QUADERNI

Università degli Studi di Siena
DIPARTIMENTO DI ECONOMIA POLITICA

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n. 332 – Novembre 2001

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November 2001

Abstract

We compare opportunity set distributions by means of set-inclusion filtral preorders (SIFPs). Some significant results of the classic theory of income inequality are reproduced in the SIFP-framework.
Keywords: Filtral Preorders, Inequality, Opportunity Sets
JEL classification: D31, D63, I31.

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1 Introduction

The present paper is devoted to the problem of ranking distributions of opportunity sets in terms of inequality starting from a certain *partial* opportunity preorder.

There are a few good reasons for taking interest in social evaluation criteria which rely on some notion of available *freedom of choice* or *opportunities.* Reference to *opportunity* is arguably the most suitable way to introduce a requirement of *flexibility* intended as the ability of an agent to accommodate the dictates of an entire set of possible, and plausible, preferences. Moreover, there is some hope that opportunity-based criteria might provide a more robust and objective basis for interpersonal comparisons of well-being than typical preference-based criteria do. In fact, it is at least conceivable that a suitably tailored definition of freedom of choice/opportunity might eventually help to operationalize some nicely robust notion of *social and economic progress.* But of course it remains to be agreed how comparisons in terms of *freedom of choice* or *opportunity* should be precisely defined. And difficulties are compounded if the issue of *opportunity inequality* is to be addressed: while powerful methods for measuring income inequality are known, the literature on inequality-based comparisons of distributions of rights, freedom, primary goods etc. is rather thin. The problem is inherently complex, and extending the standard principles and measures from univariate (income) to multivariate cases (opportunities) is far from being a straightforward task.

As it happens, the first main result provided by the axiomatic approach as proposed by Pattanaik and Xu (1990) has shown that a simple set of mild-looking requirements dictate the cardinality preorder, which is commonly rejected as trivial, as the unique possible choice for ranking opportunity sets. Some further results on inequality rankings of opportunity profiles due to Ok(1997) and Ok and Kranich(1998) *seem to imply that -in a sense- it is essentially impossible to develop a measurement theory of opportunity inequality which is analytically analogous to that of income inequality without using the cardinality preorder.* Indeed, the foregoing results are widely regarded as 'impossibility results' on freedom-of-choice-based opportunity rankings. However, we feel that such an interpretation of the Pattanaik-Xu and related theorems should be firmly resisted.

What are then, if any, the viable alternatives to the cardinality preorder?

Generally speaking we suggest that an acceptable notion of 'freedom of choice' should be at least:

·tolerably consistent with common usage;

·amenable both to different specifications by different people *and* to suitable amalgamation procedures

·conducive to a fruitful approach to the issue of opportunity inequality.

In that connection, and following Vannucci (2001), we propose a class of minimal extensions of the set-inclusion partial order whose members we shall refer to as *(set-inclusion) filtral preorders.*[1]

A set-inclusion filtral preorder on a (finite nonempty) set X of basic alternatives/opportunities is an elementary way to augment the set-inclusion partial order with a *minimum opportunity-threshold*: under the threshold, opportunity sets are indifferent to each other and to the null opportunity set, while over the threshold the set-inclusion partial order is simply replicated. A set-inclusion filtral preorder (henceforth SIFP), embodies both the idea of 'many' degrees of freedom which is typical of those traditions which emphasize 'positive' freedom and, thanks to the threshold, the notion that the matter of freedom can also be in some respect an *all-or-nothing* issue, which is the standard creed of supporters of a 'negative' view of freedom.

Moreover, SIFPs have been shown to provide a *rich* (in the sense of Dasgupta,Hammond, and Maskin (1979)), restricted domain of preorders on the set of opportunity sets of X, and to be amenable to nice Arrowian aggregation procedures, including Simple Majority Voting, thanks to their distributive latticial structure (see Monjardet(1990), Vannucci(2001)). As a result, a SIFP can always be regarded as the outcome of a nice social choice protocol.

This paper then develops a preliminary test of filtral preorders on the assessment of *opportunity inequality.* We propose to rely on SIFPs in order to define a new method of ranking *profiles of opportunity sets* in terms of *opportunity inequality.* Then, we explore to what extent SIFPs are able to support an interesting counterpart of the classic theory of income inequality. We will show that indeed some significant fragments of the latter can be reproduced in our framework.

2 Notation and definitions

Let X denote the set of alternatives/opportunities of the population of N agents and $\wp(X)$ the power set of X, i.e. the set of its opportunity sets. Assumed that $\#X \geq 3$ (with # that indicates the cardinality of the set), in order to avoid trivial qualifications and define a binary relation $\succcurlyeq$ on $\wp(X)$ which extends the set inclusion ordering, namely $A \supseteq B$ entails $A \succcurlyeq B$ for

[1] See Vannucci (2001) for more details.

all $A, B \in \wp(X)$. We are interested in a class of opportunity rankings that arise whenever:

1. all the alternatives are "good";
2. a minimum standard (threshold) for opportunity sets is considered: that standard works as a freedom- poverty line of sorts, below which no opportunity set is valuable;
3. opportunity sets are ranked in such a way that set-inclusion is preserve.

