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Multidimensional Inequality: A Survey

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MULTIDIMENSIONAL INEQUALITY: A SURVEY

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Abstract

We review research on inequality when individuals differ for several characteristics besides income.
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1 Introduction

The standard objective of the economic literature concerning inequality measurement is to compare single-dimensioned welfare indicators, such as income. But, in order to evaluate the social state of an individual, more than one criterion often needs to be applied. In fact, economic disparity does not arise from the distribution of income alone. People are different in income, education, health, etc. and we must take into account several individual characteristics if we want to answer to the two questions posed by Sen [32]: "*Why inequality?*" and "*Inequality of what?*". As it was stressed by Sen [33], Kolm [**?**], Maasoumi [23] and many other scholars, the analysis of different individual attributes is crucial to understand and evaluate inequality among people.

Unfortunately, inequality in the context of more than one variable has seldom been studied. The literature on multidimensional inequality comparisons indeed is rather thin. The problem besides is inherently complex, then it is difficult to extend the ranking principles and measures from univariate to multivariate case. The principal reason of such a difficulty is relative to the interaction between income and non-income attributes.

In this work, our aim is to show the heuristic worth of a multidimensional analysis of the economic inequality and the more robust results concerning this topic.

We show the main theoretical results on multidimensional majorization and review the few results, in economic literature, concerning multidimensional inequality measurement.

The paper consists of two sections. In the section 2, we explain basic definitions and notation concerning multidimensional majorization analytically as well as intuitively. In the section 3, we show the pros and cons of measuring multidimensional inequality, adopting alternative classes of indices. We review the results concerning social welfare functions, that evaluate the well-being associated to a multivariate distribution. Then, a survey on measurement of multidimensional inequality, using tools of convex analysis, concludes the section. Finally, some remarks and some possible future extensions of the results on multidimensional inequality conclude the paper.

2 Multidimensional majorization

The classical literature on inequality measurement depicts the disparity of an attribute, in general income, in a given population. It has been showed by Kolm [**?**], Atkinson and Bourguignon [4] and many others that this kind of approach is very unsatisfactory, because people differ in many aspects besides income. Then, we should extend our measurement to several variables, in order to take into account the other attributes (e.g. health, education, talents, capabilities, etc.), that characterize the individuals.

Historically, economic literature has followed two different trends. The first one ranks different multivariate distributions according to a social welfare function (typically Atkinson and Bourguignon [4] and Kolm [**?**]). The second one uses evaluative summary inequality statistics (Maasoumi [23] and Tsui [35]), measuring individual attributes with a utility function. In this way, it obtains an univariate distribution vector of utilities that are valued using an inequality index. Both of the approaches present some problems as Dardanoni [9] pointed out, at least because very little is known about majorization where components of vector distributions are not in $\mathbb{R}$.

In the next subsection, we review the problem of modeling and measuring multidimensional economic disparity step by step. We introduce general definitions of partial orderings on set of rectangular matrices, discussing and interpreting the results obtained.

2.1 Notation and definitions

Following the notation and terminology introduced by Marshall and Olkin [26], we can imagine that now the components of the two distributions x and y are points in $\mathbb{R}^m$, that is these components are column vectors. In this case x, y become matrices that we will denote with capital letters as

$$X = (x_1^n, ..., x_m^n),$$

where x_i^n are all column vectors of length n.

If we call a *T-transform* a linear transformation

$$T = \lambda I + (1 - \lambda)Q,$$

with $\lambda \in [0, 1]$ and Q a permutation matrix that just interchanges two coordinates, the following definition makes precise the idea that X is *more spread out* than Y:

Definition 1 *Let X and Y be $n \times m$ matrices. Then X is said to be chain majorized by Y, written $Y \prec\prec X$, if $PX = Y$ where P is a product of finitely many $n \times n$ T-transforms.*

In other terms, the idea of transfer, introduced by Muirhead [?] and Dalton [8], also applies if the components of x and y are vectors.[1]

In fact, if we suppose that x_i and x_j are replaced by y_i and y_j in order to obtain a new vector y from x, that respects the constrains:

i) y_i, y_j lie in the convex hull[2] of x_i, x_j;

ii) $x_i + x_j = y_i + y_j$.

