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Università degli Studi di Siena
DIPARTIMENTO DI ECONOMIA POLITICA

MARCELLO BASILI
CARLO ZAPPIA

Shackle's Economic Agent and Modern Decision Theory

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Abstract – This paper assesses the rationale of George Shackle's argument against Bayesian decision making in the light of recent developments of modern decision theory. The focus is on the so-called non-additive probability approach to decision theory under uncertainty, which, not unlike Shackle's analysis, stresses the inability of agents to describe uncertain environments. A discussion of the pros and cons of the parallel between Shackle's theory and the non-additive developments is provided.

Keywords – uncertainty, decision theory, non-additive probabilities

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Marcello Basili, Dipartimento di Economia Politica, Università degli Studi di Siena
Carlo Zappia, Dipartimento di Economia Politica, Università degli Studi di Siena

1. Introduction

It is commonly understood nowadays that the economist's perspective on probability is widening. The traditional view is no longer a dogma, namely, it is disputed whether probability can be readily identified with stable relative frequencies of potential economic events, and whether probabilistic concepts, either referring to objective chances or subjectively derived, can be assumed to satisfy a well-known set of axioms for consistency. In fact, the past few years witnessed an astonishing increase in the number of studies concerned with informational asymmetries, incomplete contracts, non-expected utility theory, and so on. These studies do not amount to a mere application of formal exercises in constrained maximisation; rather, they indicate that the attitude of modern economic theory towards the question of ignorance is changing (for a comprehensive survey see Hamouda and Rowley, 1996).

It is true that the traditional formulation of the problem of decision under uncertainty is still dominant. This is mainly to be ascribed to the central role played by Bayesian individuals maximising expected utility in this formulation. It is also true that almost all applied studies are still based on probability density function, assumed to be objectively or subjectively given to the actors involved. But, at the same time, an increasing number of papers in leading mainstream journals are devoted to the study of alternative ways of formalising uncertainty. It has been recognised that the kind of ignorance individual agents face in markets is different from the "rational" ignorance usually assumed in Bayesian settings.[1] In the field of theoretical studies, the growth of scepticism has generated a number of alternative approaches intended to make room for ignorance and vagueness.[2]

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[1] Consider for example the efforts of general equilibrium theorists, who are now bent on accounting for endogenous uncertainty and information asymmetries (Magill and Quinzii 1996). The standard framework is one of missing markets and impossibility of complete insurance against future events. Indeterminacy, that is multiplicity of equilibria, is regarded as the norm, and Pareto-constrained efficiency of equilibria is not guaranteed. This kind of approach hints at a departure from traditional choice theory, a departure which is not in principle limited to exogenous uncertainty, as indicated by Hahn's (1995) conjecture about the possibility of introducing endogenous uncertainty into equilibrium theory through the notion of unawareness (see also Arrow and Hahn 1999).

[2] To put it in Hamouda and Rowley's (1997: *xx*) words: "while many textbooks retain and stress the notions of probability as established by the beginning of the 1970s, two decades of active innovation with vague and imprecise alternatives has undermined earlier myopia and complacency, widened the conventional structure of policy analyses involving uncertainty, produced some means of translating common forms of imprecision into useful ingredients for modelling frameworks, and thus generated a less hostile audience for unconventional views of uncertainty and their application to real phenomena."

This paper discusses the role of George L. S. Shackle in fostering an unconventional approach to individual decision making. Shackle's peculiar place in the history of economics is attested by the fact that, over the period going from von Newmann and Morgenstern's *Theory of Games* up to early 1970s, he was the single critic of the probabilistic approach to decision making who proposed an alternative formal corpus for dealing with uncertainty. For Shackle, any effective modelling of actual decisions and decision making requires the explicit recognition of individual mental activities, in so far as "the future is imagined by each man for himself and this process of the imagination is a vital part of the process of decision" (Shackle, 1972: 3). Shackle's non-probabilistic conceptualisation of individual decisions under uncertainty is analysed in this paper from a specific viewpoint, namely, that of a possible connection between his theory and one of the most interesting recent approaches to uncertainty, the so-called non-additive probability approach.

The first part of the paper examines how the Shackleian agent chooses among different courses of action. Shackle's main target was the assumption that the individual acted on the basis of objective probability distributions. To replace that assumption, Shackle devised the concept of potential surprise, which the paper illustrates through the example of how entrepreneurs rank alternative investment projects.

Savage's systematisation was still to come when Shackle formulated his original contribution, which appeared in his 1949 volume entitled *Expectation in Economics*. However, Shackle later contended that his argument stood even when confronted with the subjective approach to probability. Yet, his viewpoint was criticised even by critics of Savage's approach like Ellsberg. The second part of the paper discusses Shackle's main argument against the use of probability, namely that it is wrong to assume that a complete list of the possible outcomes conditioning different courses of action is known by the agent, in relation to Ellsberg's critique of Savage.

In the third part of the paper the rationale of Shackle's argument is assessed in the light of recent developments of modern decision theory. The focus is on the so-called non-additive probability approach to decision theory under uncertainty, which, not unlike Shackle's analysis, stresses the inability of agents to describe uncertain environments. A discussion of the pros and cons of the parallel between Shackle's theory and the non-additive developments is provided in the conclusion.

2. Shackle's theory of entrepreneurial action

As is well known, there have been devised approaches intended to represent radical alternatives to probability models. These approaches have drawn attention to the role of ignorance, surprise and vagueness in the analysis of choice. An increasing number of economists have taken the view that the Knightian distinction between risk and uncertainty (Knight 1921) concerns crucial economic problems which cannot usefully be treated by means of a probability distribution, even subjectively derived. Most representatives of the radically subjective approach to decision making, notably neo-Austrians, post Keynesians, institutionalists and evolutionary economists, typically argue that the way in which mainstream economic theory deals with decisions under uncertainty cannot take "genuine" uncertainty into account (as a notable example see Lawson 1985). In this respect, George Shackle provided the best formulation of a central analytical point, regularly taken up by later writers. Granted that the very construction of probability calculus relied on a certain knowledge of the structure of the world, Shackle's contention was that actual individuals did not have that knowledge. Therefore, the use of probability calculus was inappropriate, because the conditions for its application simply did not exist in actual economic contexts. In particular, Shackle argued, individuals are not capable of enumerating all possible contingencies, or states of the world. [3]

Shackle (1949 and 1961) developed a formal theory which is opposite to the Bayesian approach. In Shackle's (1972: 15) view, the standard (Bayesian) meaning of probability

> "stands for a *language for expressing judgements* as to the weight that the individual in choosing his conduct ought to give to each of a variety of rival hypotheses concerning the outcome of some one course of conduct. This language however is not merely a vessel but a mould. Form and content here, in an essential matter are one. For the language of (subjective) probability is only capable of utterance subject to an all-important mental reservation. It assumes, implicitly, that the hypotheses that have been enumerated, specified and presented for the assignment of weights are *the only relevant ones*. Thus the language of subjective probability is confined to the expression of a certain kind of meaning. And there are other meanings whose exclusion would be arbitrary and senseless."

