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The Cardinality-based Ranking of Opportunity Sets in an Interactive Setting: A Simple Characterization

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Abstract - It is argued that if opportunity sets are properly embedded in an interactive environment then the lattice of set-inclusion order filters of opportunity sets is the most suitable domain for opportunity rankings. A simple characterization of the cardinality-based preorder in terms of certain invariant valuations on the foregoing lattice is provided and contrasted with the well-known Pattanaik-Xu characterization.

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1 Introduction

In the last few decades, several attempts have been made at modelling 'freedom'-related notions such as 'freedom of choice', 'opportunity', or 'liberty rights'. Two main strands of literature can be distinguished, namely works focussing on mutual *consistency* of individual liberty rights in a social choice setting, and works addressing the issue of *ranking opportunity sets* in terms of freedom of choice, with its attendant problems. Clearly enough, the proximate sources of both these strands can be traced back to some well-known seminal contributions by Amartya Sen on Pareto efficiency under liberty rights and on modelling freedom in terms of capability, respectively (Sen (1970,1985,1988): see also Laslier et al.(1998) for more recent developments). However, such a dual perspective on freedom-modelling is arguably somehow connected to a much older debate between advocates of a so-called 'negative' notion of freedom as lack of interference from other agents or institutions, and proponents of a so-called 'positive' notion of freedom as the effective opportunity to choose or to achieve. Of course, such a debate has quite a few dimensions, and nuances, that would be inappropriate to review or discuss here. Rather, we shall limit ourselves to one tentative suggestion in that connection, namely that an important source of that opposition of 'negative' vs. 'positive' freedom might lie in the contrast between an implicit emphasis on an interactive vs. non-interactive setting. The possibly significant import of that aspect is exceedingly clear from a casual inspection of the models on Pareto efficiency under minimal liberty rights and on freedom as capability as mentioned above. Indeed, the former models–whatever their degree of game-theoretic sophistication– are mainly concerned with the joint exercise of liberty rights hence with a definitely interactive environment, whereas the latter are typically conceived of as an extension of (the consumer fragment of) standard general competitive analysis i.e. the prototypical non-interactive setting. However, it seems to me that there is no intrinsic obstacle to the joint modelling of 'positive' and 'negative' dimensions of freedom: such a task can be accomplished by embedding the opportunity-ranking issue into an interactive environment.

Be it as it may, the present paper suggests one way to model *'positive'* freedom of choice in an *interactive* setting. This is done by regarding opportunities as those outcome (sub)sets of a strategic game form G which the relevant agent is enabled to enforce under G.But then, under any possible game form G with fixed outcome set X and player set N, the opportunities open to a given player in N amount to a certain superset-closed family of outcome subsets i.e. an *order filter* of the poset $(\wp(X), \supseteq)$ of inclusion-ordered outcome subsets. Since such order filters -when ordered themselves w.r.t. inclusion- constitute a (non-Boolean) distributive lattice $L^*(X) \subseteq \wp(\wp(X))$, the present proposal to model freedom of choice in an interactive setting boils down to focusing on preordered sets of type $(L^*(X), \succcurlyeq)$ rather than $(\wp(X), \succcurlyeq)$.The rest of this paper is mainly devoted to a study of the cardinality-based ranking on $L^*(X)$.

Of course, the cardinality-based preorder of opportunity sets is widely regarded as a quite trivial rule for assessing freedom of choice, and I concur with this view. However, there are at least two reasons for taking interest into the cardinality-based ranking. First, while the general consensus is that cardinality provides at best only part of the relevant information for the assessment of freedom of choice, some authors are prepared to consider it as *one* of the significant criteria, to be amalgamated with others (see e.g. Dutta,Sen(1996)). Second, given the fact that cardinality embodies *some* relevant information but just about everyone is willing to reject the cardinality-based ranking, knowledge of several different characterizations of such ranking rule is particularly valuable in that the latter are likely to provide useful suggestions about promising alternative criteria. In that vein, the present paper offers a simple characterization of the cardinality-based opportunity ranking in the 'interactive' setting mentioned above, relying on so-called *valuations* on lattices. Then, it is observed that the foregoing characterization of the cardinality-based ranking can be easily extended to the general case of an arbitrary lattice of sets. The relationship between our valuation-based approach and the Pattanaik-Xu's characterization of the cardinality preorder is also discussed.