then, we obtain that:

Definition 2 *If X and Y are two $n \times m$ matrices, then X is said to be majorized by Y, written $Y \prec X$, if $PX = Y$, where P is $n \times n$ doubly stochastic matrix.*

Because a product of *T-transforms* is doubly stochastic, then *chain majorization* implies *majorization,* $Y \prec\prec X \Rightarrow Y \prec X$ and when $n = 1$, as when $m = 2$, the converse is true also. In general, for $n \geq 2$ and $m \geq 3$ *majorization* does not imply *chain majorization.*

Definition 2 simply says that the average is a smoothing operation, that makes the components of y less *spread out* than components of x. In fact, if we think of the components of x as incomes and the components of y as representing a redistribution of the total income $\sum_{i=1}^{n} x_i$, the average, such that $y_j = \sum\limits_{i=1}^{n} x_i p_{ij}$, with $p_{ij} \geq 0$ and $\sum_{i=1}^{n} p_{ij} = 1$ for all j, makes the components of y surely less spread out than components of x. Further, because y is a redistribution of incomes in x, it must be that $\sum_{j=1}^{n} p_{ij} = 1$, that means that the matrix $P = (p_{ij})$ is doubly stochastic.

Denoted as:

$$H = co\left\{\left(x_i^1, ..., x_i^n\right), i = 1, ..., m\right)\right\}$$

the convex hull of a generic matrix X, i.e. a convex combination of the row vectors of matrix[3], an equivalent definition of the *majorization* $\prec$ is the following:

Definition 3 *Let $X, Y \in \mathbb{R}^{n \times m}$ be two matrices, then we say Y contains a lower level of disparity with respect to X, if Y lies in the convex hull of all permutation of X.*

We have said that several attributes are considered in order to describe and evaluate the social state of a society. In the next section, we review how economic literature provides solutions to such a problem.

[1] See Marshall and Olkin [26] chapter 1 and 2 for more details on the *T-transforms.*

[2] See Rado [28].

[3] See Bolker [7].

3 Measuring multidimensional inequality

3.1 Ranking matrices by using social welfare functions

As individuals vary in income, needs, education, sex, age, ability etc., the welfare comparisons must be based on applications of evaluation functions depending on the multiattribute endowment of people.

We start by considering the seminal paper of Kolm [16], who proposes the notion of *majorization* $\prec$ defined above, interpreting it under an economic point of view. His merit is that of having introduced in economics the question: "*When a given multivariate distribution is less spread out than another one?*". Kolm registers the notion of multidimensional inequality by using a social welfare function (SWF)

$$W : \mathbb{R}_{+}^{n\times m} \to \mathbb{R}$$

defined on the set of all semidefinite rectangular matrices.

In general, a SWF is an ordering preserving transformation, provided of some suitable properties like symmetry or homogeneity. Kolm includes among properties of a SWF that of *commodity neutrality*, i.e. individuals have not *a priori* assumptions about tastes for the commodities.

Moreover, Kolm introduces two notions of majorization.

Let us denote with $x_1^R, ..., x_m^R$ the rows of a generic matrix X:

Definition 4 *For each $X, Y \in \mathbb{R}_{+}^{n\times m}$, Y is said to be rowwise majorized by X, denoted $Y \prec_{row} X$ if and only if there exists a doubly stochastic matrix P such that $Px_i^R = y_i^R$ for $i = 1, ..., m$.*

Of course, $Y \prec_{row} X$ can be written as:

$$\begin{bmatrix} P & 0 & ... & 0 \\ 0 & P & ... & 0 \\ . & . & . & . \\ 0 & 0 & ... & P \end{bmatrix} (x_1^R, ..., x_m^R) = (y_1^R, ..., y_m^R) \qquad (1)$$

This implies that:

$$y_i^R \prec x_i^R \text{ for each } i = 1, ..., n \qquad (2)$$

in the sense of ordinary majorization.[4] [5]

[4]See Marshall and Olkin [26] chapter 1 for definition of ordinary majorization.

