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# On Multidimensional Inequality: Ordering and Measurement

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**Abstract -** We study inequality in a context of more than one variable. We show an alternative ordering between matrices representing individuals endowed with several commodities. Then, we represent such an ordering by using convexity and polyhedral theory.

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## 1 Introduction

The standard objective of the economic literature concerning inequality measurement is to compare single-dimensioned welfare indicators, such as income. But, in order to evaluate the social state of an individual, more than one criterion often needs to be applied. In fact, economic disparity does not arise from the distribution of income alone. People are different in income, education, health, etc. and we must take into account several individual characteristics if we want to answer to the questions posed by Sen [24]: "Why inequality?" and "Inequality of what?". As it was stressed by Sen [25], Kolm [10], Maasoumi [15] and many other scholars, the analysis of different individual attributes is crucial to understand and evaluate inequality among people. The classical literature on inequality measurement instead depicts the disparity of an attribute, in general income, in a given population. It has been showed by Kolm [10], Atkinson and Bourguignon [2] and many others that this kind of approach is very unsatisfactory, because people differ in many aspects besides income. Then, we have to extend our measurement to several variables, in order to take into account the other attributes that characterize individuals.

Unfortunately, inequality in the context of more than one variable has seldom been studied. The literature on multidimensional inequality comparisons indeed is rather thin. The problem besides is inherently complex and it is difficult to extend the ranking principle and measures from univariate to multivariate case. The principal reason of such a difficulty is relative to the interaction between income and non-income attributes.

In this paper, our aim is to show the heuristic worth of a multidimensional analysis of the economic inequality and the more robust results concerning this topic.

We review the few results, in economic literature, concerning multidimensional inequality measurement and we show a theoretical result on multidimensional majorization. We will interpret our theorem under a economic point of view, considering the problem of a decision-maker (e.g. a public authority) interested in the distribution of specific goods and commodities among people.

This paper consists of two sections. In the section 2, we explain basic definitions and notation concerning multidimensional majorization analytically as well as intuitively. We review the results concerning social welfare functions that evaluate the well-being associated to a multivariate distribution. We shows the pros and cons of measuring multidimensional inequality, adopting alternative classes of indices. Finally an alternative approach to the measurement of multidimensional inequality is reviewed. The third section contains a very general result. We show a theorem on matrix majorization and we interpret it under an economic point of view, stressing the fact that a matrix represents a population of individuals endowed with several commodities or "goods". Finally, some remarks on some possible extensions of our research in this unexplored field conclude the paper.

## 2 Multidimensional majorization

Historically, economic literature has followed two different trends. The first one ranks different multivariate distributions according to a social welfare function (typically Atkinson and Bourguignon [2] and Kolm [10]). The second one uses evaluative summary inequality statistics (Maasoumi [15] and Tsui [28]), measuring individual attributes with a utility function. In this way, it obtains an univariate distribution vector of utilities that is valued by using an inequality index. Both of the approaches present some problems as Dardanoni [6] pointed out, at least because very little is known about majorization where components of vectors are not in $\mathsf{R}$.

In this section, we review these two trends, modeling the problem of measuring multidimensional economic disparity step by step. We introduce general definitions of partial orderings on set of rectangular matrices discussing and interpreting the results obtained by different scholars. Then, we conclude by discussing an approach which applies convex analysis tools in order to compare alternative multivariate distributions.

