h}_{\Im} \circ \mathtt{i}_{\Im})$ respectively (i.e. $(D, Y) \in \mathsf{C}(\mathsf{G})$ iff $D = \mathtt{i}_{\Im}(\mathsf{h}_{\Im}(D))$ and $Y = \mathsf{h}_{\Im}(\mathtt{i}_{\Im}(Y))$).

The following proposition is a straightforward corollary to the fundamental theorem of formal concept analysis (see e.g. Davey and Priestley(1990), Ganter and Wille(1998)):

Proposition 12 *Let* $\mathsf{G} = (\mathcal{C}, \mathcal{Z}, \Im)$ *be a CGC. Then,* $\mathsf{L}(\mathsf{G}) = (\mathsf{C}(\mathsf{G}), \succeq)$ *as defined above is indeed a complete lattice .*

The following refined result also holds for special classes of CGCs:

Proposition 13 *Let* $\mathsf{G} = (\mathcal{C}, \mathcal{Z}, \Im)$ *be a semi-bounded CGC. Then,*
i) $\Im$ *is Nonbottom-Valued entails that* $\mathsf{L}(\mathsf{G})$ *is a* codense *complete lattice;*
ii) $\Im$ *is Nonimposed entails that* $\mathsf{L}(\mathsf{G})$ *is a* dense *complete lattice;*
iii) $\Im$ *is Ferrers entails that* $\mathsf{L}(\mathsf{G})$ *is a* chain *(i.e.* $\succeq$ *is total).*

Furthermore, a converse result to the previous propositions can also be readily established, namely:

Proposition 14 *Let* L *be a complete lattice. Then there exists a CGC* $\mathsf{G} = (\mathcal{C}, \mathcal{Z}, \Im)$ *such that* $\mathsf{L}(\mathsf{G}) \cong \mathsf{L}$*. Moreover, if* L *is a chain (is complete and codense, complete and dense, respectively) then there exists a (semi-bounded) CGC* $\mathsf{G} = (\mathcal{C}, \mathcal{Z}, \Im)$ *such that* $\Im$ *is Ferrers, (Nonbottom-Valued, Nonimposed, respectively) and* $\mathsf{L}(\mathsf{G}) \cong \mathsf{L}$.

Remark 15 *In Vannucci(1999b) it is also shown that –within the class of EF-induced finite CGCs– the CGCs whose concept lattice is a chain are precisely those which are representable by means of a pair of capacities as defined respectively on coalition and outcome-subset spaces, while the (larger) subclass of EF-induced finite CGCs with topological closure operators consist precisely of those EF-induced finite CGCs whose singleton-generated closed sets are meet-irreducible.*

Thus, it follows from the foregoing observations and results that a (complete) lattice –the concept lattice–can be attached in a most 'natural' way to each CGC. This fact opens up the opportunity to introduce 'new' classifications of CGCs from a number of interesting perspectives, relying on suitable concept lattice parameters. Of course, those parameters (such as *width, length, size, number of join and/or meet irreducibles*) provide some complexity-evaluation criteria concerning the structure of the underlying CGCs.

I recall here some relevant order- and lattice-theoretic notions. The *width* $w(\mathsf{P})$ of a poset $\mathsf{P} = (P, \geq)$ is the (common) size of its largest antichains (an *antichain* of P is a set of pairwise $\geq -$incomparable elements). The *length* $\mathfrak{l}(\mathsf{P})$ of a poset $\mathsf{P} = (P, \geq)$ is the least upper bound of the set of lengths of chains included in P (a chain is a totally ordered set; the length of a chain of $m+1$ elements is m).

Thus, the width of the concept lattice of a CGC provides some summary information on the maximum 'degree' of specialization of decision tasks that is allowed by the given CGC. By contrast, the length of the concept lattice of a CGC provides information on the number of layers of decision power induced by the latter.