We shall rely heavily on the following

Definition 1 (Order filter of a poset) *Let $(X, \succcurlyeq)$ be a non-empty preorder set (poset). An order filter of $(X, \succcurlyeq)$ is a set $F \subseteq X$ such that for any A, B, if $A \in F$ and $B \succcurlyeq A$ it entails $B \in F$.*

Remark 1 *Whenever X is finite, an order filter F of $(X, \succcurlyeq)$ is uniquely determined by a finite family $\mathcal{B}_F = \{B_1, .., B_l\}$ of subsets of X such that, in fact, $F = \{C \subseteq X :$ there exists $i \in \{1, .., l\}$ such that $C \supseteq B_i\}$.*

The family $\mathcal{B}_F$ is also called the basis *of F, and the B_is are said to be the* generators *of F. It should also be remarked that $\mathcal{B}_F$ is an* antichain *of $(\wp(X), \supseteq)$ namely for any $B_i, B_j \in \mathcal{B}_F$ if $i \neq j$ then $B_i \# B_j$. Moreover, there is a one-to-one correspondence between order filters and antichains of $(\wp(X), \supseteq)$.*

Definition 2 (Set-Inclusion Filtral Preorder (SIFP)) *A binary relation $\succcurlyeq_F$ on $\wp(X)$ is* set-inclusion filtral preorder (SIFP), *if*

$$A \succcurlyeq_F B \quad \text{for any } A, B \in \wp(X) \text{ and some order filter } F$$

means that either $A \supseteq B$ or $B \notin F$.[2]

In particular, we may conveniently consider X as a set of discrete items/goods, while $A, B, C \in \wp(X)$ represent opportunity sets of such "goods". Introducing a *filtral extension* of the set-inclusion poset amounts to the requirement of a minimum level of freedom which is regarded as essential for the well-being of agents. Below such a minimum standard threshold of opportunities, we can no longer think of the relevant opportunity sets in terms of individual *freedom*.

[2] See Vannucci (2001) for more details concerning motivation and interpretation of SIFPs.

As mentioned in the Introduction, the main aim of the present paper is to propose a *SIFB*-based method of ranking profiles of opportunity sets in terms of opportunity inequality (see e.g. Kranich (1996), Ok (1997), Ok and Kranich (1998), Herrero, Iturbe-Ormiaetxe and Nieto (1998), Arlegi, Nieto (1999) and Weymark (2001) for some alternative approaches).

In order to accomplish the foregoing task we have to introduce the following

Definition 3 *Let F be an order filter of $(\wp(X), \supseteq)$ and $\succcurlyeq_F$ the filtral preorder induced by F. Then, the $\succcurlyeq_F$-*induced height function $h_{\succcurlyeq_F} : P(X) \to \mathbb{Z}_+$ *is defined as follows: for any $A \subseteq X$,*

$$h_{\succcurlyeq_F}(A) = \max \left\{ \begin{array}{c} \#\mathcal{C} : \mathcal{C} \text{ is a } \succcurlyeq_F \text{-chain, such that} \\ A \in \mathcal{C} \text{ and } A \succcurlyeq_F B \text{ for any } B \in \mathcal{C} \end{array} \right\}.$$

In words, the height function assigns to each opportunity set A a rank-number of sorts as measured by the longest chain consisting of A and other opportunity sets which are ranked below it.

We are now ready to outline a description of our approach to the issue of inequality-ranking of opportunity profiles.

The procedure we propose goes as follows:

1. take a SIFP on $(\wp(X), \succcurlyeq_F)$;
2. consider the $\succcurlyeq_F$-induced height function $h_{\succcurlyeq_F}$
3. define a generalized $\succcurlyeq_F$-induced majorization preorder $\succcurlyeq_F^M$ on the set $(\wp(X))^N$ of N-profiles of opportunity sets.

   Afterwards, we shall define and discuss a related transfer operator.