## 2.1 Notation and definitions

Following the notation and terminology introduced by Marshall and Olkin [18], we can imagine that now the components of $x$, $y \in \mathsf{R}^n$ are points in $\mathsf{R}^m$, that is these components are column vectors. In this case $x, y$ become matrices that we will denote with capital letters as $X = (x_1, ..., x_n)$, where $x_i$ are all column vectors of length $m$. The element $x_{i,j} \in X$, represents the quantity of the $j$-th commodity or "good" belonging to the $i$-th individual. To make the idea that $X$ is "less spread out" than $Y$ precise, we introduce the following definition:

**Definition 1** Let $X$ and $Y$ be $n \times m$ matrices. Then $Y$ is said to be chain majorized by $X$, written $Y \prec\prec X$, if $PX = Y$ where $P$ is a product of finitely many $n \times n$ T-transforms.[1]

In other terms, the idea of transfer, introduced by Muirhead [20] and Dalton [5], also applies if the components of $x$ and $y$ are vectors. In fact,

[1] A T-transformation is a special kind of linear transformation whose matrix has the form $T = \lambda I + (1-\lambda)Q$, with $\lambda \in [0,1]$ and $Q$ a permutation matrix that just interchanges two coordinates.

if we suppose that $x_i$ and $x_j$ are replaced by $y_i$ and $y_j$ in order to obtain a new vector $y$ from $x$, that respects the constrains:

i) $y_i$, $y_j$ lie in the convex hull[2] of $x_i$, $x_j$;

ii) $x_i + x_j = y_i + y_j$.

we can have an alternative partial ordering between rectangular matrices:

**Definition 2** Let $X$ and $Y$ be two $n \times m$ matrices. Then $Y$ is said to be majorized by $X$, written $Y \prec X$, if $PX = Y$ where the $n \times n$ matrix $P$ is doubly stochastic.

The definition 2 simply says that the average is a smooth-operation, that makes the components of $Y$ more "spread out" than components of $X$.

Because a product of T-transformations is doubly stochastic, then chain majorization implies majorization ($Y \prec\prec X \Rightarrow Y \prec X$) and when $n = 1$ the converse is true also, as when $m = 2$. In general, for $n \geq 2$ and $m \geq 3$ majorization does not imply chain majorization.

Let us define what a convex hull of a matrix is:

**Definition 3** The convex hull of a generic matrix $X$, denoted as

$$H = co\left\{\left(x_i^1, ..., x_i^m\right), i = 1, ..., n\right)\}$$

[2]See Rado [21].

is a convex combination of the row vectors of matrix. It constitutes the simple polyhedron of $X$.[3]

Then, an equivalent definition of matrix majorization $\prec$ is the following one:

**Definition 4** Let $X, Y \in \mathsf{R}^{n\times m}$ be two matrices, then we say $Y$ contains a lower level of disparity with respect to $X$, if $Y$ lies in the convex hull of all permutation of $X$.

## 2.2 Ranking matrices by using social welfare functions

Several attributes are considered in order to describe and evaluate the social state of a society. Individuals vary in income, needs, education, sex, age, ability etc. and the welfare comparisons are based on applications of social evaluation functions depending on the multiattributed endowment of all individuals. In his seminal paper, Kolm [10] proposes the well-known notion of matrix majorization $\prec$ defined above, re-interpreting it under an economic point of view. His merit is that of having introduces the question: "When a given multiattribute distribution is "less spread out" than another one?". Kolm registers the notion of multidimensional inequality using a social welfare function $W : \mathsf{R}_{+}^{n\times m} \rightarrow \Re$, defined on the set of all

[3] See Bolker [4].

semidefinite rectangular matrices. In general, a SWF is an ordering preserving transformation, provided of some suitable properties like symmetry or homogeneity.

Moreover, Kolm introduces the fundamental notion of price majorization, borrowed from an open problem posed by Marshall and Olkin [18]. Let us quote Kolm:

> "We shall say that distribution $Y$ is more equal than distribution $X$, if each Lorenz curve of $Y$ lies nowhere under that of each one of $X$ for all price vectors (which can be restricted to nonnegative prices), and if they are not permutations of each other. This is equivalent to saying that all the properties, applied to the unidimensional distribution case, hold between the income distributions derived from $Y$ and $X$, whatever the prices used for this aggregation".