[5]As P could be a product of a finite number of *T-transform*, it is expected that *chain majorization* does imply 1 and 2, nonetheless it only implies that there exist doubly stochastic matrices $P_1, ..., P_m$ such that $P_i x_i^R = y_i^R, i = 1, ..., m$, but it does not guarantee that $P_i = P_j \forall i \neq j$, (see Marshall and Olkin [26] chapter 15).

Another very important notion of matrix majorization, borrowed from an open problem posed by Marshall and Olkin [26], is that of *price majorization.*

> "*We shall say that distribution Y is more equal than distribution X, if each Lorenz curve of Y lies nowhere under that of each one of X for all price vectors (which can be restricted to nonnegative prices), and if they are not permutations of each other. This is equivalent to saying that all the properties, applied to the unidimensional distribution case, hold between the income distributions derived from Y and X, whatever the prices used for this aggregation*"(Kolm [16], pg.).

What Kolm [16] called price majorization is named by Joe and Verducci [15] *majorization through linear combination* and by Bhandari [5] *directional majorization* as in Marshall and Olkin [26].

Formally:

Definition 5 *For two matrices X and Y, Y is said to be price majorized by X, written $Y \prec_d X$, if $aY \prec aX$ for all $a \in \mathbb{R}^n$.*

Marshall and Olkin [26] showed that $Y \prec X$ implies $Y \prec_d X$, in a more general setting, where $Y \prec_d X$ means $YA \prec XA$ for all $A \in \mathbb{R}^{m \times k}$ (for fixed k). They posed the open question whenever $Y \prec_d X$ implies $Y \prec X$ and Bhandari [5], in an important paper, gives the sufficient conditions under which price majorization implies majorization. The notion of price majorization is very useful when we compare non-monetary quantity. In such a case, we can compare (matrices with) qualitative components (e.g. health, education, etc.), simply giving a price to each component.[6]

The main criticism to such an approach, that uses SWF for evaluating multidimensional disparity, is that the obtained results generally hold for the case where interrelations between welfare components are assumed to be irrelevant in the inequality comparisons. These interrelations instead are very important as Atkinson and Bourguignon [4] and Rietveld [29] have shown.

[6]Unfortunately, in this way, we reduce all individual characteristics to monetary quantities, i.e. income, losing the information we have when we analyze a multidimensional distribution. This kind of critique (of losing relevant information) also applies to the measurement of inequality through a SWF. A SWF is a synthetic index of equality that expresses by a number the disparity associated to a multivariate distribution.

Atkinson and Bourguignon [4] study the inequality comparisons by means of stochastic dominance.[7] Generalizing the results on unidimensional Lorenz ordering, they analyse how different forms of deprivations (such as low income, low education, low standard of living, etc.), tend to be associated. They evaluate, through a SWF,[8] the welfare associated to different *named vectors* x^i [9], i.e. vectors that represent the percentage of the total quantity of the i-th commodity allocated to the j-th individual, and investigate the implications of different assumptions about the form of the SWF and the different degrees of interdependence between the elements of x^i.

Technically, the comparison between bivariate distributions, in the work of Atkinson and Bourguignon [4], occurs on the base of the difference in their expected utility.

Rietveld [29] instead deals with issues of inequality decompositions of income factor components. He investigates the correlations between the various components of income by means of the Lorenz curve criterion and by means of an inequality measure. Rietvield, in fact, shows that if we define the Lorenz curves for each components of individual income characteristics, we find that inequality in the total income is no greater than that of the most unequal component. It derives from this result that the Lorenz curve of total income is, in general, above the weighted mean of Lorenz curves of income components. Therefore, there exists a sort of *aggravation effect* of considering a correlation between the different components of income.

Moreover, measuring the inequality of a given distribution by a function $f : \mathbb{R}^n \to \mathbb{R}$ homogeneous and Schur-convex, Rietvield shows that the joint consideration of income components leads to a *mitigation* of inequality of total income for a broad class of inequality measure, homogeneous of degree 0, which can be written as the sum of convex functions. Nevertheless, this does not hold in general. As we consider, for example, the Gini coefficient, it is possible to show it has the inequality mitigation property, but it cannot be written as a sum of convex functions. Rietveld then claims that homogeneity and Schur-convexity are not sufficient conditions for the inequality mitigation property, in a multidimensional context, concluding

[7] They are especially concentrated on the two-dimensioned case and apply some results on multivariate stochastic dominance in portfolio theory to the measurement of inequality.