In order to proceed, let us now define the counterpart of the classic majorization preorder in our framework:

Definition 4 *Let $\mathsf{A}, \mathsf{B} \in (\wp(X))^N$ be two opportunity profiles, F an order filter of $(\wp(X), \supseteq)$, $\succcurlyeq_F$ the corresponding filtral preorder on $\wp(X)$, and $h_{\succcurlyeq_F}$-induced height function on $\wp(X)$. Then A $F-$majorizes B, written $\mathsf{A} \succcurlyeq_F^M \mathsf{B}$, if*

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$$

and

$$(h_{\succcurlyeq_F}(A_1), ..., h_{\succcurlyeq_F}(A_n)) \succcurlyeq^M (h_{\succcurlyeq_F}(B_1), ..., h_{\succcurlyeq_F}(B_n)), \qquad (1)$$

where $\succcurlyeq^M$ *denotes the standard majorization preorder, i.e. for any* $\mathsf{z} = (z_1, .., z_n)$, $\mathsf{y} = (y_1, .., y_n) \in \mathsf{Z}_+^N$:

$$\mathsf{z} \succcurlyeq^M \mathsf{y}$$

if and only if

$$\sum_{h=1}^{k} (\mathsf{z}\uparrow)_h \leq \sum_{h=1}^{k} (\mathsf{y}\uparrow)_h, \qquad h = 1, .., n-1$$

and

$$\sum_{h=1}^{n} (\mathsf{y}\uparrow)_h = \sum_{h=1}^{n} (\mathsf{z}\uparrow)_h,$$

where, for any $\mathsf{v} = (v_1, .., v_n) \in \mathsf{Z}_+^N$: $\mathsf{v}\uparrow = (v_{\sigma(1)}, .., v_{\sigma(n)})$ *with* $\sigma : N \longrightarrow N$ *a permutation such that* $\sigma(i) \geq \sigma(j)$, *entails* $v_i \geq v_j$ *).*

Remark 2 *Noticed that in general the foregoing procedure and the aggregation procedure, e.g. simple majority, possibly underlying* $(\wp(X), \succcurlyeq_F)$ *do* not *commute. In particular, SIFPs arising from different* $(\wp(X), \succcurlyeq_F)$ *need not be amenable to 'nice' aggregation procedures.*

One characteristic of $(P(X)^N, \succcurlyeq_F^M)$ is that it works by mapping the space of opportunity profiles into a 'small' set of integer points in Z_+^N i.e. the space of height vectors. This set will of course depend on the relevant order filter F and is therefore denoted as the (height) *span* of $\succcurlyeq_F$, written $H_{\succcurlyeq_F}$. In what follows we shall be mainly interested in the *positive (height) span* of $\succcurlyeq_F$i.e. $H^+_{\succcurlyeq_F} = H_{\succcurlyeq_F} \cap \mathsf{Z}_{++}^N$. Such a span is amenable to a quite simple characterization.

Definition 5 *Let* F *be an order filter of* $(\wp(X), \supseteq)$ *and* $\mathcal{B}_F = \{B_1, .., B_l\}$ *the basis of* F*. Then, the* capacity number $c(F)$ of F *is the maximum number of pairwise disjoint sets in* $\mathcal{B}_F$ *i.e.*

$$c(\mathcal{B}_F) = \max \left\{ \begin{array}{c} \#I : I \subseteq \{1, ..l\}, \ B_i \in \mathcal{B}_F \text{ for all } i \in I, \\ \text{and } B_i \cap B_j = \emptyset \text{ for any } i, j \in I \text{ with } i \neq j \end{array} \right\}.$$

The capacity number $c(\mathcal{B}_F)$ *of* $\mathcal{B}_F$ *is also said to be the* capacity number of F.

Now, the following proposition can be immediately established:

Proposition 1 *Let F be an order filter of $(\wp(X), \supseteq)$ with basis $\mathcal{B}_F$ and capacity number $k \geq \# N \geq 3$, and $(\wp(X), \succcurlyeq_F)$ the corresponding filtral preorder. Then,*

$$H^+_{\succcurlyeq_F} = \left\{ \begin{array}{c} \mathbf{z} \in \mathbf{Z}^N_{++} : \textit{there exists a pairwise disjoint subbasis} \\ \mathcal{B}' = \{B_1, .., B_n\} \subseteq \mathcal{B}_F \textit{ such that} \\ \sum_{i \in N} z_i = n + \#(X \setminus \bigcup_{i=1}^n B_i) \end{array} \right\}.$$

Proof. Let $\mathbf{z} \in H^+_{\succcurlyeq_F}$ with

$$F = \{A \subseteq X : \text{there exists } B \in \{B_1, .., B_l\} \text{ such that } A \supseteq B\}$$

i.e. $\mathcal{B}_F = \{B_1, .., B_l\}$.

Then, by definition, it must exist an ordered partition $(A_1, .., A_n)$ of X such that

$$\mathbf{z} = (h_{\succcurlyeq_F}(A_1), .., h_{\succcurlyeq_F}(A_n)) \in \mathbf{Z}^N_+.$$

Now, since $h_{\succcurlyeq_F}(A_i) \geq 1$ for each $i \in N$ and by definition of h and $\succcurlyeq_F$there exists a function

$$t : \{1, .., n\} \rightarrow \{1, .., l\}$$

such that $A_i \supseteq B_{t(i)}$, $i = 1, .., l$. Thus, $B_{t(i)} \cap B_{t(j)} = \emptyset$ for any $i, j \in N$ with $i \neq j$.