What Kolm [10] called price majorization is named by Joe and Verducci [9] majorization through linear combination and by Bhandari [3] directional majorization as in Marshall and Olkin [18].

Formally:

**Definition 5** For two matrices $X$ and $Y$, $Y$ is said to be directionally majorized by $X$, written $Y \prec_{\mathsf{d}} X$, if $aY \prec aX$ for all $a \in \mathsf{R}^{\mathsf{n}}$.

Marshall and Olkin [18] showed that $Y \prec X$ implies $Y \prec_{\mathsf{d}} X$, in a more general setting, where $Y \prec_{\mathsf{d}} X$ means $YA \prec XA$ for all $A \in \mathsf{R}^{\mathsf{m}\times\mathsf{k}}$ (with $k$ fixed). They posed the open question whenever $Y \prec_{\mathsf{d}} X$ implies $Y \prec X$ and Bhandari [3], in an important paper, gives the sufficient conditions under which directional majorization implies multivariate majorization.

We can guess that the notion of price majorization is very useful when we compare non-monetary quantity. In such a case, we can compare matrices whose components are of qualitative type simply giving a price to each component of the column vector. Unfortunately, in this way, we reduce all individual characteristics to monetary quantities, losing a part of the information that we have.

This kind of critique also applies to the measurement of inequality through a SWF. A SWF is a synthetic index of equality that expresses by a number the disparity associated to a given multivariate distribution. As Kolm and other scholars use a SWF, in order to register multidimensional inequality, they are losing all information about individual attributes. Overall, their results hold only if the interrelations between welfare components are assumed to be irrelevant for the inequality comparisons. These interrelations instead are very important as Atkinson and Bourguignon [2] and Rietveld

[22] show.

An alternative approach to consider several attributes in describing individual social states is due to Mosler [19]. In his paper, the welfare comparisons are based on simultaneous applications of a given set of social evaluation functions, according with several evaluation criteria and depending on the multiattributed endowments of all individuals. Mosler's framework approach is axiomatic. Some partial multidimensional welfare orderings are introduced and a selected class of social evaluation functions is shown to be consistent with such orderings.

### 2.3 Multidimensional inequality indices

We now review the properties of evaluative inequality statistics in a multidimensional context. According to this approach, people are first represented by an aggregate utility function of all attributes they received by chance. An univariate distribution of utilities is then obtained. Afterwards, a standard inequality index is applied to the utility distribution in order to obtain a measurement of multidimensional inequality.

Such an exercise involves two kinds of issues. First of all we have to choose a utility function. This is an arbitrary choice. To select a function instead of another one means to stress some individuals' preferences and do not take care of other evaluative spaces that could be very important. Second, we have to aggregate the vector of individuals' utilities into a real valued inequality index. This is a too extreme and information losing exercise.

Nonetheless, appealing to a criterion from information theory, Maasoumi [15] argues that when the distribution of welfare is the primary concern of the analysis, a class of utility functions (that contains many of the popular utility functions employed in economics), emerges as the best solution to the first issue quoted above. The class of indices that Maasoumi considers is that of the General Entropy. Maasoumi claims that if we multiply a matrix $X$ by a bistochastic one, then it obtains a new matrix $Y$ that should be declared more equal by any summary inequality index. Such a claim is based on an argument discussed by Kolm [10], who notices that a doubly stochastic transformation is a necessary and sufficient condition for an unambiguous improvement in the welfare relative to a multivariate distribution. Unfortunately, this is not the conclusion that arises from the application of whatever multivariate inequality index. Dardanoni [6] has provided a counterexample where the social inequality is increased after a Schur transformation (S-transform). In several situations, we in fact register a increasing of inequality after a rearrangement due to an S-transform.