[3] Shackle's argument, however, goes beyond realism. Shackle argued that, once the time dimension of choice was taken into account, accurate knowledge of the structure of the world became logically impossible, because only "seriable" experiments can justify probability weights. Decisions under uncertainty are not related to experiments repeated in a set of identical circumstances. Of course, this argument does not hold in a subjective interpretation of probability *a la* Savage, as noted from the outset by Arrow (1951, p. 15) and recalled more recently by Runde (2000, p. 222).

The fact that probability distributions are an additive measure distributed over an exhaustive list of hypotheses precludes, in Shackle's view, the analysis of decision making in the context of an investment decision. Typically the entrepreneur is not able to list all the relevant possibilities associated to the investment decision, for some of these cannot be foreseen beforehand.

As a result, Shackle's proposal was to substitute some novel concepts like potential surprise, epistemic intervals, and focus values for probability distributions. These concepts were intended to capture both the mental processes and the non-repetitive, often irreversible nature of actual economic decisions. Shackle emphasised the typically imprecise domain and regularly fuzzy character of actual decisions and focused on the subjective, idiosyncratic nature of human judgements (for a concise assessment see Hamouda and Rowley 1996, Ch. 4).

To illustrate the essentials of Shackle's theory, consider an entrepreneur's choice between uncertain prospects. For instance, he/she does not know which of two pieces of equipment he/she should acquire.[4] Due to the fact that the investment will not become profitable until a certain date in the future, an analysis of expectations regarding the competing strategies of investment is needed. Shackle maintains that the choice of investment cannot be made through a comparison between the expected discounted value of the streams of returns (after cost of purchase) of the two pieces of equipment. It is impossible, in fact, to maximise an expectation function based on probability distributions over future states of the world. This is because, in Shackle's words (1972, p.20),

> "statistical probabilities are knowledge … in regard to the wrong sort of question, when our need is for weights to assign to *rival* answers. The various hypotheses or contingencies to which frequency ratios are assigned by statistical observation are not *rivals*. On the contrary, they are members of a team. All of them are true, each in a certain proportion of the case with which, *all taken together as a whole*, the frequency-distribution is concerned. … But in answer to the question about a single trial, the frequency-ratios are not knowledge."

In Shackle's view, the same argument holds with respect to subjective probability. The basis for probability expressing personal knowledge can be indefinitely various in composition and it can include knowledge of all sorts; yet "upon a structure which can consist of knowledge, reasonably so-called in a practical sense, there must, for a decision-maker faced with

4 This is Shackle's favourite example. See for instance Shackle 1953, pp. 48-55. This section follows Ford (1990, pp.21-30).

uncertainty, rest something which can by no means be proven: which does not follow from this knowledge but is only suggested by it; something including an element of origination" (Shackle 1972, p.20).

Suppose that the entrepreneur is asked to make an exhaustive list of the specified distinct events which can affect the value of alternative investments, as required by the application of subjective probability. He/she

> "will in the end run out of time for its compiling, will realize that there is no end to such a task, and will be driven to finish off his list with a residual hypothesis, an acknowledgement that any one of the things he has listed can happen, and also any number of other things unthought of and incapable of being envisaged before the deadline of decision have come: a Pandora's box of possibilities beyond reach of formulation" (Shackle 1972, p. 22).

In Shackle's view, probability distributions are not apt for this kind of analysis because in a probabilistic framework the possible emergence of one "residual" unimagined hypothesis is excluded. In fact, the emergence of a residual hypothesis would imply a change in the degree of belief assigned to the hypotheses considered from the outset.

For this reason, Shackle assumed that a process of "editing" expectations, different in kind from probabilistic reasoning, was necessary. This editing process was based on two premises, regarded as psychological insights. The first premise is that decision-makers are concerned with separating out gains from losses. The entrepreneur imagines that a range of outcomes is feasible in view of the gains and losses which each investment equipment is likely to occasion. The second premise is that decision-makers edit the set of possible gains and losses relating to each investment equipment by taking into account only the best and the worst outcomes associated with the investment. The outcomes associated with each investment equipment cannot be summarised by an expected value, because the conditions for probability calculus to be applied do not exist. As a result, Shackle saw as "logically irrelevant" the procedure of attributing to each alternative option an index, like a weighted average of outcomes (Shackle 1953, p. 39).

As recalled earlier, Shackle intended to devise a formal corpus alternative to the all-embracing (subjective) probability approach to decision making. The question arises: if probabilities cannot be used, what is left for the purpose of analysis? Shackle proposed a schema in which, first, probabilities are replaced with the notion of potential surprise, and, second, mathematical expectation is replaced with the best-and-worst-outcomes device, with the two extremes called focus gain and focus loss respectively. As regards potential surprise, given that probability cannot be used for expressing "subjective judgements," Shackle assumed that the entrepreneur attributes "degrees of disbelief" to possible outcomes; the entrepreneur is supposed to measure the degree of implausibility of each outcome by this scale; a degree of disbelief is termed potential surprise. Measuring the degree of belief by probabilities implies that there must be an event, or a set of events, which will occur with perfect certainty. But, Shackle argues, this event, to which probability one is attributed, is meaningless in truly uncertain contexts. Potential surprise, on the contrary, measures the possibility of an event. Zero degree of potential surprise, that is, absence of disbelief, means that the event is perfectly possible.[5] Moreover, Shackle's framework includes potential surprise functions in which zero degree of potential surprise is attached to rival hypotheses: "there is, in general, no limit to the number of mutually exclusive hypotheses to all of which simultaneously a person can, without logical contradiction, attach zero potential surprise" (Shackle 1952, p. 31). One of the reasons supporting Shackle's use of potential surprise is of the utmost importance for our argument.[6] As stated by Shackle in one late assessment of his own theory (1986, p. 287), the fact that potential surprise is an inverted expression of epistemic standing is crucial, because "by this inversion of measurement we rid ourselves of the crippling additive character of probability, inherited from its origin in games of chance."

Thus, the entrepreneur ranks the outcomes of each investment equipment, with reference to both gain and losses, by means of subjective judgements concerning the degrees of potential surprise involved. All the possible outcomes are included, each with the associated potential surprise index, and the occurrence of a new, previously unanticipated outcome can be accommodated without necessarily reducing the degree of potential surprise attached to other outcomes. By means of a complicated process the entrepreneur is now supposed to telescope gains and losses into one pair of gain and loss, called focus-gain and focus-loss. This process involves the use of an "ascendancy function" that maps gains/losses and the associated potential surprise value into an index intended to represent the dominance of one expectational element over another in the entrepreneur's mind (see Shackle 1953 and Ford 1990). The Shackle's entrepreneur, however, does not carry forward all the expectational elements related to each investment equipment to the final stage wherein he selects the best investment equipment. By means of the maximisation of the ascendancy function, the expectational elements involved in the investment decision are reduced to two monetary

[5] On the significance of possibility as an epistemic state, and on Shackle's distinction between probability and possibility, see Levi 1984.