Moreover, $h_{\succcurlyeq_F}(A_i) = z_i$ entails that $\#(A_i \setminus B_{t(i)}) = z_i - 1$, for any $i \in N$. It follows that

$$\sum_{i \in N} z_i = n + \#(X \setminus \bigcup_{i=1}^n B_{t(i)})$$

as required.

Conversely, let $\mathbf{z} \in \mathbf{Z}^N_{++}$ and $\mathcal{B}' = \{B_1, .., B_n\} \subseteq \mathcal{B}_F$ a pairwise disjoint subbasis of F such that $\sum_{i \in N} z_i = n + \#(X \setminus \bigcup_{i=1}^n B_i)$.

Now, let us define an ordered partition $(A_1, .., A_n)$ as follows: (i) define a profile $(C_i \subseteq X)_{i \in N}$ such that for any $i \in N$, $\# C_i = z_i - 1$ and for any $i, j \in N$ with $i \neq j$ $C_i \cap C_j = \emptyset$. Such a profile must exist since $\#(X \setminus \bigcup_{i=1}^n B_i) = \sum_{i \in N} z_i - n)$.

(ii) posit $A_i = B_i \cup C_i$ for each $i \in N$. Then, observe that by definition $h_{\succcurlyeq_F}(A_i) = z_i$ for all $i \in N$. ■

The foregoing Proposition shows that the positive height span of $\succcurlyeq_F$is far from being as regular as one might perhaps like. This is however hardly an unexpected phenomenon. In fact, that is essentially due to the threshold effect which is by definition embodied in the very notion of a filtral preorder.

3 Equalizing Transfer Operators on Opportunity Distributions Ranked by means of a Filtral Preorder

We shall now address the problem of defining a suitable notion of *transfer* with respect to our ranking of opportunity distributions. As shown by Ok (1997) a well-behaved *weakly equalizing transfer operator* can be defined w.r.t. the *cardinality-based* total preorder. However, as mentioned above, the cardinality preorder is usually rejected as trivial, which implies that the Ok's result is also usually regarded as an impossibility-type result. Thus, in order to develop a theory of opportunity inequality along the lines of the classic theory of income inequality, we explore a particular *partial* preordering of opportunity sets which extends the Pigou-Dalton transfer principle to the case of opportunity distributions or profiles. Our aim is to define in the present context a notion of transfers as general as possible.

We imagine the opportunities under consideration as abstract *items.* We distinguish among elements of X on the basis of their *rivalry.* For instance, universal access to vote is not a rivalrous opportunity, because to extend the possibility of voting to an additional individual would not decrease the possibility of any other agent. If a transfer from individual 1 to 2 gives access to agent 2 to a new set of opportunities while excluding these opportunities from individual 1, who previously possessed them, we can consider that situation on a par with discrete money transfers: a certain similarity to the setting of income inequality measurement can be immediately noticed.

We define the set of all admissible opportunity distributions in a n-agent society as $(\wp(X))^N$. We denote with $\mathsf{A} = (A_i)_{i\in N}$ a generic opportunity profile or distribution belonging to $(\wp(X))^N$. Hence, for any $i \in N$, A_i denotes the set of opportunities of the agent (or group of agents) i in the distribution A.

Definition 6 *A transfer operator on* $(\wp(X))^N$ *is a nonempty correspondence* $\Im : (\wp(X))^N \rightrightarrows (\wp(X))^N$ *such that*

$$\forall (\mathsf{A}, \mathsf{B}) \in (\wp(X))^N \times (\wp(X))^N,\ \mathsf{B} \in \Im(\mathsf{A}) : \left[\bigcup_i A_i = \bigcup_i B_i\right]$$

A transfer operator then is a transformation which leaves the set of total opportunities in A and B unchanged. By similarity with the Pigou-Dalton principle, we furthermore require that the transfer of opportunity sets must not be so large as to more then reverse the relative positions of the receivers. Moreover, as X is the set of all alternatives/objects, we are going to transfer indivisible items. Let $\mathsf{A}, \mathsf{B} \in (\wp(X))^N$ be two discrete opportunity distributions and $A_k >_F A_j$ two components of A, ranked by an order filter F, then

$$\begin{aligned} B_k &= A_k \backslash \{x\} \\ B_j &= A_j \cup \{x\} \\ B_i &= A_i \quad i \neq k, j \end{aligned} \tag{2}$$

is called a *simple transfer* from k to j, for $\{x\}$ a available option.

In such a case the distribution B is obtained from the opportunities distribution A simply by a transfer of an item from a richer (in terms of opportunities) group of individuals to poorer ones. It is well-known, thanks to a a classic result due to Muirhead(1903) , that it is possible to obtain the opportunity distribution B from A throughout a finite sequence of simple transfers as defined above, which minimally alter the initial distribution (see Marshall and Olkin(1979)).