Dardanoni [6] have furthermore proved that we require a very extreme restriction on the class of allowed utility functions in order to ensure that the social welfare is decreasing after an unfair redistribution: the aggregate utility functions must be additively separable. A requirement which does not always represent individuals' preferences. Moreover, it is in contradiction with the evaluation of individuals' welfare when there exist correlations about the personal attributes.[4]

Tsui [28] follows the footsteps of Maasoumi and suggests an axiomatic approach to the design of multidimensional inequality measures. Tsui's framework differs from that of Maasoumi by first postulating different sets of axioms and then deriving admissible classes of social evaluation functions and their corresponding inequality indices. The paper extends, to the multidimensional case, the Atkinson-Kolm-Sen inequality approach, generalizing the axioms used in the unidimensional context. The result is a complete characterization of a specific class of social evaluation functions and of the corresponding multidimensional inequality indices. Furthermore, Tsui [29] studies the class of Generalized Entropy inequality measures generated by a non-additive separable evaluation function. He generalizes, to the multidimensional case, the class of functions studied by Shorrocks [26]. In this way,

[4] See Rietvield [22].

he obtains a non-welfaristic approach to the measurement of multidimensional inequality and a useful tool for empirical investigations of economic disparity.

## 2.4 Multivariate Lorenz majorization and Gini index

A special mention is for the joint work of Koshevoy and Mosler. They, following the approach of Rado [21], introduce the "convex analysis" in the field of multivariate majorization. In his seminal paper, Koshevoy [11] generalizes the notion of Lorenz curve through that of a convex body, i.e. a center symmetric convex polyhedron in $\mathbb{R}_+^m$. The multivariate generalization of the Lorenz curve takes the name of Lorenz zonotope, that is defined as follows:

**Definition 6** The Lorenz zonotope $LZ(X)$ of a matrix $X$ is the sum of segments $[0, \overline{x}_i]$, i.e.

$$LZ(X) = \{\theta_1 \overline{x}_1 + ... + \theta_n \overline{x}_n : \theta_i \in [0,1] \text{ for all } i = 1, ..., n\}.$$

Then, Koshevoy defines the multivariate version of univariate Lorenz criterion as follows:

**Definition 7** Let $X$ and $Y$ be two matrices, then $Y$ is said to be Lorenz majorized by $X$, denoted as $Y \preceq_L X$, if $LZ(Y) \subseteq LZ(X)$.

Finally, Koshevoy compares the notion of Lorenz majorization with those of matrix majorization and price majorization. As he proves that the Lorenz majorization is equivalent to the price majorization, we may conclude that the chain of equivalences among matrix majorization $\prec$, directional majorization $\prec_{\mathsf{d}}$ and Lorenz majorization $\prec_{\mathsf{L}}$ holds. But, this is not true. Majorization implies Lorenz majorization, but, in general, for the multidimensional case, the contrary does not hold.

Disparity in several attributes and its relation to multivariate orderings are investigated also in Koshevoy [12]. In this work he develops a geometric approach to order multivariate distributions.

Linked to these pioneering papers, there are two brilliant works of Koshevoy and Mosler [13], [14].

In the first [13] one, they study extensions of the Gini mean difference and Gini index[5] to measure the disparity of a population with respect to several attributes. This is an important result besides under a theoretical point of view even for empirical applications.

In the last of this companion work, Koshevoy and Mosler [14] extend the notions of the Lorenz curve (and the Lorenz order) to several attributes of a multivariate empirical distribution. In order to generalize the usual

[5] Notice that the Gini index is the most known and applied measure of disparity.

Lorenz curve to the multivariate situation, they use the notion of zonoid[6]. Their main result is that the inclusion of Lorenz zonoids is equivalent to directional majorization.

## 3 Matrix majorization: a result

In this section, we develop a result on multidimensional inequality using some tool from convexity and polyhedral theory, which generalizes several findings reviewed above. We let $M_{n,m}$ denote the set of all column-stochastic $n \times m$ matrix, such that each of the $m$ column vectors lie in the standard simplex $S_m = \left\{x \in R_+^m : \sum_{i=1}^m x^i = 1\right\}$. This means that we normalize the quantity of commodities endowed by individuals.