[6] For the rationale of the other reasons, see in particular Shackle (1952, pp. 33-35).

values, a gain and a loss. Pairs of gain and losses, each one identifying a different investment equipment, then can be ranked by means of a map of indifference curve in a gain and loss plane which summarises the entrepreneur's preferences over investment decisions, termed by Shackle the gambler-preference map. Eventually an alternative formal corpus for dealing with uncertainty is provided.

Shackle's theory was widely discussed in the 1950s. Arrow (1951) singled it out as one of the only two approaches alternative to von Neumann and Morgenstern axiomatisation, the other alternative being "the modern theory of statistical inference" of Neyman and Pearson and its (foreshadowed) axiomatisation in the line of De Finetti (1937) and Savage (1954).[7] In 1959, an entire issue of *Metroeconomica* was dedicated to a symposium on Shackle's work. But the formal apparatus was regarded both as lacking the virtue of simplicity and arbitrary in the process of eliciting focus-gains and focus-losses.[8] Moreover an axiomatic foundation for the theoretical structure was not offered. Eventually, while the methodological arguments of Shackle continued to be referred to by critics of the mainstream, the formal theory sunk to oblivion.[9]

3. Ellsberg's critique of subjective probability and Shackle's argument

The mainstream approach to decision under uncertainty rests mostly on Savage's (1954) expected utility theory. Basically, Savage made it possible to apply all rules of probability theory to individual beliefs. This approach centres around two fundamental assumptions. First, a complete list of possible future states of the world is available to the individual – a list which, in an interpersonal context, is common knowledge to all individuals. The individual is endowed with subjective beliefs over the state space. These beliefs are represented by a well-defined (additive) probability function. More precisely, the subjective probability distribution of the individual is derived from axioms on a preference ordering over uncertain prospects.

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7 After reviewing Shackle's construction of the gambler-preference map, Arrow (1951, p.38) commented: "This theory is not based on consideration of rational behaviour, which Shackle specifically rejects, but on an alleged inability of the mind to consider simultaneously mutually exclusive events"

8 As note by Ford (1990, p. 40) the fact that the ascendancy function does not encapsulate the information contained in the whole range of potential surprise function implies that "when two strategies are being evaluated by the gambler-preference function on the bases of focus-values, it is possible for a strategy to be chosen that is, as it were, almost dominated stochastically by the other."

That is, to adopt Savage's expression, individual probability distributions originate from "acts" defined on the state space. As a result, individuals in uncertain settings are supposed to be able to undertake expected-cost/expected-benefit analysis in information gathering, and hence to reach an informational optimum. This is why Savage's individuals are termed "probabilistically sophisticated" individuals (Machina and Schmeidler 1992). As shown in the previous section, this first assumption is a main target of Shackle's critique.

The second fundamental assumption of probabilistic decision making has to do with the cognitive capabilities of the decision-maker. The processing of information consists in a Bayesian process of updating individual beliefs (prior probability distribution), when he/she receives a signal on the realisation of the state. This assumption follows from the implicit hypothesis that individuals are rational in a strong sense, namely, that they manage to deduce all logical propositions contained in the axioms of the theory. Of course, this second assumption is highly questionable after Simon (1982), and it has been abandoned in bounded rationality and evolutionary models (see in particular Nelson and Winter 1982). However, computational and cognitive problems will not be discussed here, since these were outside Shackle's research field, at least as far as the formal structure of his analysis was concerned.

It hardly needs saying why Savage's representation theorem is of the utmost importance in the development of modern decision theory. Von Neumann and Morgenstern represented uncertainty by means of explicit probabilities, so that the objects of individuals' choices consisted of well-defined probability distributions over outcomes. However, real-world uncertainty mostly consists of alternative events (or states), and choices are typically bets which assign outcomes to these events. Savage's approach derives the principle of expected utility maximisation from a number of axioms over acts. In the tradition of the choice-theoretic approach to subjective probability developed by Ramsey and De Finetti, individual subjective probabilities are elicited from choices. But an implication of this approach is worth considering. The appropriateness of a subjective interpretation of probability rests on the claim that subjective probability satisfies the rule of additivity. De Finetti argued that an individual's betting rates could not violate this rule, because if it could another individual would take advantage of the situation. This argument is known as the "Dutch book" argument: a Dutch book is a sequence of bets with a non-positive outcome in every state of the world and a negative outcome in at least one state. It can be shown that an

9 For a notable exception, see Ford's (1983) attempt to axiomatise Shackle's theory.

individual acting on the basis of a unique, additive probability cannot be induced to accept a Dutch book.[10]

A first, partial step to move on towards a more convincing representation of uncertainty can be sketched as follows. It is helpful to retain the Bayesian assumption that, in principle, individuals are able to formulate a single subjective probability distribution which enables them to cope with any possible uncertain situation. What is questioned is the reliability of this distribution when individuals are aware that an unlisted event can happen in the future, or when they take decisions conditioned by a non-repetitive event. The issue of the reliability of the subjective probability distribution emerged as a result of experimental evidence, which revealed systematic violations of Savage's "sure-thing principle."[11] These violations were inconsistent with the hypothesis of expected utility maximisation. The most discussed of such violations are the Allais Paradox (Allais 1953) and the Ellsberg Paradox (Ellsberg 1961).

The Allais Paradox is a seminal counter-example concerning the validity of the expected utility theory. However, this paradox does not necessarily undermine the concept of probability per se: it can be considered of limited significance from a normative point of view because it involves payoffs that are out of the ordinary and probabilities that are very small or very large.[12] On the contrary, the challenge posed to the expected utility theory by the

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[10] De Finetti (1974) assumed that the individual should be offered odds not only on certain events or proposition, but also on the set of events covering the entire space state. The individual should also be ready to accept either side of a bet: that is, he/she should be prepared to be either for or against any given event at given odds. While it seems more plausible that individuals are willing to take only one side of a bet, the argument for additivity of the subjective probability distribution does not apply to one-side bets (Shafer 1986).

[11] In the standard representation, an individual maximising his/her expected utility, either in the case in which probabilities are objectively (von Neumann and Morgenstern 1944) or subjectively held (Savage 1954), weights consequences through a single probability measure on the set of states of the world, in such a way as to induce the linearity of the preference function. Linearity in probabilities is a direct consequence of two very similar axioms, namely, the "independence axiom" in von Neumann-Morgenstern's framework, and the "sure-thing principle" in Savage's theory. The independence axiom states that, given two alternatives (lotteries in technical language), with each of them composed of an action and a certain common act, preferences between them should be independent of any common consequence with identical probability (common act). The sure-thing principle assumes that the decision-maker, when he/she chooses between actions, ignores states in which actions yield the same consequences.