In the context of income inequality, we requires that the transfer must not reverse the relative positions of the donor and receiver. Here we introduce a further limiting condition. The transfer of an *item* must not be such that the donor crosses the threshold induced by the relevant order filter F.

Definition 7 *Let $\Im$ be a transfer operator on $\wp(X)^N$. $\Im$ is called a* filter concerned transfer *wrt $\succcurlyeq_F$ if, for $\mathsf{A}, \mathsf{B} \in (\wp(X))^N$,*

$$\forall \mathsf{B} \in \Im(\mathsf{A}) \text{ and } \forall i \in N : A_i \in F \text{ implies } B_i \in F \tag{3}$$

(It should be recalled here that in our framework the relevant threshold consists in the (set-inclusion) antichain which form the basis of the (set-inclusion) order filter F :see Remark 1 above in the text).

Remark 3 *Definition 7 above implies properties 3.3[(i); (ii)] of weakly equalizing transfers as defined by Ok (1997), while the converse does not hold. $\mathsf{B} \in \Im(\mathsf{A})$ indeed implies $[A_k \geq A_j \Rightarrow \{B_j \geq A_j$ and $A_k \geq B_k\}]$, but as a total preorder $\succcurlyeq$ on $(\wp(X))^N$ need not be a filtral preorder $\succcurlyeq_F$, a weakly equalizing transfer in Ok's sense may not be a weakly equalizing transfer in our own sense.*

Remark 4 *Definition 7 does not guarantee the connectedness property of definition 3.3 [(iii)] in Ok (1997) as the following example shows. Suppose $X = \{a, b, c, d, e, f\}$; $F = \{\{a, b\}, \{c, d\}, \{a, c, e\}\}$. Let us take two* *opportunity profiles* A *and* B *such that* $\mathsf{A} = (\{a,b\},\{c,d\},\{e,f\})$ *and* $\mathsf{B} = (\{b\},\{d\},\{a,c,e,f\})$*. We can obtain* A *from* B *through a sequence of two* $\Im$-transforms*, but we cannot make distribution* B *perfectly egalitarian as the items in* X *are rivalrous in consumption.*

In the rest of this paper we shall consider transfer sequences, which are defined in an obvious way as follows. Let $\Im$ be a transfer operator, then, for any positive integer t and any $\mathsf{A} \in (\wp(X))^N$ we define inductively $\Im^{(t)}(\mathsf{A}) = \Im(\Im^{(t-1)}(\mathsf{A}))$.

Definition 8 *A simple transfer operator* $\Im$ *is* weakly rank-monotonic *w.r.t* $\succcurlyeq_F$ *if and only if it does not cause height-reversals i.e. for any* $\mathsf{A}, \mathsf{B} \in (\wp(X))^N$ *and any* $i, j \in N$*, if*

$$\mathsf{B} \in \Im(\mathsf{A}), B_i \neq A_i, B_j \neq A_j$$

and

$$h_{\succcurlyeq_\mathsf{F}}(A_i) \geq h_{\succcurlyeq_\mathsf{F}}(A_j)$$

then

$$h_{\succcurlyeq_\mathsf{F}}(B_i) \geq h_{\succcurlyeq_\mathsf{F}}(B_j).$$

Definition 9 *A simple transfer operator* $\Im$ *is* weakly equalizing *w.r.t.* $\succcurlyeq_F$*if and only if for any* $\mathsf{A}, \mathsf{B} \in (\wp(X))^N$*:*

$$\mathsf{B} \in \Im(\mathsf{A}),\ B_i \supset A_i \text{ and } A_j \supset B_j$$

entails that

$$h_{\succcurlyeq_\mathsf{F}}(A_j) \geq h_{\succcurlyeq_\mathsf{F}}(A_i).$$

Definition 10 *A simple transfer operator* $\Im$ *is said to be* Daltonian w.r.t. $\succcurlyeq_F$ *if it is both weakly rank-monotonic w.r.t.* $\succcurlyeq_F$*and weakly equalizing w.r.t.* $\succcurlyeq_F$*.*

Proposition 2 *Let* $\mathsf{A}, \mathsf{B} \in (\wp(X))^N$ *be two opportunity profiles such that* $\{h_{\succcurlyeq_\mathsf{F}}(\mathsf{A}), h_{\succcurlyeq_\mathsf{F}}(\mathsf{B})\} \subseteq H^+_{\succcurlyeq_\mathsf{F}}$ *and* $\mathsf{A} \succ^M_F \mathsf{B}$*. Then there exist a* $\succcurlyeq_F$*-Daltonian transfer operator* $\Im$ *and a finite integer* k *such that* $\mathsf{B} \in \Im^{(k)}(\mathsf{A})$.