**Definition 8** Let $X \in R^{n\times m}$ and $Y \in R^{p\times m}$ be two matrices. Then $Y$ is said to be E-majorized by $X$, denoted $X \gg Y$, if there exists a matrix $P \in M_{p,n}$, such that $PX = Y$.

Note that the number of rows in the two matrices $X$ and $Y$ may be different. This is in contrast to the orderings $\succ$ and $\succ\succ$ mentioned in the section 2 (where the matrix $P$ was respectively a doubly stochastic matrix

[6] A zonoid is the set of all point between the graph of the dual multivariate Lorenz function and the graph of the multivariate Lorenz function.

and the finite product of $n$ T-transforms). Denote with $e$ a vector (suitably dimensioned) of all ones, since $eP = e$ whenever $P \in M_{p,n}$, we see that if $X \gg Y$ then $ePX = eX = eY$, so the $i$-th columnsum in $X$ and $Y$ coincide for $i = 1, ..., m$. This is a suitable property of this preordering because it allows to compare set of individuals with different cardinality.

Other suitable properties of the binary relation $\gg$ are the following ones:

1. $\gg$ is a preorder, i.e. a reflexive and transitive binary relation;

2. Selected $I \subseteq \{1, ..., n\}$ we denote with $X_I$ the submatrix of $X$ induced by the rows indexed by the elements in $I$. Therefore, if $X \gg Y$, then $X_I \gg Y_I$ for each $I \subseteq \{1, ..., n\}$;

3. If $X \gg Y$ and $H \in \mathbb{R}^{m \times m}$, then $XH \gg YH$;

4. If $X \gg Y$ and $P \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{p \times p}$ are two permutation matrices, then $PX \gg QY$;

5. If $X \gg Y$, then $cone\,(X) \supseteq cone\,(Y)$;

6. If $X \gg Y$, then $rank\,(X) \geq rank\,(Y)$

All this properties are obtained using elementary arguments. They could be assumed like axioms. Moreover, notice that we can deduce such characteristics of matrix majorization directly from the definition of $\gg$, considering that the set of column-stochastic matrices is closed under matrix products. Property 2 is a sort of **decomposibility** property, which allows this ordering to be coherent with an inequality measurement via an additive SWF. The third property says that if we premultiply the matrices $X$ and $Y$ for the same square matrix, then the ordering between $X$ and $Y$ does not change. We require, in other words, that the ordering $\gg$ respect a sort of **homogeneity** property, i.e. multiplying the endowment of every individual for the same coefficient the disparity between two different populations does not change. Property 4 is a kind of **symmetry** requirement, that guarantees that inequality does not depend on who occupies (by chance) a certain position in the distribution (**anonimity**).[7]

The fifth property, even if just quoted in Koshevoy [12], is less intuitive. It is equivalent to say that there are nonnegative numbers $p_{i,j}$ for $i \leq n$ and $j \leq p$ such that $y_j = \sum_{i=1}^{n} p_{i,j} x_i$ for each $j \leq p$, i.e. $PX = Y$ where $P := [p_{i,j}]$. We could consider each number $p_{i,j}$ as a weight associated with the vector $x_j$, one for each row $y_j$ of $Y$. Thus, $X \gg Y$ occurs precisely when the sum of all these weights associated with $x_j$ add up to 1, for each $i \leq j$, or equivalently when $cone\,(X) \supseteq cone\,(Y)$.

[7] Furtherly, it notes that there are examples where $X >> Y$, but $XP >> Y$ for some suitable permutation matrix $P$.

In general, $X \gg Y$ reflects the fact that the columns of $X$ are less spread out than the columns of $Y$, i.e. in $Y$ there are more columns which are linear dependent. In the following, a very intuitive result will show the heuristic appeal of such partial ordering.