2 Model and results

Let X the finite basic outcome set, with $\#X = m \geq 2$. We also implicitly fix a certain player i out of a finite set N of players. We shall be concerned with the problem of ranking opportunity sets for i under all possible game forms on (N, X).

To begin with consider any strategic game form on (N, X), namely $G = (N, X, (S_i)_{i \in N}, h)$ where $h : \prod_{i \in N} S_i \to X$ denotes the (surjective) outcome function.In order to represent in a most succinct manner the decision power accruing to each coalition under the interaction structure represented by G,the *effectivity function(s)* attached to G can be aptly introduced.

Generally speaking, an *effectivity function* (EF) on (N, X) is a function $E : P(N) \to P(P(X))$ such that :

$EF1)$ $E(N) \supseteq P(X) \setminus \{\emptyset\}$; $EF2)$ $E(\emptyset) = \emptyset$; $EF3)$ $X \in E(S)$ for any S, $\emptyset \neq S \subseteq N$;

$EF4)$ $\emptyset \notin E(S)$ for any S , $\emptyset \subset S \subseteq N$.

Finally, an EF E on (N, X) is *monotonic* if for any $S, T \subseteq N$ and any $A, B \subseteq X$

$$[A \in E(S) \text{ and } S \subseteq T \text{ entail } A \in E(T)] \text{ and}$$
$$[A \in E(S) \text{ and } A \subseteq B \text{ entail } B \in E(S)] .$$

In what follows we shall confine ourselves to *monotonic* EFs.

At least two monotonic well-behaved EFs may be attached to any strategic game form, according to two different specifications of apriori decision power among coalitions.

The allocation of "guaranteeing power" under strategic game form G is suitably represented by the $\alpha - EF$ of G - denoted by $E_\alpha(G)$- as defined by the following rule: for any non-empty $S \subseteq N$,

$$(E_\alpha(G))(S)=\left\{\begin{array}{c} A\subseteq X\text{: a } t^S\in\prod_{i\in S}S_i \text{ exists such that } (t^S,s^{N\setminus S})\in D \text{ and}\\ G(t^S,s^{N\setminus S})\subseteq A\\ \text{for any } s^{N\setminus S}\in\prod_{i\in N\setminus S}S_i, \end{array}\right\}.$$

Conversely, the allocation of "counteracting power" under strategic game form G is more aptly represented by the $\beta-EF$ of G, denoted by $E_\beta(G)$ and defined as follows :

for any non-empty $S\subseteq N$

$$(E_\beta(G))(S)=\left\{\begin{array}{c} A\subseteq X:\text{ for any } s^{N\setminus S}\in\prod_{i\in N\setminus S}S_i \text{ some } t^S\in\prod_{i\in S}S_i\\ \text{exists such that } (t^S,s^{N\setminus S})\in D\\ \text{and } G(t^S,s^{N\setminus S})\subseteq A \end{array}\right\}.$$

In any case, and for any given, EF the apriori decision power of each player or coalition turns out to represented by an order filter of the set-inclusion poset $(\wp(X),\supseteq)$ as defined below.

An *order filter* of $(\wp(X),\supseteq)$ is a set $F\subseteq\wp(X)$ such that for any $A,B\subseteq X$, if $A\in F$ and $B\supseteq A$ then $B\in F$. We recall that a *lattice* $(L,\geq)$ may be regarded as an *antisymmetric* preordered set –or *poset*– that is both a *join-semilattice* –i.e. for any $a,b\in L$ there exists a $\geq -$*least upper bound* $a\vee b\in L$– and a *meet-semilattice* i.e. for any $a,b\in L$ there exists a $\geq -$*greatest lower bound* $a\wedge b\in L$. A lattice is *distributive* if $a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)$ for any $a,b,c\in L$.It is well-known that the set of order filters of any poset is a distributive lattice (see e.g. Klain,Rota(1997)).

Let $(L^*(X),\cup,\cap)$ be the distributive lattice of order filters of $(\wp(X),\supseteq)$. For any $A\subseteq X$ we shall denote by $A\uparrow$ the *principal order filter* of $(\wp(X),\supseteq)$ generated by A i.e. $A\uparrow=\{B\subseteq A : B\supseteq A\}$.The *cardinality-based preorder* $\succcurlyeq_\#$on $L^*(X)$ is defined in the obvious way namely for any $F_1,F_2\in L^*(X)$, $F_1\succcurlyeq_\# F_2$ if and only if $\#F_1\geq\#F_2$.