In particular, the notion of *order-dimension* is made available for CGCs through their concept lattices. Indeed, let $\mathsf{L} = (L, \geq)$ be a lattice. Then, the *order dimension* $d_{\mathsf{O}}(\mathsf{L})$ of L is given by the minimum positive integer h such that there exist h chains $(L, \geq_1), .., (L, \geq_{\mathsf{h}})$ with $\geq = \cap_{\mathsf{i}=1}^{\mathsf{h}}$. Therefore, for any CGC G one may also posit $\dim \mathsf{G} = d_{\mathsf{O}}(\mathsf{L}(\mathsf{G}))$. Then, the following fact –which is easily established as an immediate corollary of a well-known result of formal concept analysis (see Ganter and Wille(1998))– entails that the order dimension of any finite CGC G can be in principle detected by direct inspection of G:

Claim 16 *Let* $\mathsf{G} = (\mathcal{C}, \mathcal{Z}, \Im)$ *be a finite CGC. Then its order dimension is given by its so-called* Ferrers dimension *i.e.*

$$\dim(\mathsf{G}) = \min \left\{ \begin{array}{c} k \in \mathsf{Z}_+ : \textit{ there exist} \\ \{\Im_{\mathsf{i}} \subseteq C \times Z, \Im_{\mathsf{i}} \textit{ is Ferrers} : i = 1, .., k\} \\ \textit{such that } \Im = \bigcap_{\mathsf{i}=1}^{\mathsf{k}} \Im_{\mathsf{i}} \end{array} \right\}$$

Summing up, concept-latticial parameters such as *width* and *length* or *order dimension* provide in a most succinct way some basic information on the characteristic degrees of decentralization, specialization and hierarchization of decision tasks among coalitions that are induced by a given distributed mechanism.

I also submit that this last circumstance might be of particular significance for some possible future developments of an *artificial-agent-supported implementation theory* : indeed, suppose one is interested in

a) implementing a certain choice correspondence F (e.g. a cooperative bargaining solution, or any other prescribed social choice rule as defined on a domain of profiles of nonverifiable individual characteristics) via a distributed mechanism, under

b) the additional constraint that the distributed mechanism is to 'faithfully' replicate the allocation of decision power embodied in the choice correspondence itself, and with

c) the opportunity to take advantage of suitably designed artificial agents (e.g. artificial 'mediators' or 'arbitrators'). Now, replicating some (standard) effectivity function of choice correspondence F within the similar effectivity function of a mechanism with extra added agents is of course hopeless. Replicating the *concept lattice of the relevant effectivity function of* F , however, is not – and seems indeed to be a sensible and attainable goal for 'artificial-agent-augmented' mechanisms.

Be it as it may, the intuitive meaning of concept latticial parameters of CGCs as outlined above suggests an analysis of the relationship of such parameters to core-stability and related properties of coalitional game forms, which are the focus of a large part of the extant literature on coalitional games. This task is best accomplished by asking– and answering– a few questions concerning *spectral properties of concept lattices of CGCs* , namely questions of the following form:

• what are the possible values of a certain integer parameter t of the concept lattice of a CGC G, when G is allowed to vary among the CGCs satisfying a given property p?

In view of the well-known fact that preconvex EF-induced CGCs are strongly core-stable, we address issues concerning certain spectral properties of their concept lattices. We have the following results:

Proposition 17 *Let N, X be finite sets such that $t = \min\{\#N, \#X\}$. Then, for any positive integer $k \leq \sum_{h=1}^{t-1}(t-h)$ there exists a $\mathcal{Z}$-monotonic preconvex $\mathsf{G} = (\mathcal{C},\mathcal{Z},\Im)$ –induced by an EF on $(P(N), P(X))$– such that $\mathfrak{l}(\mathsf{L}(\mathsf{G})) = k$*

Clearly enough, the previous proposition only provides a lower bound on the maximum concept-latticial length of a $\mathcal{Z}$-monotonic preconvex CGC. It is not known to the author whether this lower bound can be ameliorated. A neater 'positive' result is embodied in the following:

Proposition 18 *Let N, X be finite sets such that $t = \min\{\#N, \#X\}$ is odd, and let $U \in \{N, X\}$ with $\#U = t$. Then,*

for any positive integer $k \leq \#\left\{S \subseteq U : \#S = \frac{1}{2}[(\#U) + 1]\right\}$there exists a $\mathcal{Z}$-monotonic preconvex CGC G =$(\mathcal{C}, \mathcal{Z},\Im)$ –induced by an EF on $(P(N), P(X))$– such that $w(\mathsf{L}(\mathsf{G})) = k$.

Notice that for a finite set Y of odd cardinality, $\left\{S \subseteq Y : \#S = \frac{1}{2}[\#(Y) + 1]\right\}$ is an antichain of maximum size of $(P(Y), \supseteq)$ (this is indeed the content of Sperner's theorem: see e.g. Anderson(1987)). Thus, the foregoing proposition establishes that the requirement of preconvexity (hence of strong core stability) on a CGC does not entail any structural constraint on the width of the corresponding concept lattice. From the point of view of mechanism design, that proposition amounts of course to an interesting positive result. In that connection it would be interesting to explore the possibility to extend the result to the case of *maximal* $\mathcal{Z}-$monotonic preconvex CGCs.