By first, let us define what a Markotope is:

**Definition 9** For a matrix $X \in \mathsf{R}^{n,m}$ and a positive integer $k$, the set

$$Z(X,k) = \{SX : S \in M_{k,n}\}$$

is called the Markotope associated with $X$.

A matrix $R \in M_{n,k}$ is sometimes called a Markov matrix. Thus, by definition of the binary relation $\gg$ we have that

$$Z(X,k) = \left\{ Y \in \mathsf{R}^{k,m} : X \gg Y \right\}.$$

A Markotope is a special polytope and it is also defined as the convex hull of matrices obtained from $X$.[8]

Now, denote with $\langle X, G \rangle = \sum_{i,j} x_{i,j} g_{i,j} = Tr(X^{\mathsf{T}} G)$ the inner product between two $k \times m$ matrices $X = [x_{i,j}]$ and $G[g_{i,j}]$, then define a function that plays an important role in the study of matrix majorization:

[8] See Webster [30] chapter 3.

**Definition 10** Let $C \subset \mathsf{R}^{k,m}$ a nonempty compact set. A function $\varphi_C : \mathsf{R}^{k,m} \to \mathsf{R}$ defined as:

$$\varphi_C(X) = \max\{\langle X, G\rangle : G \in C\} \quad \text{for } X \in \mathsf{R}^{k,m}$$

is sublinear.

A function sublinear is a function positively homogeneous (i.e. $\varphi(\lambda x) = \lambda\varphi(x)$ for each $x \in \mathsf{R}^m$and a real number $\lambda \geq 0$) and convex. We denote with $\Phi_k(\mathsf{R}^m)$ the set of sublinear functionals that may be written as a maximum of at most $k$ linear functionals.

The following theorem summarizes some equivalent conditions for matrix majorization:[9]

**Theorem 1** Let $X = [x_{i,j}] \in \mathsf{R}^{n,m}$ and $Y = [y_{i,j}] \in \mathsf{R}^{p,m}$. Then the following three statements are equivalent:

1. $X \gg Y$;
2. $Z(X,k) \supseteq Z(Y,k)$ for each positive integer $k$;
3. For each $k \geq 1$ and $\varphi \in \Phi_k(\mathsf{R}^m)$ we have that $\sum_{j=1}^{n} \varphi(x_j) \geq \sum_{j=1}^{n} \varphi(y_j)$.

[9] Notice that an inequality $\leq$ between vectors means that the inequality holds for each component.

**Proof.** $(1 \Rightarrow 2)$ Let $k \geq 1$ and assume that $X \gg Y$, so there is a matrix $P \in M_{p,n}$, with $PX = Y$. For each $N \in M_{k,p}$, we get $NPX = NY$. Here $NP \in M_{k,n}$, so we conclude our statement.

$(2 \Rightarrow 3)$ Let $k \geq 1$ and $L \in R^{k,m}$. We see that $\max\{\langle X', L\rangle : X' \in Z(X,k)\} \geq \max\{\langle Y', L\rangle : Y' \in Z(Y,k)\}$, that is, by definition, tantamount to $\varphi_{Z_X} \geq \varphi_{Z_Y}$. Analogously, denoted with $l_1, ..., l_k$ $k$ linear functionals,

$$\max\left\{\left\langle X', L\right\rangle : X' \in Z(X,k)\right\} \geq \max\left\{\left\langle Y', L\right\rangle : Y' \in Z(Y,k)\right\}$$

is equivalent to

$$\max\left\{\sum_j \sum_t p_{t,j} \sum_i l_{i,t} x_{i,j} : P \in M_{p,n}\right\} \geq$$

$$\max\left\{\sum_j \sum_t p_{t,j} \sum_i l_{i,t} y_{i,j} : P \in M_{p,n}\right\}$$

