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Facoltà di Scienze Economiche e Bancarie

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QUADERNI DELL'ISTITUTO DI ECONOMIA

Richard Goodwin

THE USE OF GRADIENT DYNAMICS IN LINEAR GENERAL DISEQUILIBRIUM THEORY

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Richard Goodwin

THE USE OF GRADIENT DYNAMICS IN LINEAR GENERAL DISEQUILIBRIUM THEORY

Siena, luglio 1985

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I wish to use potential theory and gradient dynamics as a way of arriving at some insights into the nature of economic systems. Perhaps the most important aspect of this procedure is that it brings us into the area of qualitative, as opposed to quantitative dynamics. Thus without actually having to solve our impossibly large number of equations, we can, nonetheless, say some things. It is not that I am hostile to econometrics, indeed I began life as an econometrician and I shared the bright hopes of Frisch, Schumpeter and Tinbergen that this approach would revolutionize our discipline. But in spite of the extraordinarily massive accumulation of data in the last century, we must still admit that we do not have what natural scientists have, reliable constants to work with.

I begin with the assumption of the existence of a square matrix of given coefficients which embody all the systematic variations of expenditure in the entire economic system. Thus I am assuming a basic linearity in the transfer function. It is not that I believe that everything is linear, but I do not see how

a) we can explain the behaviour of the economy without disaggregation, and

b) that being the case, I do not see any hope of solving n non-linear dynamical equations, and

c) I do not see how we can understand the structure of the economy, other than by studying its motions, i.e. its dynamics.

Therefore I define an Economic Potential as a very gross, net national product, i.e. the total value of all output less all the systematic variations of expenditure in producing the output.

$$V(p, q) = p\,q - p\,a\,q = \langle p \rangle [\,I - [\,a]]\,\{q\}$$

Thus, unlike classical mechanics, we have a bilinear form with a matrix that is not symmetrical. The system is non-linear and autonomous, or homogeneous.

First a problem: all existing economies must produce more than they use up in producing, so that there must always be a positive surplus. Von Neumann kept the system homogeneous, with arbitrary scale, by his famous minimum common profit rate and maximum common growth rate, and proved the existence and uniqueness of this solution. However, he did not discuss stability and with good reason since, as Harrod perceived, though in a different context, the system is unstable. It is of little interest to know that an unstable equilibrium exists, since it will never be realized in practice. Thus in a one dimensional system if producers set their growth rate, g, so as to adapt output to demand,

$$\Delta g = -\bar{\beta}[\,1 - (1+g(t))\,\bar{a}]\;q(t)$$

If $(1+g)\,\bar{a} > 1$, then Δg will increase without limit, thus perpetually accelerating output, and the opposite for $(1+g)\bar{a} < 1$. The same analysis holds

for an n dimensional system, where there is no way in which producers can find the exact value, $(1+g^{x})\,a = 1$ which will yield dynamical equilibrium. So von Neumann's solution exists but is useless.

Instead I find it more illuminating to treat this as an unhomogeneous system, with fixed capital and labour as exogenous to the systematic expenditure function. Consider, rather, the potential with a partitioned matrix:

$$V(p, q) = p\,q - \langle p\,;\,w \rangle \begin{bmatrix} a & a_c \\ a_f & a_{ff} \end{bmatrix} \begin{Bmatrix} q \\ \bar{f} \end{Bmatrix} - \langle B \rangle \{q\} - \langle p \rangle \{A\}$$

where

$\langle a_l \rangle$ are inputs of labour per unit of output

$\{a_c\}$ are consumption per unit of labour employed

w is money wage

$\langle B \rangle$ are fixed money costs per unit of expected normal output

$\{A\}$ are all other real demands not dependent on the level of output (e.g. investment, government, etc.)

To keep the model simple, I am ignoring taxes, foreign trade, etc..