[8] The SWFs are assumed to be addively separable and symmetric with respect to the individuals.

[9] The term "named good" is employed by F.H. Hahn (Econometrica 39, 1971), in the analysis of transaction costs. Hahn notes: "[...] *households face a sequence of budget constraints and there may be no unique set of discount rates applicable to all households which allows one to "amalgamate" all those constraints into a single present value budget constraint*".

that the interrelations between welfare components are relevant in inequality comparisons.

On the same side is the work of Mosler [27], that we quote for the sake of completeness. In this paper, Mosler considers several attributes in describing individual social states and several criteria of evaluation. Welfare comparisons are based on simultaneous applications of a given set of social evaluation functions, depending on the multiattributed endowments of all individuals. Mosler uses social evaluation functions which can be represented as sum of evaluation functions of individual states, in order to compare, in a purely ordinalistic framework, individual welfare levels. The approach is axiomatic and some partial multidimensional welfare orderings are introduced and a selected class of social evaluation functions is shown to be coherent with such orderings. The originality of this work stays in the fact that, in view of the limited comparability of social states under an ordinalistic setup, Mosler proposes a new approach to multidimensional welfare orderings through a comparison of individual endowments with respect to a critical level, i.e. a minimum endowment in commodities (i.e. a threshold), respect to which comparisons between different distributions of attributes take place.

3.2 Multidimensional inequality indices

We study the properties of evaluative inequality statistics in a multidimensional context. According to this approach, people are first represented by an aggregate utility function of all attributes they received by chance. An univariate distribution of utilities is then obtained. Afterwards, a standard inequality index is applied to the utility distribution in order to obtain a multidimensional inequality evaluation.

Definition 6 *A multidimensional inequality index can be written as a function of the real valued vector*

$$U_1 (x_1), ..., U_n (x_n),$$

where $U_i (\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}$ *denotes an individual utility function and* x_j *a row of* X*.*

Such an exercise involves two kinds of issues. First of all we have to choose a utility function. This is an arbitrary choice. To select a function instead of another one means to stress some individuals' preferences and do not take care of other evaluative spaces that could be very important.

Second, we have to aggregate the vector of individuals' utilities into a real valued inequality index. This is an information losing exercise.

Appealing to a criterion from information theory, Maasoumi [23] argues that when the distribution of welfare is the primary concern of the analysis, a class of utility functions (that contains many of the popular utility functions employed in economics), emerges as the best solution to the first issue quoted above. He therefore studies the class of the General Entropy indices. Following the Kolm's approach [16], he, in fact, considers a matrix X that represents a society of n individuals, endowed with m commodities.[10] Then, he claims that if we multiply a matrix X by a bistochastic matrix, we obtain a new matrix Y that should be declared more equal by any summary inequality index. Such a claim is based on an argument discussed by Kolm [16], who notices that a doubly stochastic transformation is a necessary and sufficient condition for an unambiguous improvement in the welfare of the multivariate distribution. Nevertheless, as Dardanoni [9] noticed:

> "[...] *The analysis, in a multidimensional context, is however quite different when we consider the effects of a bistochastic transformation on the amount of inequality between the individuals composing a society*".

Let us suppose to have a society of three people endowed with three commodities like in the following example:

$$X = \begin{bmatrix} 10 & 10 & 10 \\ 10 & 90 & 10 \\ 90 & 10 & 10 \end{bmatrix}$$

and to multiply X by the bistochastic matrix P defined as:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

in order to obtain:

$$Y = \begin{bmatrix} 10 & 10 & 10 \\ 50 & 50 & 10 \\ 50 & 50 & 10 \end{bmatrix}$$

[10] Notice that Maasoumi [24] extends this approach by considering the case where several attributes are continuously distributed and Maasoumi and Zandvakili [25] apply this framework to the measurement of mobility.