A *valuation* on $L^*(X)$ is a real-valued function $v:L^*(X)\to\mathbb{R}$

such that $v(F\cup G)=v(F)+v(G)-v(F\cap G)$ for any $F,G\in L^*(X)$,

and $v(\emptyset)=0$.

Thus, a valuation essentially preserves the structure of the relevant lattice within the natural order of real numbers.

Moreover, a valuation v on $L^*(X)$ is *isotonic* if for any $F,G\in L^*(X)$, $F\supseteq G$ entails $v(F)\geq v(G)$, and *invariant* if for any permutation $\pi:X\to X$, and any $F\in L^*(X)$, $v(\pi[F])=v(F)$, where, with slight abuse of notation, $\pi[F]=\{\pi(A):A\in F\}=\{\{\pi(x):x\in A\}:A\in F\}$.

We recall the following basic

Proposition 1 *(see e.g. Klain,Rota(1997)): A valuation $v:L^*(X)\to\mathbb{R}$ extends uniquely to a valuation $v^*:\wp(\wp(X))\to\mathbb{R}$ on the entire lattice $(\wp(\wp(X)),\cup,\cap)$ of subsets of $\wp(X)$.In particular, for any $A\subseteq X$,*

$v^*(\{A\})=v(A\uparrow)-v(\bigcup\{F_i : B_i\uparrow=F_i\subset A\uparrow$, for some $B_i\subseteq X$ and for no $C\subseteq X$, $F_i\subset C\uparrow\subset A\uparrow\}$.

As a consequence of the foregoing Proposition the following requirements on valuations can also be introduced.

A valuation v on $L^*(X)$ is said to satisfy *Singleton-Uniformity (SU)* if there exists $k \in \mathbb{R}$ such that $v^*(\{A\}) = k$ for any $A \subseteq X$, and *Positive Singleton-Uniformity (PSU)* if in particular $k > 0$. Similarly, a valuation v satisfies *Singleton-Quasi-Uniformity (SQU)* if there exist $k \in \mathbb{R}$ and $A' \subseteq X$ such that $v^*(\{A\}) = k$ for any $A \subseteq X$ such that $A \neq A'$, and *Positive Singleton-Quasi-Uniformity (PSQU)* if in particular $k > 0$. Moreover, a valuation v on $L^*(X)$ will be said to be $\{X\}$*-Normalized* if $v(\{X\}) = 0$.

It should be remarked that the foregoing properties are not mutually independent. In particular, the following holds

Claim 2 *Let $v : L^*(X) \to \mathbb{R}$ be a SU valuation or a* {X}*-Normalized SQU valuation. Then, v is also invariant.*

Proof. Let $F = \{A_1, .., A_h\} \in L^*(X)$, and $\pi : X \to X$ a permutation such that $v(\pi[F]) \neq v(F)$.If v is a SU valuation then in particular $v(\{A_i\}) = v(\{\pi(A_i)\})$ for any $i = 1, .., h$, whence $v(F) = v(\pi[F])$ (since obviously, for any $A, B \subseteq X$, $\{A\} \cap \{B\} = \{A\} = \{B\}$ if $A = B$, and $\emptyset$ otherwise). Moreover, $\pi(X) = X$, hence $\{X\} \in F$ if and only if $\{X\} \in \pi[F]$.Thus, if v is a $\{X\}$-Normalized SQU valuation, then $v(F) = v^*(F) = \sum_{A \in F \setminus \{X\}} v^*(\{A\}) = \sum_{A \in \pi[F] \setminus \{X\}} v^*(\{A\}) = v^*(\pi[F]) = v(\pi[F])$, a contradiction. ■

Let us then consider the following requirements for a binary relation $\succcurlyeq$ on $L^*(X)$:

Valuation Consistency (VC): A relational system $(L^*(X), \succcurlyeq)$ is said to be *(strictly) valuation-consistent* iff there exists a valuation $v : L^*(X) \to \mathbb{R}$ that *(strictly) reflects* $\succcurlyeq$ i.e. for any $F, G \in L^*(X)$, $v(F) \geq v(G)$ entails $F \succcurlyeq G$ (and $v(F) > v(G)$ entails $F \succ G$).