Then we measure in deviations from the equilibrium values

$$\hat{p} = \langle w \langle a_l \rangle + \langle B \rangle \rangle [\,I - a\,]^{-1}$$

and

$$\hat{q} = [\,I - a\,]^{-1} \left\{ \{a_c\}\,f + \{A\} \right\}$$

so that we recover the simple form

$$V = p\,[I - a]\;q\,.$$

At equilibrium, V = 0 and is minimal with p = q = 0. We can then dissolve this into the dual linear systems:

$$\text{Grad } V_p = [I - a]\;q \sim \text{supply less demand;}$$

and

$$\text{Grad } V_g = p[I - a] \sim \text{price less cost.}$$

With the potential zero ('entropy' maximum) all gradients are zero, equilibrium is achieved and the economic problem solved with all prices equal to cost and supply equal to demand. Starting with this pair of linear vector fields, we can then derive them from a pair of classical nxn quadratic forms.

Taking equilibrium values (p and q zero) as the desired values, actual p_i and q_i represent errors. Seen in this light, the price-market system is an automatic control device based on error control: the great tradition of laissez-faire theory from Adam Smith through Walras and Marshall to its final resting place in Chicago, taught that the study of equilibrium was the essence, since the economy, being stable, would always be at or near it. This nineteenth century liberal programme was, appropriately, capitalist apologetics. From the study of complicated systems (though far less complicated than the economy) modern control theory tells us that this is too simplistic a view. Auto-control may work well, poorly, or very badly indeed. The problem is particularly serious when the mechanism (e.g. a ship or a torpedo)

is subject to repeated disturbances. The economy is subjected to unending, large, non-random, shocks, e.g. international trade, the weather, wars, technical progress, strikes, government policy, etc.

But first, in order to simplify the analysis, I wish to transform to linear generalized coordinates. The matrix a, being in principal empirical, none of the special problems arise.

Therefore we can make a unique, principal axis transformation to diagonal canonical form with the n distinct eigenvalues on the diagonal, n eigenprices and n eigenoutputs, each with its associated eigenvector.

Putting observed values with primes

$$p = p'h \text{ and } q = h^{-1}q', \text{ so that}$$

$$h[I - a]\;h^{-1} = \lceil I - \lambda \rfloor, \text{ diagonal, where}$$

$$\lambda = \begin{bmatrix} \lambda_A & & & & 0 \\ & \ddots & & & \\ & & \lambda_k & & \\ & & & \lambda_{k+1} & \\ 0 & & & & \ddots \; \lambda_n \end{bmatrix}$$

with λ_1, to λ_k real and λ_{k+1} to λ_n as block diagonals, consisting of conjugate complex pairs of eigenvalues, $\gamma \pm i\beta$; β determines the oscillations and γ the stability.

This is not a proper gradient system, which commonly assumes a symmetric matrix and hence only real eigenvalues. Nonetheless, we may still consider the more general case where the gradient of the system determines the dynamics, but no longer on the simple analogy of a 'potential basin' subject to the action of gravity. Thus any conjugate pair of roots with

$$\begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix}$$

will have a flow, no longer along the gradient but orthogonal to it, which will result in elliptical trajectories along a 'level set'. Though the flow is no longer along the gradients, it is still entirely determined by the gradients (being orthogonal to the gradients which are in turn orthogonal to the level surfaces, so that the solution flows along the level surface (See Figure 1b).

The analysis now becomes transparently simple: recommence with dual potentials in simplest quadratic form:

$$V_p(p) = 1/2p\,[1 - \lambda p]; \quad V_q(q) = 1/2q\,[1 - \lambda]\,q,$$

or

$$V(p, g) = 1/2\,p\,[1 - \lambda]\,p + 1/2\,g\,[1 - \lambda]\,g;$$

so that

$$\text{Grad } V_p = \langle (1 - \lambda_1)\,p_1 \ldots (1 - \lambda_n)\,p_n \rangle$$

and

$$\text{Grad } V_q = \begin{Bmatrix} (1 - \lambda_1)\,q_1 \\ (1 - \lambda_n)\,q_n \end{Bmatrix}$$

If we regard conjugate pairs as a single variable, the variables are separated and we have also separated dynamics from interdependence. These are two elliptic basins with gradients always positive except for the singular, equilibrium points. The gradients are always orthogonal to the level surfaces for any V = K, a costant. The level of V is arbitrary, leaving gradients unaltered, and since - Grad V (x) = Grad (-V(x)), negative gradients turn basins into hills.