[12] Morgensten's (1979, p. 180) reaction, as is well-know, was totally dismissive of Allais' findings: "the [expected utility] theory is "absolutely convincing" which implies that men will act accordingly. If they deviate from the theory, an explanation will cause them to readjust their behaviour." But Allais (1990, p. 8) insisted that the paradox "does not reduce to a mere counter-example of purely anecdotal value … [but] is fundamentally an illustration of the need to take account not only of mathematical expectation of cardinal utility, but also of its distribution as a whole about its

Ellsberg Paradox is crucial as regards our reconstruction so far, in view of the fact that it focuses on the belief side of the decision problem and involves considerations of ambiguity and degree of confidence. The following experiment was formulated by Ellsberg (1961). An individual faces an urn containing 30 red balls and 60 balls in some combination of black and yellow; there is no information whatsoever about the respective number of black and yellow balls in the urn (unknown proportion). A ball is drawn from the urn. There are two pairs of acts, X and Y, and X' and Y.' Acts have consequences of 1 or 0 as follows: choosing X one gets 1 if the ball is red and 0 if it is black or yellow; choosing Y one gets 0 if red or yellow and 1 if black, choosing X' one gets 1 if red or yellow and 0 if not; choosing Y' one gets 0 if red and 1 if black or yellow. Ellsberg reported that, among those asked,[13] most people chose X instead of Y, and Y' instead of X,' thus revealing a remarkable preference for betting on known probabilities of winning. That is, it appeared that confidence in estimates of subjective probabilities was taken into account. The results of this experiment are inconsistent with Savage's sure-thing principle. In fact, both pairs of acts have different consequences only when the yellow state occurs, and these consequences are the same both for X and Y (the individual gets 0) and for X' and Y' (the individual gets 1).[14]

Moreover, the beliefs of the individual exhibiting these preferences cannot be represented through an additive probability distribution. Suppose p(r), p(b) and p(y) are the subjective probabilities of each possible draw, with $p(r)=1/3$. Setting $U(0)=0$, Savage's subjective expected utility implies that X is to be preferred to Y if and only if $p(r)U(1)> p(b)U(1)$ or $p(r)>p(b)$. Likewise Y' is preferred to X' if and only if $p(b\cup y)>p(r\cup y)$. This contradicts the assumption that probabilities are additive: in fact, given $p(b\cap y)=0$, if $p(b\cup y)=p(b)+p(y)$ then to prefer Y' to X' implies $p(b)>p(r)=1/3$, which conflicts with what is implied by preferring X to Y, that is, $p(b)<p(r)=1/3$.

As a result, these preferences contradict not only the expected utility theory, but also every other theory of rational behaviour under uncertainty that assumes a unique additive

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average, basic elements characterizing the psychology of risk." For a reconstruction of Allais' influence more sympathetic than the one sketched here, see Jallais and Pradier (2003)

13 Ellsberg proposed his choice experiment to a number of well-known decision theorists, including Savage himself.

14 Ellsberg recalled in his paper that Knight (1921) had performed a similar thought experiment. Knight used an urn containing red and black balls in an uncertain proportion to discuss the so-called principle of insufficient reason. But Ellsberg should also have referred to Keynes's *Treatise on Probability* (1921), which features the same thought experiment. For Ellsberg'S acknowledgement, see his Ph.D. thesis recently published as Ellsberg 2001.

probability measure underlying choices. In Ellsberg's (1961: 654) words, "it is impossible, on the basis of such choices, to infer even qualitative probabilities for the events in question … [and] to find probability numbers in terms of which these choices could be described – even roughly or approximately – as maximising the mathematical expectation of utility." The violations of both the "complete ordering of actions" and the "sure-thing principle", pointed out by Ellsberg in his hypothetical experiments, have been confirmed by many laboratory experiments replicated in recent years (Camerer and Weber 1992). These experiments suggest that most agents prefer making unambiguous choices rather than ambiguous ones. Individual choices can be affected by the nature of one's information concerning the relative likelihood of events. What is at issue, Ellsberg (1961: 657) clarified, "might be called the *ambiguity* of this information, a quality depending on the amount, type, reliability and unanimity of information, and giving rise to one's degree of confidence in an estimate of relative likelihood."

As already noted, the traditional formulation of the problem of decision under uncertainty is still dominant, especially because of the central role it ascribes to Bayesian individuals maximising expected utility. But, mainly as a result of Ellsberg's contribution, an increasing number of papers in leading mainstream journals have been devoted to the study of alternative ways of formalising uncertainty.[15] Ellsberg's experiment has been interpreted as a demonstration that an accurate representation of preferences should include not only probability assessments, but also a measure of the degree of confidence in those assessments (Mukerji 1997, and Eichberger and Kelsey 1999).

It may appear at first that Ellsberg's reference to ambiguity is not related to Shackle's view that subjective probability cannot accommodate a residual hypothesis. As recently recalled by Runde (2000, p. 236 n.) in his assessment of the inability of decision theory to account for Shackle's critique, studies related to the Ellsberg Paradox in fact deal with a problem "of ambiguous probabilities" rather than with one "of providing an exhaustive list of states." But there exist studies originated from the Ellsberg Paradox which seem to imply a different interpretation of the issue at stake. It must be noted, in the first place, that Ellsberg intended to challenge Savage's view that individuals' beliefs about unknown states of the

[15] This point is best summarised by Hamouda and Rowley (1996, p. 49) as follows: "After three decades of the presentation of his paradox, we should acknowledge the commonsense in Ellsberg's two modest suggestions that we avoid E(xpected) U(tility) axioms in the 'certain, specifiable circumstances where they do not seem acceptable' and that we should, in these circumstances, give more attention to alternative decision rules and non-probabilistic description of uncertainty."

world can be adequately represented by additive probability distributions. In fact, the behaviour reported by Ellsberg is consistent with $p(b) = 1/3 - \varepsilon$, and $p(r \cup y) = 2/3 - \varepsilon$, that is with beliefs that do not satisfy additivity, for small ε. Shackle's point that probability is a non-distributional variable hints at the same question, with ε representing uncertainty. But, contrary to Shackle's view, the dismissal of additivity does not entail the abandonment of probability to represent beliefs, as the next section of the paper intends to show.