Positive Singleton-Uniform Valuation-Consistency (PSU-VC) A relational system $(L^*(X), \succcurlyeq)$ is said to be *(strictly) PSU-valuation consistent* iff there exists a positive singleton-uniform -hence invariant- valuation $v : L^*(X) \to \mathbb{R}$ that (strictly) reflects $\succcurlyeq$ i.e. such that for any $F, G \in L^*(X), v(F) \geq v(G)$ entails $F \succcurlyeq G$ (and $v(A) > v(B)$ entails $A \succ B$).

It should be noticed here that while valuation-consistency of a binary relational system clearly entails that it is a totally preordered set, the converse is not true. To see this, consider the following example:

Example 3 *A preordered set $(\wp(X), \succcurlyeq)$ is a filtral extension of $(\wp(X), \supseteq)$ -or a set-inclusion filtral preorder (see Vannucci (1999))- if and only if there exists an order filter F of $(\wp(X), \supseteq)$ such that for any $A, B \subseteq X : A \succcurlyeq B$ if and only if $A \supseteq B$ or $B \notin F$.Now, consider the* height function *h of $(\wp(X), \succcurlyeq)$, namely for any $A \subseteq X$, $h(A) = \max\{\#\mathcal{C} : \mathcal{C}$ is a $\succcurlyeq$-chain such that $A \in \mathcal{C}$ and $A \succ B$ for any $B \in \mathcal{C} \setminus \{A\}\}$. Next, consider a principal filter $F = \{A \subseteq X : A \supseteq C\}$ with $C \subseteq X$ such that $\#C \geq 2$, and take the* total *extension $(\wp(X), \succcurlyeq_h)$ of $(\wp(X), \succcurlyeq)$ induced by h, i.e. for any $A, B \subseteq X : A \succcurlyeq_h B$ if and only if $h(A) \geq h(B)$. Let us then suppose that there exists a valuation v that strictly reflects $(\wp(X), \succcurlyeq_h)$, and let $A, B \subseteq X$, such that $A \neq \emptyset \neq B, A \cap B = \emptyset, C = A \cup B$. It follows that*

$v(C) = v(A \cup B) = v(A) + v(B)$.However, by definition, $A \sim B \sim \emptyset$, whence $h(A) = h(B) = h(\emptyset) = 0$, and therefore $A \sim_h B \sim_h \emptyset$.Moreover, by definition of $\succcurlyeq$, $C \succ \emptyset$, hence in particular $C \succ_h \emptyset$.Since -by our hypothesis- v strictly reflects $(\wp(X), \succcurlyeq_h)$, it must be the case that $v(A) = v(B) = v(\emptyset) = 0$,and $v(C) \geq 0$.As a result, we have $0 \leq v(C) = v(A) + v(B) = 0$, i.e. $v(C) = 0 = v(\emptyset)$ whence $\emptyset \succcurlyeq_h C$,a contradiction.

We are now ready to state the main result of this paper, namely

Theorem 4 *Let $(L^*(X), \succcurlyeq)$ be a binary relational system. Then $\succcurlyeq = \succcurlyeq_\#$ i.e. $\succcurlyeq$ is the cardinality-based total preorder on $L^*(X)$ if and only if $(L^*(X), \succcurlyeq)$ is* strictly PSU-valuation-consistent.

Proof. Let $\imath(.)$be the natural embedding of the natural numbers into the reals, and $\#(.)$ the cardinality function on $L^*(X)$. Then, it is easily checked that the composition $\imath(\#(.))$ is indeed a positive singleton-uniform valuation on $L^*(X)$ that strictly reflects $\succcurlyeq_\#$.

Conversely, let $v : L^*(X) \to \mathbb{R}$ be a positive singleton-uniform valuation such that $F \succcurlyeq G$ whenever $v(F) \geq v(G)$, and $v(F) > v(G)$ whenever $A \succ B$. Hence, in particular, there exists $k \in \mathbb{R}, k > 0$ such that for any $A \subseteq X : v^*(\{A\}) = k$.