Last inequality can even be written as:

$$\sum_j \max_{t \leq k} \sum_i l_{i,t} x_{i,j} \geq \sum_j \max_{t \leq k} \sum_i l_{i,t} y_{i,j}$$

that is equivalent to

$$\sum_j \varphi(x_j) \geq \sum_j \varphi(y_j);$$

$(3 \Rightarrow 1)$ As $\sum_j \varphi(x_j) = \sum_j \max_{t \leq k} \sum_i l_{i,t} x_{i,j}$, we obtain, repeating the same argument in $(2 \Rightarrow 3)$, that $Z(X,k) \supseteq Z(Y,k)$. Then, for $k = p$ and because the identity matrix $I_p \in M_{p,p}$, there exists a $P \in M_{p,n}$ such that $PX = I_p Y = Y$, i.e. $X \gg Y$. ■

The three statements of the theorem represent a theoretical generalization of the previous results reviewed in section 2 above.

The binary relation $\gg$ is a more general notion of matrix majorization as well as the equivalent notion of Markotope, that, at the same time, is a specification of the zonotope used by Koshevoy [11], which maintains the suitable characteristics of it. Finally, the last equivalent condition $\sum_{j=1}^{n} \varphi(x_j) \geq \sum_{j=1}^{n} \varphi(y_j)$ could be interpreted as an utility function, that measures the welfare or, by converse, the inequality, of each individual in the distribution.

## 4 Conclusion and further possible extensions

We have reviewed how to rank matrices, that represent the distribution of goods and commodities among people, by using a SWF. We have noted as, in general, this operation either is information losing or implies a strong restriction on the class of evaluation functions. Then, we have surveyed some results on multidimensional inequality indices. As an inequality index is a synthetic measure of the degree of disparity among individuals, we loose the goal of our exercise. To investigate multidimensional inequality means to take into account several attributes, besides income, which characterize people. Then forcing all variables into a scalar is an unsuitable practice that is arbitrary and very restrictive. Finally, we have considered the attempt of Koshevoy and Mosler to capture the notion of majorization and Lorenz order in a multidimensional context. The outcome obtained is unsatisfactory. Despite the analytical sophistication used by these two scholars, the results are not so far from those well-known in literature of Theory of Majorization, while a lot of work remain to do. We have then examined a more general definition of multidimensional inequality ordering, providing a complete characterization and representation.

In what follows, we summarize the contents of our future research in this field.

Studying multidimensional inequality, our aim is to generalize the T-transforms to a more general class of transformations induced by linear groups, whose name is G-majorization.[10] In particular, we want to analyze, in a multidimensional context, the meaning of the composite transfer, i.e. what Shorrocks and Foster [27] call a favorable composite transfer (FACT), namely a kind of transfer that decreases the inequality of the distribution,

[10] A vector $x$ is said to be G-majorized by a vector $y$ if $y$ lies in the convex hull of the orbit of $x$ under a group G of linear transformations.

through a progressive and regressive transfer.[11] How inequality changes when different transfers take place between the individual characteristics is far to be known and surely it is worth to pursue. In other words, we want to investigate how to induce a partial ordering on the set of nonnegative matrices when the Lorenz hypersuperfaces intersects several times.

Another aspect of the multidimensional inequality that we want to deepen concerns the class of functions that preserves the matrix ordering. Our aim is to generalize a well-known result due to Schur and Ostrowki to a wider class of functions than those which are continuously differentiable on a vector space of real $n \times m$ matrices.

# References


[1] Atkinson, A.B. (1970) - "On the measurement of Inequality" - Journal of Economic Theory, 2, 244-263.

[2] Atkinson, A.B. and F. Bourguignon (1982) - The comparison of multidimensioned distributions of economic status - Review of Economic Studies 39, 183-201.


[11] See Rothschild and Stiglitz [23] for a definition of what a regressive transfers is.

[3] Bhandari, S.K. (1988) - Multivariate majorization and directional majorization: positive results - Sankhyā: The Indian Journal of Statistics, vol.50 pp.199-204.