Additionally, while it is true that the Ellsberg Paradox does not allude to a lack of clarity in the description of events and consequences of actions, Ellsberg's challenge to Savage's approach is originated by the idea that decisions under uncertainty imply a fuzzy perception of the likelihood of events on the part of individual agents. The decision-maker which Ellsberg's analysis refers to is an individual whose subjective knowledge about the environment is consistent with more than one probability distribution; furthermore, he/she is not endowed with a unique second-order probability distribution over the set of possible probabilities.[16] It is possible that there lies here one of the reasons why Shackle refused to adopt an index summarising expectations, like the maximisation of expected utility. His use of consequences in pairs (focus-gain and focus-loss) instead of an average of them is justified by the idea that a decision maker is forced to simplify the expectational elements he confronts by the fact that a single, crucial decision cannot be taken as if the (subjective) probabilities associated with possible events conditioning this decision were reliable. In a situation in which many, in principle infinite, priors are possible the Shacklean entrepreneur does not accept to single out one prior, as subjectivists like Ramsey and Savage would do, and reacts by reducing the possibilities he/she faces to a kind of point estimate for the imagined gain and

[16] Even if he did not tackle it, Savage was aware of the relevance of this issue when he stated: "there seem to be some probability relations about which we feel relatively "sure" as compared with others. When our opinions, as reflected in real or envisaged action, are inconsistent, we satisfy the unsure options to the sure ones. The notion of "sure" and "unsure" introduced here is vague, and my complaint is precisely that neither the theory of personal probability, as developed in this book, nor any other device known to me renders the notion less vague. There is some temptation to introduce probabilities of a second order … But such a program seems to meet insurmountable difficulties … [because] once second order probabilities are introduced, the introduction of an endless hierarchy seems inescapable" (Savage 1954, pp. 57-58). Ellsberg (1961, p. 658) conceded that "Savage himself alludes to this sort of judgement and notes as a difficulty with this approach that no recognition is given to it." However, Ellsberg also compared Savage's point with Knight's perspective. According to Ellsberg, Knight "asserts what Savage's approach tacitly denies, that such overall judgment may influence decisions."

loss associated with the strategy. The next section reports an alternative both to Savage and Shackle procedure.[17]

4. Shackle and modern decision theory

Shackle's (1961: 49-50) distinction between distributional uncertainty variables, which can be used if "the list [of suggested answers to a question] is *complete without a residual hypothesis*," and non-distributional variables, which should be used when "the list in order to attain formal completeness must be rounded off with a residual hypothesis," reflects an essentially non-additive characteristic of his theory. This was clear to Shackle from the very beginning of his research. In a response to some critics of his first work on this topic, *Expectation in Economics* (1949), he made clear that his system was non-additive. Granted that entrepreneurial decisions are, in Shackle's view, an example of "crucial" decisions, he argues that "when an experiment, a question about the future, is unique, isolated, or crucial, it does not make sense to add together its rival hypothetical outcomes or answers," because they are mutually exclusive outcomes. In these instances, "to decide between rival courses of action … he [the entrepreneur] cannot add the rival hypotheses, he must choose among them." Thus, Shackle distinguishes between the cases in which "it makes sense to use frequency-ratios probabilities as a measure of acceptance" and hence the appropriate procedure is the additive one, and crucial decisions, which require a different approach. To tackle the latter, the traditional procedure of adding up the probabilities of alternative outcomes is "an irrational compromise," Shackle contends.

As he put it, what was needed in order to describe "mental states of uncertainty" was "a measure of acceptance of a hypothesis proposed in answer to some question, that shall be independent of the degrees of acceptance simultaneously accorded to rival hypothesis," that is, "a measure of acceptance by which the individual can give to new rival hypotheses, which

[17] It is worth mentioning that Ellsberg (1961, p. 644) viewed Shackle as a representative of "the Knightian position that statistical information on frequencies within a large, repetitive class of event is strictly irrelevant to a decision whose outcome depends on a single trial." Ellsberg (1961, 646) rejected Shackle's explanation for dispensing with probability calculus in the case of single decisions, but concluded that, in the class of situations of choice in which people do not conform to Savage's postulates, "certain proposals for alternative decision rules and nonprobabilistic descriptions of uncertainty might prove useful." In this respect, Ellsberg referred to various theories including Shackle's.

did not at first occur to him, some degree, and even the highest degree, of acceptance without reducing the degrees of acceptance accorded to any of those already present in his mind." This measure of acceptance, as has been shown in the second section, is represented by Shackle by means of the concept of potential surprise. The defining characteristic of potential surprise, namely, that the degrees in which it is assigned to the various members of an exhaustive set of rival hypotheses are not required to make any particular total, represents "the first of the ways in which my system is non-additive." (Shackle 1949-50: 70-71).

But Shackle recognises that there is a second way in which his theory is non-additive. The traditional procedure allows "all, or some large subset, of the hypotheses which the decision-maker has considered and has not absolutely excluded as impossible, to have some way in influencing his decision." On the contrary, Shackle proposes to concentrate attention on "those two or few of them [hypotheses] which out-rival all the others in their power of stimulus," that is, he recommends concentrating on focus-values.[18]

This section of the paper focuses on the non-additive character of probabilistic judgements made to deal with crucial, unique decisions. This is the topic of some recent models of individual decision making. In particular, reference will be made to the so-called non-additive probability theory of decision under uncertainty (Schmeidler 1989 and Gilboa and Schmeidler 1994). This theory starts with the view that ignorance is an inherent feature of every decision regarding future events, and moves on to propose an axiomatic foundation for decision making intended to dispense with the assumption that the individual has complete structural knowledge of the environment, that is a generalisation of expected utility which accommodates Ellsberg's evidence. In this respect, this theory calls to mind the Shacklean assertion that the future is the unpredictable consequence of creative choices made by individual agents. However, this theory shows that the reliance of standard Bayesian theory on probabilistic judgements based on point-probability estimate, a reliance that Shackle intended to oppose, cannot be viewed as a justification for dispensing with probability calculus *tout court*, once room for non-additive probability distributions is made.

[18] Interestingly, Shackle was also aware that his proposal may have failed to gain acceptance because the adoption of a non-additive procedure was really unconventional: even "people who have never asked themselves how frequency-ratio probability can possibly have any relevance or meaning for the analysis of uncertainty (i.e. ignorance) about the outcome of a non-divisible isolated experiment, are predisposed towards some sort of additive solution, because it is by an additive procedure that we reach knowledge of the outcome of divisible experiment." But, Shackle wondered, "if we discard the notion of frequency ratio probabilities, ought we not also to discard our

So far, the main task emerging from our reconstruction is how to represent the individual agent's confidence in a probability assessment. In situations where some events are ambiguous, the weights of the priors can depend on the attitudes of decision-makers towards their probability assessments, that is, on their confidence in the specifications of states of the world those assessments provide. The point here is that, even if uncertainty encompasses intuitive concepts like ambiguity and vagueness, it is possible to formulate a more precise notion of uncertainty (as different from risk) through an accurate description of the way in which the individual misses knowledge of the states of the world.