Let us choose any symmetric concave utility function $U(\cdot)$, so that $U_2(x_2) = U_3(x_3) \leq U_3(y_3) = U_2(y_2)$. It is possible to show that a Schur-convex inequality index will be increased after such a transformation, because the two richer individuals in the society are now better off. This result contradicts proposition 3 in Maasoumi [23], as the index of General Entropy is not everywhere increasing, as it should be.

After having shown that the Maasoumi's claim is wrong, Dardanoni poses the following crucial question: "*Are there any transformation we could use to impose restrictions on the class of allowed summary utility functions?*"

Let us consider two different matrices X and Y which, *more solito,* represent a society of three individuals endowed with three commodities:

$$X = \begin{bmatrix} 100 & 1 & 1 \\ 1 & 90 & 100 \\ 90 & 100 & 90 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 1 & 1 & 1 \\ 90 & 90 & 90 \\ 100 & 100 & 100 \end{bmatrix}$$

We can suppose that Y is obtained from rearranging the elements of X, such that the first individual gets the lowest amount of all three commodities, the second individual gets the second lowest amount of all three commodities and the third one gets the highest.

A decision-maker concerned for equity agrees on the fact that the social inequality is increased after such a rearrangement. Unfortunately, this is not the conclusion that arises from the application of whatever multivariate inequality index. There exist several cases where, applying the Maasoumi's General Entropy class of functions, we can register a decreasing of inequality after a rearrangement like that above. We have to pay much more attention to the use of an aggregate utility function in the multidimensional inequality context. The strong result provided by Dardanoni [9], in order to ensure that the social welfare, evaluated by a function $U(\cdot)$, is decreasing after an unfair rearrangement, implies a very extreme restriction on the class of allowed utility functions. It actually requires that aggregate utility functions must be additively separable. Unfortunately, such a requirement does not, in general, represent individuals' actual preferences[11] and it is in contradictions with the evaluation of individuals' welfare when there exists correlations about the attributes.[12]

The Maasoumi's two-stage approach to design a class of multidimensional inequality measures is also applied by Tsui [35]. This scholar, following the footsteps of Maasoumi, suggests an axiomatic approach to the

[11] See Fishburn [?].

[12] See Rietvield [29].

derivation of classes of social evaluation functions and their corresponding inequality indices. The paper provides a complete characterization of the class of Atkinson-Kolm-Sen's inequality indices for the multidimensional case.

Tsui considers n individuals endowed with m attributes. He denotes with $X = [x_{ik}]$ the matrix, where x_{ik} represents the amount of the k-th attribute possessed by the i-th individual. If D is some subset of the set of $n \times m$ matrices with positive real elements, define with $I(\cdot) : D \to \mathbb{R}_+$ a real-valued function that represents a multidimensional inequality index. Tsui assumes that such an index has the well-behaved properties of continuity and symmetry. The underlying evaluation function $W(\cdot) : D \to \mathbb{R}$ ranks different distributions of the m individual attributes and it is continuous, monotone, symmetric and quasi-concave.

Then, Tsui assumes that there exists a non-singleton set of individuals $S \subset \{1, 2, ..., N\}$, such that a kind of additive separability axiom, which insures that $W(\cdot) = \sum_i U(x_i)$, where $U(\cdot) : \mathbb{R}^m_{++} \to \mathbb{R}$ is an increasing and strictly concave function and a ratio-scale invariance axiom, that represents a generalization of homotheticity property, hold for any$W$ continuous, strictly increasing, symmetric, quasi-concave social evaluation function. W satisfies axioms of symmetry and ratio-scale invariance if and only if $W(X)$ is ordinally equivalent to $\sum_i U(x_i)$, where U is a strictly increasing concave function.

Note the proof of such a theorem lies on a result due to Blackorby, Donaldson and Auesperg [6], and on a standard solution for a class of functional equation due to Aczel *et alii [2]*.

In another work, Tsui [36] generalizes, to the multidimensional case, the class of functions studied previously by Shorrocks [**?**], obtaining a non-welfaristic approach to the measurement of multidimensional inequality and useful tools for empirical investigations concerning multidimensional disparity.