[4] Bolker, E.D. (1969) - A class of convex bodies - Transactions for American Mathematical Society 145, 323-346.

[5] Dalton, H. (1920) - The measurement of inequality - Economic Journal, 348-361.

[6] Dardanoni, V. (1992) - On Multidimensional Inequality Measurement - mimeo.

[7] Dasgupta, P., A.K. Sen, and D. Starret (1973) - "Notes on the measurement of inequality" - Journal of Economic Theory, 6, 180-187.

[8] Hardy, G.H., J.E. Littlewood, and G. Polya, (1934, 1952) - Inequalities - Cambridge University Press, London.

[9] Joe, H. and J. Verducci (1993) - Multivariate majorization by positive combination - In: Stochastic Inequalities IMS Lecture Notes: Monograph Series, vol.22, 159-181.

[10] Kolm, S.C. (1977) - Multidimensional Egalitarianisms - Quarterly Journal of Economics 91, 1-13.

[11] Koshevoy, G. (1995) - Multivariate Lorenz majorization - Social Choice and Welfare 12, 93-102.

[12] Koshevoy, G. (1998) - The Lorenz zonotope and multivariate majorizations - Social Choice and Welfare 15, 1-14.

[13] Koshevoy, G. and K. Mosler (1999) - Multivariate Gini indices - Journal of Multivariate Analysis 53, 112-126.

[14] Koshevoy, G. and K. Mosler (1996) - The Lorenz zonoid of a multivariate distribution - Journal of American Statistical Association 91, 873-882.

[15] Maasoumi, E. (1986) - The measurement and decomposition of multidimensional inequality - Econometrica 54, 991-997.

[16] Maasoumi, E. (1989) - Continuously distributed attributes and measures of multivariate inequality - Journal of Econometrics 42, 131-144.

[17] Maasoumi, E. and S. Zandvakili (1990) - Generalized entropy measures of mobility for different sexes and income levels - Journal of Econometrics 43, 121-134.

[18] Marshall, A.W., and I. Olkin, (1979) - Inequalities: Theory of Majorization and its Applications - Academic Press: New York.

[19] Mosler, K. (1991) - Multidimensional Welfarism - In Eichhorn, W. (Ed.), Models and Measurement of Welfare and Inequality - Springer-Verlag, 808-820.

[20] Muirhead, R.F. (1903) - Some methods applicable to identities and inequalities of symmetric algebraic functions of $n$ letters - Proceeding Edinburgh Mathematical Society 21, 144-157.

[21] Rado, R. (1952) - An inequality - Journal of London Mathematical Society 27, 1-6.

[22] Rietveld, P. (1990) - Multidimensional inequality comparisons - Economics Letters 32, 187-192.

[23] Rothschild M. and J.E. Stiglitz (1973) - Some further results on the measurement of inequality - Journal of Economic Theory 6, 188-204.

[24] Sen, A.K. (1997) - "On Economic Inequality" - (Second eds), extended edition with J. Foster, Oxford: Clarendon Press.

[25] Sen, A.K (1980) - "Equality of What?" - in McMurrin, S. ed., Tunner Lectures on Human Values; vol.1, Cambridge: Cambridge University Press.

[26] Shorrocks, A.F.

[27] Shorrocks, A.F. and J. Foster (1987) - "Transfer sensitive inequality measures" - Review of Economic Studies, 54, 485-497.

[28] Tsui, K.Y. (1995) - Multidimensional Generalizations of the Relative and Absolute Inequality Indices: The Atkinson-Kolm-Sen Approach - Journal of Economic Theory 67, 251-265.

[29] Tsui, K.Y. (1999) - Multidimensional inequality and multidimensional generalized entropy measures: an axiomatic derivation - Social Choice and Welfare 16, 145-157.

[30] Webster, R. (1994) - Convexity - Oxford Science Publication, Oxford University Press.