Proof. Let us consider $\mathsf{A}, \mathsf{B} \in (\wp(X))^N$ such that

$$\{h_{\succcurlyeq_\mathsf{F}}(\mathsf{A}), h_{\succcurlyeq_\mathsf{F}}(\mathsf{B})\} \subseteq H^+_{\succcurlyeq_\mathsf{F}}$$

and $\mathsf{A} \succ_F^M \mathsf{B}$. Then, consider the arrangements in increasing order of $h_{\succcurlyeq_\mathsf{F}}(\mathsf{A})$ and $h_{\succcurlyeq_\mathsf{F}}(\mathsf{B})$ i.e. $(h_{\succcurlyeq_\mathsf{F}}(\mathsf{A}))\uparrow$ and $(h_{\succcurlyeq_\mathsf{F}}(\mathsf{B}))\uparrow$, respectively. We also denote by $\mathsf{A}\uparrow, \mathsf{B}\uparrow$ the corresponding arrangements of A and B.

Of course,

$$\sum_{i=1}^{k}((h_{\succcurlyeq_\mathsf{F}}(\mathsf{B}))\uparrow)_i \geq \sum_{i=1}^{k}((h_{\succcurlyeq_\mathsf{F}}(\mathsf{A}))\uparrow)_i$$

for any $k \in \{1,..,n\}$ and there must exist one largest integer $j^* \in \{1,..,n\}$ such that

$$\sum_{i=1}^{j^*}((h_{\succcurlyeq_\mathsf{F}}(\mathsf{B}))\uparrow)_i > \sum_{i=1}^{j^*}((h_{\succcurlyeq_\mathsf{F}}(\mathsf{A}))\uparrow)_i.$$

Hence

$$((h_{\succcurlyeq_\mathsf{F}}(\mathsf{A}))\uparrow)_{j^*+1} > ((h_{\succcurlyeq_\mathsf{F}}(\mathsf{B}))\uparrow)_{j^*+1}.$$

Moreover, there must exist one largest integer $i^* \leq j^*$ such that

$$((h_{\succcurlyeq_\mathsf{F}}(\mathsf{B}))\uparrow)_{i^*} > ((h_{\succcurlyeq_\mathsf{F}}(\mathsf{A}))\uparrow)_{i^*}.$$

It follows that:

$$((h_{\succcurlyeq_\mathsf{F}}(\mathsf{A}))\uparrow)_{j^*+1} > ((h_{\succcurlyeq_\mathsf{F}}(\mathsf{B}))\uparrow)_{j^*+1} \geq ((h_{\succcurlyeq_\mathsf{F}}(\mathsf{B}))\uparrow)_{i^*} > ((h_{\succcurlyeq_\mathsf{F}}(\mathsf{A}))\uparrow)_{i^*}$$

hence

$$((h_{\succcurlyeq_\mathsf{F}}(\mathsf{A}))\uparrow)_{j^*+1} - ((h_{\succcurlyeq_\mathsf{F}}(\mathsf{A}))\uparrow)_{i^*} \geq 2$$

which in turn implies that there exist $B', B'' \in \mathcal{B}_F$ such that

$$(\mathsf{A}\uparrow)_{j^*+1} \supseteq B', (\mathsf{A}\uparrow)_{i^*} \supseteq B''$$

and

$$\#((\mathsf{A}\uparrow)_{j^*+1} \setminus B') - \#((\mathsf{A}\uparrow)_{i^*} \setminus B'') \geq 2.$$

Next take $x \in ((\mathsf{A}\uparrow)_{j^*+1} \setminus B') \setminus ((\mathsf{A}\uparrow)_{i^*} \setminus B'')$. By definition, $x \in ((\mathsf{A}\uparrow)_{j^*+1} \setminus (\mathsf{A}\uparrow)_{i^*})$ since A is an (ordered) partition of X. Then take a simple transfer operator $\Im$ such that

$$\mathsf{A}^* = (A_i^*)_{i\in N} \in \Im(\mathsf{A})$$

where

$$\mathsf{A}^*{}_i = (A\uparrow)_{j^*+1} \setminus \{x\}$$

for some i with

$$A_i = (\mathsf{A} \uparrow)_{j^*+1}\,,\; A_j^* = (\mathsf{A} \uparrow)_{i^*} \cup \{x\}$$

for some j with $A_j = (\mathsf{A} \uparrow)_{i^*}$ and $A^*{}_h = A_h$ for each $h \in N \setminus \{i, j\}$.

Clearly, by construction, $\Im$ is $\succcurlyeq_F$-Daltonian.