It is then immediately checked by an easy induction on the cardinality of $H \subseteq \wp(X)$, that for any $H \subseteq \wp(X)$:

$$(*) \quad v^*(H) = k \cdot \imath(\#H).$$

Indeed,

let $(*)$ hold true for any $H \subseteq \wp(X)$ such that $\#H \leq m - 1$.

and let $H' \subseteq \wp(X)$ be such that $\#H' = m$.

Then obviously for any $B \in H'$

$H' = H'' \cup \{B\}$ where $H'' = H' \setminus \{B\}$

Also, by the induction hypothesis

$$v^*(H'') = k \cdot \imath(\#H'')$$

whence

$$v^*(H') = v^*(H'' \cup \{B\}) = v^*(H'') + v^*(\{B\}) - v^*(\emptyset) = k \cdot \imath(\#H'') + k = \\ = k \cdot \imath(\#H'). \qquad \blacksquare$$

It should be remarked that the foregoing Theorem is indeed tight. To check this, let us consider the following examples.

To begin with, we provide a preordered set that is indeed strictly-valuation-consistent with respect to an invariant singleton uniform valuation, but is not PSU-valuation consistent.

Example 5 *Indeed, take the preordered set $(L^*(X), \succcurlyeq')$ defined as follows: for any $F, G \in L^*(X)$, $F \succcurlyeq' G$ if and only if $\#G \geq \#F$. Then, take $v' : L^*(X) \to \mathbb{R}$ as defined by the following rule: $v'(F) = -\imath(\#(F))$.Clearly, v' is a valuation that strictly reflects $\succcurlyeq'$ .Also, $v'$is obviously invariant and uniform (with $k = -1$). Now, let us assume that $v : L^*(X) \to \mathbb{R}$ is a positive singleton-uniform valuation that reflects $\succcurlyeq'$ .Then, for any $G \in L^*(X)$ such that $\{X\} \subset G, v(G) = v^*(G \setminus$*

$\{X\}) + v(\{X\}) = \sum_{A \in G \setminus \{X\}} v^*(\{A\}) + v(\{X\}) > v(\{X\}) = k$ *while* $\{X\} \succ' G$, *a contradiction.*

Next, we define a preordered set that is strictly-valuation-consistent with respect to an invariant PSQU valuation, but is not strictly *PSU*-valuation consistent.

Example 6 *Let us consider the preordered set* $(L^*(X), \succcurlyeq^\circ)$ *where* $\succcurlyeq^\circ$*is defined as follows: for any* $F, G \in L(X), F \succcurlyeq^\circ G$ *if and only if* $[\#(F \cap (\wp(X) \setminus \{X\})) \geq \#(G \cap (\wp(X) \setminus \{X\}))]$. *It is easily checked that* $\succcurlyeq^\circ$*is strictly reflected by the PSQU (hence in particular invariant) valuation* $\mu = \sum_{i=1}^m \mu_i$ *on* $L^*(X)$ *where* $\mu_i(F) = \#\{A \subseteq X : A \in F \cap \wp(X), \#A = m - i\}, i = 1, .., m$. *Indeed, by definition,* $\mu(\{X\}) = \sum_{i=1}^m \mu_i(\{X\}) = 0$ *hence* μ *is not* PSU *(rather,* v *is a* {X}*-Normalized PSQU valuation) . Let us now suppose that* $\succcurlyeq^\circ$*is also strictly reflected by a* PSU *valuation* v_1 *on* $L^*(X)$.*It follows that* $v_1(\{X\}) > 0 = v_1(\emptyset)$, *hence* $\{X\} \succ^\circ \emptyset$, *a contradiction since, by definition,* $\{X\} \sim^\circ \emptyset$.

Finally, we give an example of a preordered set that is PSU-valuation-consistent but not *strictly* PSU-valuation-consistent.

Example 7 *Let us consider the preordered set* $(L^*(X), \succcurlyeq'')$ *as defined by the following rule: for any* $F, G \in L^*(X)$, $F \succcurlyeq'' G$ *if and only if* $[\#F \geq \#G$ *or* $\#F \geq k]$, *where* $0 < k < 2^m$.*Now, it is easily checked that* $\succcurlyeq''$ *is reflected by the PSU valuation* $v(.) = \imath(\#(.))$: *indeed* $v(F) \geq v(G)$ *implies* $\#F \geq \#G$ *hence* $F \succcurlyeq'' G$. *However, if* $F \in L^*(X)$ *is such that* $\#F = k$ *and* v *is a PSU valuation that* strictly *reflects* $\succcurlyeq''$*then* $v(\wp(X)) > v(F)$ *whence* $\wp(X) \succ'' F$, *a contradiction since* $F \sim'' \wp(X)$.