3.3 Multivariate Lorenz majorization and Gini index

A special mention goes to the joint work of Koshevoy and Mosler. They, following the approach of Rado [28], introduce *convex analysis* in the field of multivariate majorization. In his seminal paper, Koshevoy [17] considers a population with n agents among which a vector of goods is distributed. He takes a distribution matrix $X = (x_{ij}) \in \mathbb{R}^{n \times m}_+$ which assigns to the i-th agent its annual vector of goods and poses the following question: "*Given two distribution matrices X, Y, which one contains the lower level of dis-*

parity?". In order to answer to such a question, Koshevoy generalizes the notion of Lorenz curve through that of a convex body, precisely that of a center symmetric convex polyhedron in $\mathbb{R}_+^m$, which constitutes a multivariate generalization of the Lorenz curve, called *Lorenz zonotope*, and denoted as $LZ(X)$. Then, the multivariate version of univariate Lorenz criterion is the following:

Definition 7 *Let X and Y be two matrices, then X is said to be Lorenz majorized by Y, denoted as $Y \preceq_{LO} X$, if $LZ(Y) \subseteq LZ(X)$.*

Finally, Koshevoy compares the notion of Lorenz majorization with those of (matrix) majorization $\prec$ and price majorization, obtaining that the Lorenz majorization is equivalent to the price majorization.

According to the result of Bhandari [5], we may conclude that the chain of equivalences among matrix majorization $\prec$, price majorization $\prec_d$ and Lorenz majorization $\prec_{LO}$ holds. But, this is not true. Majorization implies Lorenz majorization, but, *in general*, for the multidimensional case, the contrary does not hold. The argument used to show this is similar to that of Dardanoni [9], but reformulated in a framework of convex analysis. Koshevoy, indeed, shows that the Lorenz ordering on X and Y does not hold for any restriction to their submatrices.

Disparity in several attributes and its relation to multivariate orderings are investigated also in another fundamental paper of Koshevoy [18], where he develops a geometric approach to order multivariate distributions

A multivariate distribution is defined as a matrix allocating the percentage of the total quantity of the k-th commodity ($k = 1, 2, ..., m$) to the i-th household ($i = 1, 2, ..., n$). A multivariate distribution can be replaced by a convex set and can be ordered with respect to the inclusion of the corresponding convex sets, analogously to the requirement that a Lorenz curve lies above another. Koshevoy [18] shows that different multivariate distributions have different Lorenz zonotopes, and that when $m = 1$ the definition of zonotope collapses in the generalization of the univariate Lorenz curve. Then, given a cone of directions (coordinates of a direction can be interpreted as weights or prices of individual attributes), the cone Lorenz majorization is the order defined as the inclusion of the cone extension of the Lorenz zonotope. Koshevoy establishes the equivalence between the cone Lorenz majorization and the cone price majorization, where a distribution is said to be cone directional majorized by another if the expenditures of households at any prices in a cone with the first distribution are less dispersed than with the other.

This should apparently seem an analogous result of that obtained in previous paper [17], simply in a more general setting. But, actually, it is not.

> "*The advantage of studying cone majorizations is that the set of matrices which are majorized by a given matrix can be described by a finite number of inequalities.*"

As a consequence, in the case of a cone with a finite number of extreme rays, checking for a cone directional majorization is equivalent to verifying the univariate dispersions of the households' expenditures for a finite number of prices (directions).

Linked to these pioneering papers, there are two brilliant works of Koshevoy and Mosler [19], [20] we go to review.

In the first [19] of these two papers, they study extensions of the Gini mean difference and Gini index to measure the disparity of a population with respect to several attributes. Koshevoy and Mosler investigate two approaches, one based on the distance of the distribution from itself, the other on the volume of a convex set in $(m+1)$-space, named the *lift zonoid* of the distribution. A lift zonoid is a multivariate generalization of the generalized Lorenz curve. When $m = 1$, the lift zonoid collapses to the area between the Lorenz curve and the line of perfect equality, up to a scale factor. The main result of this paper [19] consists in proving the equivalence among the lift zonoid and *relative dilation* and *directional relative* and *absolute dilation*, defined as the generalizations, in the continuum, of matrix majorization and price majorization for $n \times m$ matrices to the case of continuous multivariate distributions.