Also, it is easily checked that $\mathsf{A} \succ_F^M \mathsf{A}^* \succ_F^M \mathsf{B}$. Thus, the thesis follows by a repeated application of the foregoing procedure. ■

Remark 5 *Notice that the foregoing Proposition is strictly related– but does not reduce – to the classic result on integer majorization due to Muirhead (see e.g. Marshall and Olkin(1979)) . Here transfers involve 'objects' or 'opportunities' and are only indirectly reflected by numbers. Hence, a remote control problem of sorts concerning transfers is to be faced: thus, one has to double check that numbers always change in the 'right' direction.*

Indeed, the foregoing result does not extend to the entire height-span $H_{\succcurlyeq_F}$,due to the characteristic threshold effect induced by $\succcurlyeq_F$. To see this, consider the following

Example 1 *Take* $X = \{a, b, c, d, e, f\}$,

$$F = \left\{ \begin{array}{c} Y \subseteq X : \textit{there exists } C \in \{\{a, b\}, \{c, d\}, \{a, c, e\}\} \\ \textit{such that } Y \supseteq C \end{array} \right\}$$

$$\mathsf{A} = (\{a, b\}, \{c, d\}, \{e, f\}),\; \mathsf{B} = (\{a, c, e, f\}, \{b\}, \{d\}).$$

Then, clearly $h_{\succcurlyeq_F}(\mathsf{A}) = (1, 1, 0)$ *and* $h_{\succcurlyeq_F}(\mathsf{B}) = (2, 0, 0)$ *hence* $\mathsf{B} \succ_F^M \mathsf{A}$*. However, it is immediately checked that any simple (weakly) equalizing transfer from* B *can only transform one of the two zeros into a positive number by bringing the height of the first opportunity set from* 2 *to* 0 *hence by violating the weak-rank-monotonicity requirement.*

Let $\mathsf{A}, \mathsf{B} \in (\wp(X))^N$ be two opportunity distributions whose components are ranked in increasing order by some filtral preorder $\succcurlyeq_F$, namely $A_n \succcurlyeq_F ... \succcurlyeq_F A_1$and $B_n \succcurlyeq_F ... \succcurlyeq_F B_1$, and for any $\mathsf{C} \in (\wp(X))^N$ and $x \in C_i$ denote by $\mathsf{C} \setminus \{x\}_i$ the opportunity profile C' such that $C'_i = C_i \setminus \{x\}$ and $C'_j = C_j$ for any $j \in N \setminus \{i\}$.

Then, the following proposition holds:

Proposition 3 *For any* $\mathsf{A}, \mathsf{B} \in (\wp(X))^N$ *and any* $x \in X$*, if* $\mathsf{B} \succ_F^M \mathsf{A}$ *for some order filter* F *and* $j \leq i$*, then*

$$\mathsf{B} \setminus \{x\}_j \succ_F^M \mathsf{A} \setminus \{x\}_i$$

Proof. The vectors of opportunities $\mathsf{A} \setminus \{x\}_i$ and $\mathsf{B} \setminus \{x\}_j$ may not have components in increasing magnitude, but if $i' \leq i$ is chosen so that

$$h_{\succcurlyeq_\mathsf{F}}\left(A_{j'}\right) = h_{\succcurlyeq_\mathsf{F}}\left(A_{j'+1}\right) = \ldots = h_{\succcurlyeq_\mathsf{F}}\left(A_i\right) \text{ and}$$
$$\text{either } h_{\succcurlyeq_\mathsf{F}}\left(A_{j'}\right) > h_{\succcurlyeq_\mathsf{F}}\left(A_{j'-1}\right) \text{ or } i' = 1$$

then $\mathsf{A} \setminus \{x\}_{i'}$ has the components of $\mathsf{A} \setminus \{x\}_i$ reordered increasingly. Similarly for $\mathsf{B} \setminus \{x\}_i$. Rather than showing $\mathsf{B} \setminus \{x\}_j \succ_F^M \mathsf{A} \setminus \{x\}_i$, it is more convenient to show the equivalent fact that

$$G \equiv \mathsf{B} \setminus \{x\}_{j'} \succ_F^M \mathsf{A} \setminus \{x\}_{i'} \equiv F.$$

For $k > \min\{i', j'\}$, we have

$$\sum_{\alpha=1}^{k} h_{\succcurlyeq_\mathsf{F}}(F_\alpha) = \sum_{\alpha=1}^{k} h_{\succcurlyeq_\mathsf{F}}(A_\alpha) \geq \sum_{\alpha=1}^{k} h_{\succcurlyeq_\mathsf{F}}(B_\alpha) = \sum_{\alpha=1}^{k} h_{\succcurlyeq_\mathsf{F}}(G_\alpha)$$

for $k \leq \max\{i', j'\}$,

$$\sum_{\alpha=1}^{k} h_{\succcurlyeq_\mathsf{F}}(F_\alpha) = \sum_{\alpha=1}^{k} h_{\succcurlyeq_\mathsf{F}}(A_\alpha) \setminus \{x\} \geq \sum_{\alpha=1}^{k} h_{\succcurlyeq_\mathsf{F}}(B_\alpha) \setminus \{x\} = \sum_{\alpha=1}^{k} h_{\succcurlyeq_\mathsf{F}}(G_\alpha);$$

and for $k = n$

$$\sum_{\alpha=1}^{k} h_{\succcurlyeq_\mathsf{F}}(F_\alpha) = \sum_{\alpha=1}^{k} h_{\succcurlyeq_\mathsf{F}}(A_\alpha) \setminus \{x\} = \sum_{\alpha=1}^{k} h_{\succcurlyeq_\mathsf{F}}(B_\alpha) \setminus \{x\} = \sum_{\alpha=1}^{k} h_{\succcurlyeq_\mathsf{F}}(G_\alpha).$$