3 Summary

Coalitional game contexts (CGCs) have been introduced and shown to encompass the standard formats of coalitional game-theoretic data structures, and extend them in two ways namely by weakening the incidence structure and generalizing the underlying coalitional and outcome space structures. Moreover, CGCs establish a precise formal connection to the mathematical logic literature in that the category of CGCs is a (nonfull) subcategory of 'classifications' or 'Chu spaces'. Concept lattices of CGCs have also been introduced and discussed. A natural notion of order-dimension of CGCs based upon their concept lattices has been presented. Some spectral properties of concept lattices of 'nice' preconvex CGCs have also been studied. In particular, it has been shown that –essentially– preconvex CGCs with a concept lattice of arbitrary width can be devised.

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References

[1] Abdou J., H. Keiding (1991): *Effectivity Functions in Social Choice.* Dordrecht: Reidel

[2] Anderson I. (1987): *Combinatorics of Finite Sets.* Oxford: Oxford University Press

[3] Andjiga N.G., J.Moulen (1989): Necessary and Sufficient Conditions for l-Stability of Games in Constitutional Form. *International Journal of Game Theory* 18, 91-110

[4] Barwise J., J. Seligman (1997): *Information Flow. The Logic of Distributed Systems.*Cambridge: Cambridge University Press

[5] Blau J.H., D.J. Brown (1989): The Structure of Neutral Monotonic Social Functions. *Social Choice and Welfare* 6, 51-61

[6] Bonanno G. (1991): Extensive Forms and Set-Theoretic Forms. *Economics Letters* 37, 363-370

[7] Davey B.A., H.A. Priestley (1990): *Introduction to Lattices and Order.* Cambridge: Cambridge University Press

[8] Ferejohn J.A., P.C. Fishburn (1979): Representations of Binary Decision Rules by Generalized Decisiveness Structures. *Journal of Economic Theory* 21, 28-45

[9] Fristrup P., H. Keiding (2001): Strongly Implementable Social Choice Correspondences and the Supernucleus. *Social Choice and Welfare* 18, 213-226

[10] Ganter B., R.Wille (1998): Applied Lattice Theory: Formal Concept Analysis, Appendix H of Grätzer(1998)

[11] Gilboa I., E. Lehrer (1991): Global Games. *International Journal of Game Theory* 20, 129-147

[12] Grätzer G. (1998): *General Lattice Theory* (Second edition). Basel: Birkhäuser Verlag

[13] Greenberg J. (1990): *The Theory of Social Situations. An Alternative Game-Theoretic Approach*. Cambridge: Cambridge University Press

[14] Ichiishi T. (1989): On Peleg's Theorem for Stability of Convex Effectivity Functions. *European Journal of Political Economy* 5, 149-160

[15] Kalai E., E.A. Pazner, D. Schmeidler (1976): Collective Choice Correspondences as Admissible Outcomes of Social Bargaining Processes. *Econometrica* 44, 233-240

[16] Moulin H., B.Peleg (1982): Cores of Effectivity Functions and Implementation Theory. *Journal of Mathematical Economics* 10, 115-145

[17] Moulin H.(1983): *The Strategy of Social Choice.* Amsterdam: North Holland

[18] Packel E.W.(1981): Semisimple Games, Enactment Power and Generalized Power Indices. *International Journal of Game Theory* 10, 137-146

[19] Peleg B. (1984): *Game Theoretic Analysis of Voting in Committees.* Cambridge: Cambridge University Press

[20] Pratt V. (1999): *Chu Spaces*. Notes for the School on Category Theory and Applications, University of Coimbra

[21] Rosenthal R.W. (1972): Cooperative Games in Effectiveness Form. *Journal of Economic Theory* 5, 88-101

[22] Shubik M. (1982): *Game Theory in the Social Sciences. Concepts and Solutions.* Cambridge MA: MIT Press

[23] Vannucci S. (1999a): On a Lattice-Theoretic Representation of Coalitional Power in Game Correspondences, in H.de Swart (ed.): *Logic, Game Theory and Social Choice.* Tilburg: Tilburg University Press

[24] Vannucci S. (1999b): On the Galois Lattice of an Effectivity Function. University of Siena DEP Discussion Paper 242