Koshevoy and Mosler show that several properties, that hold for the univariate case, follow easily from the definitions of the distance-Gini mean difference and distance-Gini index for the multivariate case. For example, they prove that if to a distribution in m attributes an $(m+1)$-th attribute is added which does not vary in the population, then the disparity index remains essentially unchanged: it multiplies by a factor which depends only on m (*ceteris paribus property*).

Then, as the definition of the univariate Gini index is twice the area between the Lorenz curve and the diagonal, Koshevoy and Mosler apply a very known result in convex analysis[13], in order to obtain the multivariate volume-Gini index.

[13] We are talking about the Minkowski volume. See, for an extensive discussion on this topic, Webster [**?**].

They furthermore show that the standard suitable properties of the disparity indices, like the *ceteribus paribus property,* hold even in a multidimensional context of volume-Gini index. These results are important under a theoretical point of view and for the empirical applications. Gini index, in fact, is the most known and applied measure of disparity and its application to samples with several attributes are crucial in order to establish the degree of inequality in a multidimensional context.

In the last but not least of this companion work, Koshevoy and Mosler [20] extend the notions of the Lorenz curve and the Lorenz order to several attributes of a multivariate *empirical* distribution. They generalize the usual Lorenz curve to the multivariate situation, by using the notion of zonoid, namely the set of all point between the graph of the dual multivariate Lorenz function and the graph of the multivariate Lorenz function. The Lorenz zonoid is a closed convex subset of the unit hypercube in $\mathbb{R}^{m+1}$, that becomes a convex polytope for a discrete distribution. It coincides with the main diagonal of the hypercube for an egalitarian distribution. Comparing two alternative multivariate distributions, Koshevoy and Mosler extend the notion of Lorenz order to the multidimensional case, defining what is the inclusion of Lorenz zonoids. The main result obtained is that the inclusion of Lorenz zonoids is equivalent to a well defined notion of price majorization. Further, Koshevoy and Mosler show that if a distribution F has less multivariate disparity than G, then the same thing holds for their marginal distributions. The reverse is also true, when F and G are product distributions, i.e. when the attributes are stochastically independent. A result very close to the argument discussed by Dardanoni [9].

4 Conclusion and further possible extensions

For concluding, we have reviewed as ranking matrices, that represent the distribution of goods and commodities among people, by using a SWF. We have noted as, in general, this operation either is information losing or implies a strong restriction on the class of evaluation functions.

Then, we have surveyed some results on multidimensional inequality indices. As an inequality index is a synthetic measure of the degree of disparity among individuals, we loose the goal of our exercise. To investigate multidimensional inequality means to take into account several attributes, besides income, which characterize people. Then forcing all variables into a scalar is an unsuitable practice that is arbitrary and very restrictive.

Finally, we have considered the attempt of Koshevoy and Mosler to capture, by using tools of Convex Analysis, the notion of majorization and Lorenz order in a multidimensional context. The outcome obtained is unsatisfactory. Despite the analytical sophistication used by these two scholars, the results are not so far from those well-known in literature of Theory of Majorization, while a lot of work remain to do.

The future research in this field, will have to analyse the different possible kinds of transfers between matrices of individuals-characteristics. It must generalize the *T-transforms* to a more general class of transformations. In particular, it has to study, in a multidimensional context, the meaning of the composite transfer, i.e. what Shorrocks and Foster [34] call a *favorable composite transfer* (FACT), namely a kind of transfer that decreases the inequality of the distribution, through a progressive and regressive transfer[14]. How inequality changes when different transfers take place between the individual characteristics is far to be known and surely it is worth to pursuing.

A second aspect of the multidimensional inequality that it has to be deepen concerns the class of functions that preserves the matrix ordering.

In the multidimensional context, let $\Im(\prec\prec)$ and $\Im(\prec)$ be the family of functions $\phi : \mathbb{R}^{m\times n} \rightarrow \mathbb{R}$ that preserve the ordering $\prec\prec$ and $\prec$, Rinott [**?**] provided stringent conditions under which φ preserves such two orderings. An attempt of relaxing these requirements is worth to pursuing as we notice that, substantially, all social welfare functions are order-preserving functions.

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