Consider a decision problem in which the states of the world included in the model do not exhaust the actual ones. A description of the world can be considered as a mis-specified model whenever certain states are not explicitly included in the model. When an individual agent does not know how many states are omitted, his/her beliefs can be represented by either a non-necessarily-additive measure or a set of additive probability distributions on the set of events. Gilboa and Schmeidler (1994) represent preferences by a Choquet integral of expected utility with respect to a (convex) capacity, that is, a not necessarily additive probability distribution.

A convex capacity, μ, is monotone measure that is normalised to 1 on the full set and 0 on the null set, like probability, but, unlike probability, the sum of the capacities of two subsets may be strictly less than the capacity of the union of these sets.[19] The convexity of the capacity is a property suggested by the Ellsberg Paradox where, as shown in the previous section where the belief of the unambiguous event of drawing a black or a yellow ball strictly exceeded the sum of the beliefs for the ambiguous events that a black ball is drawn or that a yellow ball is drawn. Since μ is a non-additive measure, the integration of a real-valued function with respect to μ is impossible in the Lebesgue sense. It has been shown that the proper integral for a capacity is the Choquet integral. The Choquet integral with respect to a capacity is a generalisation of the Lebesgue integral, which requires that states of the world have been ranked from the most to the least favourable ones, or vice versa, with respect to

predisposition towards additive solutions and start afresh with an open mind?" (Shackle 1949-50, p.74).

[19] In a more formal way, let $\Omega=\{w_1,\ldots,w_n\}$ be a non empty set of states of the world and let $S=2^{\Omega}$ be the set of all events. A function $\mu:S\rightarrow R_+$ is a non-necessarily-additive probability measure, or capacity, if $\mu(\emptyset)=0$ and $\mu(\Omega)=1$, and if for all $s_1,s_2\in S$ such that $s_1\supset s_2$, $\mu(s_1)\geq\mu(s_2)$. A capacity is convex if for all $s_1,s_2\in S$ such that $s_1\cup s_2$, $s_1\cap s_2\in S$, $\mu(s_1\cup s_2)\geq\mu(s_1)+\mu(s_2)-\mu(s_1\cap s_2)$.

their consequences. As a result, the Choquet integral is a generalisation of the mathematical expectation usually used in expected utility models with respect to a capacity.[20]

It has also been shown that a decision model with non-additive measure on the state space can be embedded in a decision model with an additive measure (probability) in which the enlarged state space includes all the possible missing states (Mukerji 1997). As a result, it is possible to relax the non-additivity of a measure at the expense of the dimension of the decision model. In the representation of uncertainty by a non-additive measure on the space state, a relationship between the epistemic status of the individual (consisting of the awareness of incomplete knowledge and the limited reliability of likelihood assessments) and his/her choice is implicitly assumed. Mukerji (1997) clarifies this relationship by means of a "two-tiered" state space model which embeds "a space on which the individual assigns primitive beliefs and a space of payoff relevant states, i.e. states on which the available acts are directly defined" (Mukerji 1997: 25). The two-tiered state space modelled by Mukerji happens to be mathematically isomorphic to the enlarged space of Gilboa and Schmeidler. At first the individual assigns his/her beliefs (priors) on his/her state space perceived as simpler (primitive), and then he/she "infers beliefs about the events to which the outcomes of acts are directly related" (Mukerji 1997: 25). The inferred space is called derivative world. It is straightforward to interpret the primitive and derivative worlds as Savages's small and grand worlds, respectively.[21]

The primitive state space (the small world) is a set of objects of which the individual has direct experience, clear intuition and empirical knowledge; belief assessments on this state space express this confidence. Likelihood assessments on the derivative world (the grand

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20 The Choquet integral of *f* with respect to μ is

$$\int f d\mu = \int_0^{\infty} \mu(\{w|f(w) \geq t\})dt + \int_{-\infty}^{0} [\mu(\{w|f(w) \geq t\}) - \mu(\Omega)]dt$$

21 In Savage's theory the grand world is the complete list of states which are of concern to an individual. The small world is a construction derived from a partition of the grand world into subsets, or small-world states, which are subsets, or events, of the grand world. Savage (1954: 9) maintains that an individual has to confine his/her attention to a relatively simple situation in almost all his/her decision; that is, in practice, the individual is concerned with a small world, which is "derived from a larger by neglecting some distinctions between states, not by ignoring some [grand-world] states outright." By considering a small world as crucial for his/her decision, the individual describes states of the world and consequences in limited detail. It is worth noting that the individual can consider progressively a more refined and detailed small world, until he/she arrives to the grand world which takes everything into account. But to demand that "one envisage every conceivable policy for the government of his whole life (at least from now on) in its most minute details, in the light of a vast number of unknown states of the world … is utterly ridiculous" (Savage 1954: 16).

world) are deduced from an "implication mapping" which represents the individual knowledge of the association between the two worlds. As a result, "the decision-maker's knowledge about the likelihood of an event in the derivative frame is given by the sum of the beliefs assigned to those elements of the primitive frame whose implications are sub-events of the event in question (Mukerji 1997: 33).

Depending on the epistemic condition informing the agent's situation, the beliefs on the derivative frame may have a non-additive representation. In fact, if the individual transfers a likelihood assigned to an event in the small world to an event in the grand world, that is, if he/she is unable to distribute beliefs across the elements of the grand world, then the non-additivity of subjective probability measures becomes an expression of the limits of the decision-maker's understanding of the possibilities of the world as well as of his/her awareness of these limits. It is straightforward to assume that an individual behaves as if he/she has a set of priors or a non-additive measure rather than a well-defined probability if his/her perception of the grand world is fuzzy, incomplete or vague.[22]

By means of this representation we have a formal definition of uncertainty which makes the Knightian distinction in principle operational. It can be said that a decision-maker faces Knightian, radical uncertainty when either he/she has a mis-specified description of the states of the world, or is unable to assign a reliable probability distribution to states of the world, or is ignorant of the world in which he/she has to act, or attaches an interval of probabilities to each event. Furthermore, it can be said that the decision-maker expresses Knightian uncertainty aversion if he/she assigns larger probabilities to states in which consequences are unfavourable, than to states in which consequences are favourable, that is, if his/her non-additive measure is convex (Dow and Werlang 1992). Hence, the convexity of the capacity captures the decision-maker's Knightian uncertainty aversion and encompasses the conservative statement that the decision-maker acts "as though the worst were somewhat more likely than his best estimates of likelihood would indicate [and] he distorted his best

