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Dutch Book and Capacities

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Abstract - Making a book (Dutch Book) is the most prominent argument against using capacities or multiple priors in decision theory. I show that if an individual uses Choquet expected utility the strongest normative justification to reject individual decisions based on multiple priors or capacity fails. Namely, an individual who acts on the basis of capacities would not be induced to accept a Dutch Book.

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INTRODUCTION

Dutch Book is the most famous theoretical argument for using Savage expected utility to make decision under uncertainty. Ramsey (1931) and De Finetti (1937) argued that the Bayesian formalism is universally appropriate to represent the subjective probability. The fundamental idea is that the probability of an event is simply the rate at which an individual is willing to bet for or against the occurrence of that event. The betting rate follows the rule of the additivity, indeed the probability that one event and its complement is equal to the sum of their probabilities. De Finetti showed (1937) that if an individual violates the law of additivity, that is if the individual has non-additive probabilities, he could be manipulated by a bookmaker to make a sequence of bets leading to a sure loss. The standard interpretation of probability like a bet sets that an individual is willing to take both sides of a bet on the event, that is she is willing to bet on both the event and its complement. Literature about decision theory under ambiguity, which uses multiple priors or capacity approaches, has rejected a Dutch Book argument by assuming that an individual only accepts one side of the bet or refuses both of them (Kelsey and Quiggin 1992). In this paper a quite different approach against a Dutch Book argument is showed, namely it is pointed out that if the individual applies the Choquet expected utility a Dutch Book will not occur.

CHOQUET EXPECTED UTILITY AND THE DUTCH BOOK

Consider an individual who faces a given event A and its complement A^C and let $p(A)\geq 0$ and $p(A^C)\geq 0$ be the probabilities that A and A^C will occur. The individual (bettor) considers a bet on event A with betting quotient p as the amount (cents) that she bets to receive back one dollar when A occurs. The stake S gives her the gain of $(1\text{-}p)S$ if A occurs and the loss of pS otherwise, that is A^C occurs.[1] As a consequence, the net betting gains G on A and A^C, that is $S_A\, p(A)$ and $\mathrm{S}_{\mathrm{A}^c} p(A^C)$, provide the certainty equivalent of the bet on A and A^C. Suppose now that the DM is willing to bet

[1] It is worth noting that $p(A)$ might encompass a positive (negative) probability premium if the decision-maker is risk averse (loving), and similarly for $p(A^c)$.

on A and on A^c. Betting on both events, the decision-maker looses the possibility of obtaining the certainty equivalent of both bets. The expected utility of two bets are:

$$G(A) = S_A - S_A\, p(A) - S_{A^c}\, p(A^C)$$

$$G(A^C) = S_{A^c} - S_A\, p(A) - S_{A^c}\, p(A^C)$$

A bet is fair if the individual is indifferent with respect to both sides of the bet, that is if she does not perceive any advantage in acting as bettor or bookmaker. This statement is the core of a Dutch book arguments. In fact, considering the stakes as unknown variables, a system of two-linear equations is obtained. If W is the expected value of the bets, the decision problem may be represented by the following matrix:

$$G := \begin{bmatrix} 1-p(A) & -p(A^c) \\ -p(A) & 1-p(A^c) \end{bmatrix} \cdot \begin{bmatrix} S_A \\ S_{A^c} \end{bmatrix}$$

The determinant of the probability matrix is :

$$\|G\| = [1 - p(\mathrm{A}) + p(\mathrm{A}^c))].$$

If the determinant was not zero, the bookmaker could set the stakes so that the bettor's payoffs would be all negative. The bookmaker would be able do make a book against the bettor or a sure loss, if and only if $[\, p(A) + p(A^C)] \neq 1$, that is if and only if the probabilities were non-additive and the bettor was not an expected utility maximizer.

Consider a bettor facing ambiguity. She has a non-additive measure or capacity on the set of events. Let $\Omega=\{\omega_1,\ldots,\omega_L\}$ be a non empty set of states of the world and $\Sigma=2^\Omega$ be the set of all events. A set-function μ: $\Sigma \rightarrow R_+$ is called capacity or a non-additive measure if it is normalized, that is $\mu(\varnothing)=0$ and $\mu(\Omega)=1$, and is monotone, that is, for all $s_1,s_2 \in \Sigma$, $s_1 \supset s_2$ implies $\mu(s_1) \geq \mu(s_2)$. A capacity is convex or supermodular (concave or submodular) if for all $s_1,s_2 \in \Sigma$ such that $s_1 \cup s_2 \in \Sigma$

and $s_1 \cap s_2 \in \Sigma$, $\mu(s_1 \cup s_2)+\mu(s_1 \cap s_2) \geq (\leq) \mu(s_1)+\mu(s_2)$. It is superadditive (subadditive) if $\mu(s_1 \cup s_2) \geq (\leq) \mu(s_1)+\mu(s_2)$ for all $s_1, s_2 \in \Sigma$ such that $s_1 \cup s_2 \in \Sigma$, $s_1 \cap s_2 = \varnothing$ Given a real-valued function $f{:}\Omega \rightarrow R$, the natural integral of f with respect to μ is the Choquet integral, originally defined by Choquet (1954) and discussed in Schmeidler (1986). The Choquet integral of f with respect to μ is $\int_{\Omega} f(\omega) d\mu(\omega) = \int_{0}^{\infty} \mu(\{\omega \in \Omega : f(\omega) \geq t\}) dt + \int_{-\infty}^{0} [\mu(\{\omega \in \Omega : f(\omega) \geq t\}) - 1] dt$. The bettor expresses ambiguity aversion (loving) if she assigns larger probabilities to states when they are unfavorable (favorable), than when they are favorable (unfavorable), that is if her non-additive measure is convex (concave).

Under ambiguity, the bettor consider her Choquet Expected Utility (CEU, henceforth). The CEU of the bet on A is equal to $[S_A - S_A p(A) - (1 - p(A)) S_{A^c}]$, where A^c is interpreted as a "less favorable" event with respect to the bet. Analogously, the CEU of the bet on A^c is equal to $[S_{A^c} - S_{A^c} p(A^c) - S_A(1 - p(A^c))]$, where obviously now A^c is seen as more "favorable" than A by the DM. As a result, the Choquet Expected Utility of the bets, G_{CEU}, on A and A^c is

$$G_{CEU} := \begin{bmatrix} 1 - p(\mathrm{A}) & -(1 - p(\mathrm{A})) \\ -(1 - p(\mathrm{A}^c)) & 1 - p(\mathrm{A}^c) \end{bmatrix} \cdot \begin{bmatrix} \mathrm{S}_{\mathrm{A}} \\ \mathrm{S}_{\mathrm{A}^c} \end{bmatrix}$$

Considering the system of the two-linear equations with two unknown variables $S(A)$ and $S(A^C)$, the determinant is

$$\|G_{CEU}\| = [1 - p(A) - p(A^C) + p(A)\, p(A^C)] - [1 - p(A) - p(A^C) + p(A)\, p(A^C)] = 0$$

The determinant is zero if the bettor is a Choquet expected utility maximizer, that is if she attaches non-additive measures to complementary events. As a result, the bookmaker could never make a book against the Choquet expected utility bettor.

CONCLUDING REMARK

A Dutch Book argument does not apply to the Choquet expected utility decision-maker because she uses the Choquet integral to evaluate her expected net betting gain and there is no reasonable argument to reject the use of capacity or multiple priors.

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