According to the focus of the present paper on an interactive setting, the characterization theorem provided above concerns the lattice $L^*(X)$ of order filters of $(\wp(X), \supseteq)$. However, our result extends readily to the lattice of sets $(\wp(X), \cup, \cap)$, provided that the notion of positive singleton-uniformity is suitably reformulated as follows:

Definition 8 *A valuation* v *on* $\wp(X)$ *is said to satisfy* Singleton-Uniformity (SU) *if there exists* $k \in \mathbb{R}$ *such that* $v(\{x\}) = k$ *for any* $x \in X$, *and* Positive Singleton-Uniformity (PSU) *if in particular* $k > 0$.*Moreover, a relational system* $(\wp(X), \succcurlyeq)$ *is said to be (strictly) PSU-valuation-consistent if and only if there exists a PSU-valuation* $v : \wp(X) \to \mathbb{R}$ *such that* $v(A) \geq v(B)$ *entails* $A \succcurlyeq B$ *(and* $v(A) > v(B)$ *entails* $A \succ B$*).*

Theorem 9 *Let* $(\wp(X), \succcurlyeq)$ *be a binary relational system. Then,* $\succcurlyeq = \succcurlyeq_\#$*i.e.* $\succcurlyeq$ *is the cardinality-based total preorder on* $\wp(X)$ *if and only if* $\succcurlyeq$ *is strictly PSU-valuation-consistent.*

Proof. Same as the proof of Theorem 4 above. ■

3 Discussion and concluding remarks

A well-known alternative characterization of the cardinality-based total preorder of *non-empty* opportunity sets on a finite set X is provided in Pattanaik and Xu(1990). Thus a comparison between the Pattanaik-Xu's theorem and our own result is in order here.

The Pattanaik-Xu's characterization concerns *preorders* and relies on the following three conditions:

Indifference between Singletons (IS): $\{x\} \sim \{y\}$for any $x, y \in X$,

Independence (IND): for any $A, B \subseteq X, A \neq \emptyset \neq B$, and any $x \in X \setminus (A \cup B)$, $A \succcurlyeq B$ if and only if $A \cup \{x\} \succcurlyeq B \cup \{x\}$,

Strict Monotonicity (SM): for any $x, y \in X$, if $x \neq y$ then $\{x, y\} \succ \{y\}$.

A first point to be observed is that removal of the implicit domain restriction adopted by Pattanaik and Xu, namely the exclusion of the empty set from the relevant domain of opportunity sets, requires a slight adjustment of the axioms. To see this, consider the following

Example 10 *Let $(\wp(X), \succcurlyeq^*)$ be defined by the following rule: for any $A, B \subseteq X$, $A \succcurlyeq^* B$ if and only if [either $A = \emptyset$ or $A \neq \emptyset \neq B$ and $\#A \geq \#B$]. It is easily checked that $\succcurlyeq^*$is indeed a total preorder, if arguably a somehow schizofrenic one. Moreover, $\succcurlyeq^*$clearly satisfies IS, IND and SM. Therefore, a characterization of $(\wp(X), \succcurlyeq_\#)$ requires the following strenghtened version of IND, with the non-emptiness restriction removed, namely:*

(IND*): *for any $A, B \subseteq X$, and any $x \in X \setminus A \cup B$, $A \succcurlyeq B$ entails $A \cup \{x\} \succcurlyeq B \cup \{x\}$.*

It should also be remarked that both IS and SM as formulated above have no bite as far as $L^*(X)$ is concerned because $F \in L^*(X)$ is a singleton only if $F = \{X\}$. However, once IS and SM are suitably reformulated as Indifference between Join-Irreducibles and Extended Strict Monotonicity (see the definitions below) in order to cover in a non-trivial way *any* lattice of sets, it transpires that *neither* of them is implied by strict valuation-consistency as such.