[22] As Savage (1954: 82-84) is keen to claim, subjective expected theory should be applied to small worlds only, that is, when all the possibilities can be exhaustively enumerate in advance, and all the implication of all possibilities explored in detail so that they can be labelled and placed in their proper position. He stresses the "practical necessity of confining attention to, or isolating, relatively simple situations in almost all applications of the theory of decision developed" in his *Foundations*. In fact, Savage's rejection of the positive critique to his theory, namely that "real people frequently and flagrantly behave in disaccord with utility theory," is mainly based on the distinction between the grand and the small world (Savage 1954: 100-101). But Savage is forced to admit that a small world "is completely satisfactory only if it is actually a *microcosm*, that is, only if it leads to a probability measure and a utility well articulated with those of the grand world" (Savage 1954: 88).

estimates of likelihood, in the direction of increased emphasis on the less favourable outcomes and to a degree depending on his best estimate" (Ellsberg 1961: 661).[23]

To take stock of our analysis so far. For both theoretical and empirical reasons economists working in decision theory have sought to generalise the expected utility model. At the basis of this developments there is the distinction between risk and uncertainty usually attributed to Knight. Although this distinction is deemed unimportant to scholars working in a Bayesian perspective (for textbook evidence see Hirshleifer and Riley 1992), we have briefly expounded an axiomatic development which tries to incorporate such a distinction. The model discusses an individual maximising his/her expected utility with a non-additive probability, and includes the possibility that the subjective probability attributed, for instance, to two mutually exclusive events does not necessarily add up to 1, in order to reflect the individual's attitude to uncertainty. What is more, the axiomatisation provides a base for dealing with situations in which the uncertainty of the individual involves the existence of a third (or more) event, whose occurrence had no probability attached at the outset.[24]

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23 Alternatively, Knightian uncertainty can be represented by a set of possible priors instead of a single one on the underlying state space. That is, the individual knows enough about the problem at hand to rule out a number of possible distributions. In this case the agent has multiple additive probability measures P on $\Omega=\{w_1,...,w_n\}$, and his/her preferences are compatible with either the maximin or the maximax expected utility decision rule, where the maximin (maximax) expected utility postulates that an individual with multiple priors considers the least (most) value of expected utility for any act and chooses that act for which this least (most) value is greatest. Gilboa and Schmeidler (1989) and Chateauneuf (1991) have demonstrated that when an arbitrary (closed and convex) set of possible priors P is given, and one defines either a non-additive probability measure υ (convex) or ν (concave) on Ω, such that all additive probabilities measures in P majorise υ or minorise ν, the non-additive expected utility theory coincides with either the maximin or the maximax decision rule, respectively (for a detailed treatment see Basili 2001).

24 A number of important applications have been proposed in the fields of financial markets (Epstein and Wang 1994), incomplete contracts (Mukerji 1998), environmental problems concerning irreversibility (Basili and Vercelli 1998), and in game-theoretical contexts (Eichberger and Kelsey 2000). For an assessment of the potential of the non-additive approach for a conceptual generalisation of theories of decision under uncertainty, see also Vercelli 1999. It is worth stressing here that the representation of beliefs through real-valued set functions which do not necessarily satisfy additivity is not new. In particular, "belief functions" were introduced by Dempster (1967) and Shafer (1976). Although these theories were not directly related to decision under uncertainty, it turned out that "beliefs functions" were a special case of non-additive measures (or capacities) (Gilboa and Schmeidler 1994). Likewise, Zadeh's (1978) theory of fuzzy sets has been shown to be compatible with the non-additive probability approach (Wakker 1990).

5. Concluding remarks

Many critics of the mainstream have maintained that the assumption of a subjective prior probability distribution does not allow a meaningful distinction between risk, or "measurable uncertainty" in Knight's words, and proper uncertainty, or "unmeasurable uncertainty" in Knight's words (Lawson 1985). Shackle, in particular, emphasised that not only knowledge is bounded, but that the bounds are necessarily imprecise. As recently recalled by Loasby (2000: 5), Shackle's view of the use of probability in decision theory amounts to the point that "the imposition of probability distributions, whether subjected or supposedly objective, on closed sets is a pretence of knowledge."

The argument developed in this paper and outlined at the end of the previous section is worth considering for two reasons. First, the paper has dealt with an issue, the representation of individuals' assessment, that is both the main analytical point underlying Shackle's theory and the main analytical reference for writers dealing with the problem of how to represent decisions under genuine uncertainty. As previously mentioned, Shackle's legacy has been taken up especially by Keynesians, institutionalists, and Austrians. For instance, in the Austrian tradition (Langlois 1994 and Vaughn 1994) this argument plays a crucial role, in so far as it serves to distinguish between "rational ignorance" and "radical ignorance" and hence to differentiate Savage's approach from Knight's approach.[25] In this regard, this paper has documented that important developments in modern decision theory take Shackle's issue seriously, while the mainstream is no longer an exercise in Bayesian statistics to the exclusion of other viewpoints. The second ground for relevance of our argument is that we have shown that the issue of reliability of probability distributions can be dealt with through a relaxation of the hypothesis of additivity of probability distributions, contrary to Shackle's view that probability calculus should be dismissed. The use of non-additive measures opens up the possibility of dealing with situations in which an individual is aware that she misses some relevant information, and of doing so without abandoning the realm of probabilities.

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[25] On the Austrian analysis of decision under uncertainty see Zappia 1998, and Basili and Zappia 2000.

REFERENCES

Allais, M. (1953), 'Le comportemenet de l'homme rationel devant le risque: critique des postulates et axioms de l'Ecole Americaine,' *Econometrica*, 21: 503-546.

Allais, M. (1990), 'Allais Paradox,' in J. Eatwell et alt., eds., *Utility and Probability*. London: Macmillan.

Arrow, K. J. and Hahn, F. H. (1999), 'Notes on sequence economies, transaction costs and uncertainty,' *Journal of Economic Theory*, 86: 203-218.

Basili, M. (2001), 'Knightian uncertainty in financial markets: a critical assessment,' *Economic Notes*.

Basili, M. and Vercelli, A. (1998), 'Environmental option values, uncertainty and learning,' in G. Chichilnisky, G. Heal, and A. Vercelli, eds., *Sustainability, Dynamics and Uncertainty*, Kluwer Academic Publisher.

Basili, M. and Zappia, C. (2000), "Radical ignorance in individual decision making: assessing Austrian subjectivism," *Quaderni del Dipartimento di Economia Politica*, Università di Siena, 289.

Camerer, C. and Weber, M. (1992), 'Recent developments in modelling preferences: uncertainty and ambiguity,' *Journal of Risk and Uncertainty*, 5: 325-370.

Chateauneuf, A. (1991), 'On the use of capacities in modelling uncertainty aversion and risk aversion,' *Journal of Mathematical Economics*, 20: 343-369.

Choquet, G. (1953), 'Theorie des capacites,' *Annales de l'Institut Fourier*, 5: 131-233.

Dempster, A. P. (1967), 'Upper and lower probabilities induced by a multivalued mapping,' *Annals of Mathematical Statistics*, 38: 325-339.

De Finetti, B. (1974), *Theory of Probability*. New York: Wiley.