If $i' \geq j'$, then we have that for $i' \geq k > j'$,

$$\sum_{\alpha=1}^{k} h_{\succcurlyeq_\mathsf{F}}(F_\alpha) = \sum_{\alpha=1}^{k} h_{\succcurlyeq_\mathsf{F}}(A_\alpha) \setminus \{x\} > \sum_{\alpha=1}^{k} h_{\succcurlyeq_\mathsf{F}}(B_\alpha) \setminus \{x\} = \sum_{\alpha=1}^{k} h_{\succcurlyeq_\mathsf{F}}(G_\alpha).$$

It remains to show, for the case that $i' < j'$ and $j' \geq k > i'$, that

$$\sum_{\alpha=1}^{k} h_{\succcurlyeq_\mathsf{F}}(F_\alpha) \geq \sum_{\alpha=1}^{k} h_{\succcurlyeq_\mathsf{F}}(G_\alpha).$$

Notice that $\sum_{\alpha=1}^{k} h_{\succcurlyeq_F}(F_\alpha) \geq \sum_{\alpha=1}^{k} h_{\succcurlyeq_F}(G_\alpha)$ is equivalent to $\sum_{\alpha=1}^{k} h_{\succcurlyeq_F}(A_\alpha) \geq \sum_{\alpha=1}^{k} h_{\succcurlyeq_F}(B_\alpha)$. If $h_{\succcurlyeq_F}(A_{k+1}) < h_{\succcurlyeq_F}(B_{k+1})$, then $\sum_{\alpha=1}^{k}(h_{\succcurlyeq_F}(B_\alpha) - h_{\succcurlyeq_F}(A_\alpha)) > \sum_{\alpha=1}^{k+1}(h_{\succcurlyeq_F}(B_\alpha) - h_{\succcurlyeq_F}(A_\alpha)) \geq 0$, the last inequality holds because $A \prec_F^M B$. The remaining case is $h_{\succcurlyeq_F}(A_{k+1}) \geq h_{\succcurlyeq_F}(B_{k+1})$. Because $i \geq j \geq j' \geq k > i'$,

$$h_{\succcurlyeq_F}\left(A_{j'}\right) = ... = h_{\succcurlyeq_F}(A_{k+1}) \geq h_{\succcurlyeq_F}(B_{k+1}) \geq h_{\succcurlyeq_F}(B_k) \geq ... \geq h_{\succcurlyeq_F}\left(B_{j'+1}\right) > h_{\succcurlyeq_F}(B_j);$$

this yields

$$0 > \sum_{\alpha=j'}^{k}(h_{\succcurlyeq_F}(B_\alpha) - h_{\succcurlyeq_F}(A_\alpha)) \geq \sum_{\alpha=j'}^{k}(h_{\succcurlyeq_F}(B_\alpha) - h_{\succcurlyeq_F}(A_\alpha)) +$$

$$-\sum_{\alpha=j'}^{n}(h_{\succcurlyeq_F}(B_\alpha) - h_{\succcurlyeq_F}(A_\alpha)) = -\sum_{\alpha=k+1}^{n}(h_{\succcurlyeq_F}(B_\alpha) - h_{\succcurlyeq_F}(A_\alpha)) =$$

$$= \sum_{\alpha=1}^{k}(h_{\succcurlyeq_F}(B_\alpha) - h_{\succcurlyeq_F}(A_\alpha))$$

■

In plain words, the foregoing proposition states that the majorization $B \succ_F^M A$ is preserved if an opportunity x is subtracted from a component of each opportunity-set profile. As subtraction may alter the ordering of the components of the profiles, the result is by no means trivial. Notice that such a preservation of majorization need not hold without the condition that the components of the distributions involved are made up of indivisible items. In fact, Proposition 3 captures an interesting aspect of the intuitive notion of opportunity. Subtracting an option from an opportunity set contributes to reducing the amount of freedom of choice attached to the latter, independently of which individual suffers from this loss in options she had previously access to.

4 Concluding remarks

The results of the present paper on SIFP-based inequality rankings of opportunity profile are clearly preliminary and partial. For instance, we should like to say more about the 'degree' of partiality of our generalized SIFP-based majorization preorder. We think Proposition 1 might be especially helpful in that connection. Moreover an explicit analysis of the behaviour of the SIFP-based majorization preorder with respect to several normative requirements as commonly used in the extant literature would also be of some interest. We leave all that as a topic for further research.

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