To make this precise, consider the following definitions:

*Indifference between Join-Irreducibles (IJ):*Let $(L, \cup, \cap)$ a lattice of sets, with $L \subseteq \wp(\wp(X))$;then, a preordered set $(L, \succcurlyeq)$ satisfies *IJ* if $A \sim B$ and for any pair$A, B \in L$ of distinct *join-irreducibles* of L (recall that $C \in L$ is join-irreducible if does not exist $\{C_1, .., C_k\} \subseteq L \setminus \{C\}$ such that $C = \bigcup_{i=1}^k C_i$).

Extended Strict Monotonicity (ESM): Let $(L, \cup, \cap)$ a lattice of sets, with $L \subseteq \wp(\wp(X))$;then, a preordered set $(L, \succcurlyeq)$ satisfies *ESM* if $A \cup B \succ B$, for any two *join-irreducibles* $A, B \in L$ such that $\emptyset \neq A \neq B$.

Thus, we may introduce the next example, namely:

Example 11 *Let us consider the preordered set* $(L^*(X), \succcurlyeq_0)$ *defined as follows: for any* $F, G \in L^*(X)$, $F \succcurlyeq_0 G$ *if and only if* $\mu_0(F) \geq \mu_0(G)$, *where* μ_0 *is the* discrete (dual) Euler characteristic *i.e. for all* $F \in L^*(X), \mu_0(F) = \sum_{i=1}^{m}(-1)^{i+1}\mu_i(F)$ *(with* $\mu_i, i = 1, .., m$ *as defined previously under Example 6). It is easily checked that* μ_0 *is indeed an invariant valuation (see Klain,Rota (1997) for an extensive discussion of the discrete Euler characteristic). Moreover,* $\mu_0(F) > \mu_0(G)$ *entails* $F \succ_0 G$*,because otherwise it must be the case that* $F \sim_0 G$*, which in turn entails* $\mu_0(G) \geq \mu_0(F)$*, a contradiction: hence* $(L^*(X), \succcurlyeq_0)$ *satisfies strict valuation-consistency. However, generally speaking* $(L^*(X), \succcurlyeq_0)$ *does not satisfy ESM. To see this, take* $X = \{x, y, z\}$*, and consider for instance the (join-irreducible) order filters* $F = \{X, \{x, y\}\}, G = \{X, \{x, y\}, \{x, z\}, \{x\}\} \in L^*(X)$*.Obviously* $F \cup G = G$*. By definition of* μ_0*, however,* $\mu_0(F) = \mu_0(G) = 1$*, whence* $F \cup G = G \sim_0 F$*.Furthermore,*$\mu_0(\{X\}) = 0$ *hence* $F \succ_0 \{X\}$ *and* $G \succ_0 \{X\}$ *which both contradict IJ since* $\{X\}$ *is also a join-irreducible element of* $L^*(X)$.

In fact, it is the requirement that the relevant valuation be positively singleton-uniform that forces Extended Strict Monotonicity of $(L^*(X), \succcurlyeq)$ in our own characterization. All in all, and perhaps unsurprisingly, the valuation-based and the Pattanaik-Xu's characterizations exhibit some significant similarities among obvious differences. However, an advantage of the valuation-based approach is that it immediately offers a characterization of the cardinality-based ranking which can be contrasted against a background of 'similar' structures. Indeed, as mentioned above, positively singleton-uniform valuations are in particular *invariant*: this fact suggests that the cardinality-based preorder can be interpreted as an opportunity ranking which arises from a particular choice of an *invariant* valuation. Moreover, the valuation-approach to the cardinality ranking is amenable to a quite natural extension to *Euclidean opportunity spaces*, thus providing a simple characterization of the volume-induced preorder (see Klain,Rota(1997) for a thorough discussion of the parallel between the finite and the Euclidean cases, and Xu(1999) for an alternative characterization of the Euclidean volume-induced preorder which relies on strict monotonicity). Finally, valuations on lattices of sets can be regarded as (essentially) *additive* real-valued functions on coalition spaces hence as *additive* coalitional TU-games in characteristic function form. Since a prominent property of such additive games is their core-stability, a further interpretation of the cardinality-based ranking as an opportunity ranking arising from a special class of *core-stable* TU-games in characteristic function form is suggested by our results. Whether and to what extent such analogies might prove to be fruitful is however left as a topic for further research.

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