Dow, J. and Werlang, S. R. da Costa (1992), 'Uncertainty aversion, risk aversion, and the optimal choice of portfolio,' *Econometrica*, 60: 197-204.

Eichberger, J. and Kelsey, D. (1999), 'E-capacities and the Ellsberg paradox,' *Theory and Decision*, 46: 107-140.

Eichberger, J. and Kelsey, D. (2000), 'Non-additive beliefs and strategic equilibria,' *Games and Economic Behaviour*, 30: 183-215.

Ellsberg, D. (1961), 'Risk, ambiguity, and the Savage axioms,' *Quarterly Journal of Economics*, 75: 643-669.

Ellsberg, D. (2001), *Risk, Ambiguity and Decision*. New York: Routledge.

Epstein, L. G. and Wang, T. (1994), 'Intertemporal asset pricing under Knightian uncertainty,' *Econometrica*, 62: 283-322.

Ford, J. L. (1983), *Choice, Expectations and Uncertainty*. Oxford: Basil-Blackwell.

Ford, J. L. (1990), 'Shackle's theory of decision making under uncertainty: a brief exposition and critical appraisal,' in S. F. Frowen, ed., *Unknowledge and Choice in Economics*. London: Macmillan.

Gilboa, I. (1987), 'Expected utility with purely subjective non-additive probabilities,' *Journal of Mathematical Economics*, 16: 65-88.

Gilboa, I. and Schmeidler, D. (1994), 'Additive representations of non-additive measures and the Choquet integral,' *Annals of Operations Research*, 52: 43-65.

Hamouda, O. F. and Rowley, J. C. R. (1996), *Probability in Economics*, London: Routledge.

Hamouda, O. F. and Rowley, J. C. R. (1997), 'Introduction,' in *Foundation of Probability, Econometrics and Economic Games*. Aldershot: Edward Elgar.

Hirshleifer J. and Riley, J. C. (1992), *The Analytics of Uncertainty and Information*. Cambridge: Cambridge University Press.

Kelsey, D. and Quiggin, J. (1992), 'Theories of choice under ignorance and uncertainty,' *Journal of Economic Surveys*, 6: 133-153.
Keynes, J. M. (1921), *Treatise on Probability*. London: Macmillan.
Knight, F.H. (1921), *Risk, Uncertainty and Profit*. Chicago: University of Chicago Press, 1985.
Jallais, S. and Pradier, P-C. (2003), 'Changes in Expected Utility Theory and the Allais Experiment,' in P. Fontaine and R. Leonard, eds., *The Experiment in the History of Economics*. London: Routledge, forthcoming.
Langlois, R. N. (1994), 'Risk and uncertainty,' in *The Elgar Companion to Austrian Economics*, edited by P. Boettke. Cheltenham: Edward Elgar.
Lawson, T. (1985), "Uncertainty and economic analysis," *Economic Journal*, 95: 909-927.
Levi, I. (1984), "Serious possibility," in *Decisions and Revisions: Philosophical Essays on Knowledge and Value*. Cambridge, Cambridge University Press.
Loasby, B. J. (2000), 'How do we know?,' in P. E. Earl and S. F. Frowen, eds., *Economics as an Art of Thought. Essays in memory of G. L. S. Shackle*. London: Routledge.
Machina, M., and Schmeidler D. (1992), 'A more robust definition of subjective probability,' *Econometrica*, 60: 745-780.
Magill, M. and Quinzii, M. (1996), *Theory of Incomplete Markets*. Cambridge, MA: MIT Press.
Mukerji, S. (1997), 'Understanding the non-additive probability decision model,' *Economic Theory*, 9: 23-46.
Mukerji, S. (1998), 'Ambiguity aversion and incompleteness of contractual form,' *American Economic Review*: 88: 1207-1225.
Morgenstern, O. (1979), "Some reflection on utility," in M. Allais and O. Hagen, eds., *Expected Utility Hypotheses and the Allais Paradox*. Boston: Reidel.
Nelson, R. R. and Winter, S. G. (1982), *An Evolutionary Theory of Economic Change*. Cambridge, MA: Belknap Press.
Runde, J. (2000), 'Shackle on probability,' in P. E. Earl and S. F. Frowen, eds., *Economics as an Art of Thought. Essays in memory of G. L. S. Shackle*. London: Routledge.
Savage, L. J. (1954), *The Foundation of Statistics*. New York: Wiley, 1972, 2nd ed.
Schmeidler, D. (1989), Subjective probability and expected utility without additivity, *Econometrica*, 57: 571-587.
Shackle, G. L. S. (1949), *Expectations in Economics*. Cambridge: Cambridge University Press.
Shackle, G. L. S. (1949-50), 'A non-additive measure of uncertainty,' *Review of Economic Studies*, 17: 70-74.
Shackle, G. L. S. (1952), 'On the meaning and measure of uncertainty: I,' reprinted in Shackle 1955.
Shackle, G. L. S. (1953), 'On the meaning and measure of uncertainty: II,' reprinted in Shackle 1955.
Shackle, G. L. S. (1955), *Uncertainty in Economics and Other Reflections*. Cambridge: Cambridge University Press.
Shackle, G. L. S. (1961), *Decision, Order and Time in Human Affairs*. Cambridge: Cambridge University Press.
Shackle, G. L. S. (1986), "The Origination of Choice," in I. M. Kirzner (ed.), *Subjectivism, Intelligibility and Economic Understanding*. London: Macmillan.
Shackle, G. L. S. (1972), *Epistemics and Economics*. Cambridge: Cambridge University Press.
Shafer, G. (1976), *A Mathematical Theory of Evidence*. Princeton: Princeton University Press.
Shafer, G. (1986), "Savage revisited", *Statistical Science*, 1: 463-501.

Simon, H. (1982), *Models of Bounded Rationality*. Cambridge, MA: MIT Press.

Vaughn, K. (1994), *Austrian Economics in America. The Migration of a Traditio*n. New York: Cambridge University Press.

Vercelli, A. (1999), 'The recent advances in decision theory under uncertainty: a non-technical introduction,' in L. Luini (ed.), *Uncertain Decisions: Bridging Theory and Experiments*. Boston: Kluwer Academic Publisher.

von Neumann, J. and Morgenstern O. (1944), *Theory of Games and Economic Behaviour*. Princeton: Princeton University Press.

Wakker, P. (1990), 'A behavioral foundation for fuzzy Measures,' *Fuzzy Sets and Systems*, 37: 327-350.

Zadeh, L. A. (1978), 'Fuzzy sets as a basis for a theory of possibility,' *Fuzzy Sets and Systems*, 1: 3-28.

Zappia, C. (1998), "Radical subjectivism and Austrian economics," in R. Koppl and G. Mongiovi (eds.), *Subjectivism and Economic Analysis*. London: Routledge.