The variables being separated, the entire economy may be analyzed two sectors at a time, thus allowing the use of geometric visualization. Unlike the physical case, potentials may be either hills or basins, and price movements may depend on prices or on outputs, and similarly for outputs. Nodes,

$$\begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_j \end{bmatrix}$$

may be either attracting or repelling (Figure 1a). For pure imaginaries, centers,

$$\begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix}$$

the gradients are tangent to level surfaces, in either direction (Figure 1b). For complex conjugates,

$$\begin{bmatrix} \gamma & -\beta \\ \beta & \gamma \end{bmatrix}$$

the two effects are combined into inward or outward spirals, foci, in either direction (Figure 1c).

fig.1a

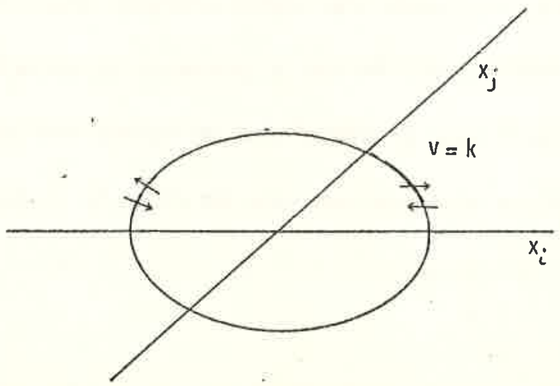


fig 1b

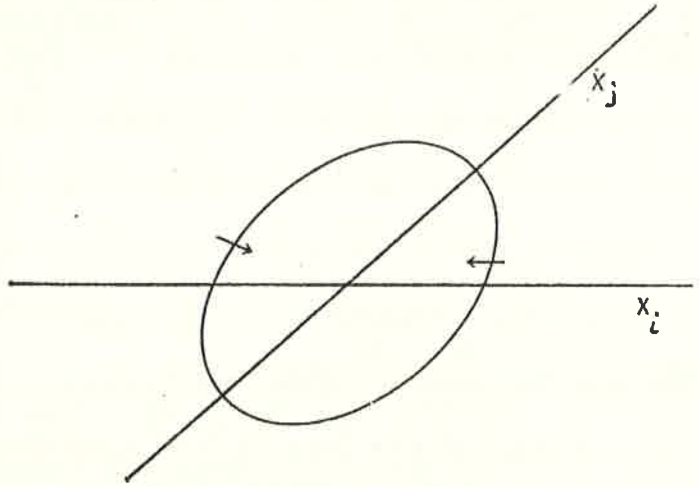
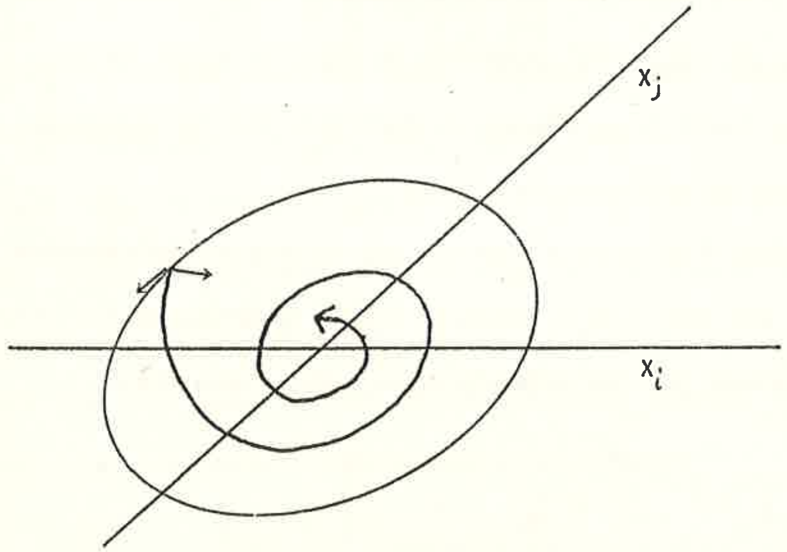


fig 1c



These complete the dynamical structure of the systematic linear transfer function of the system: there are no other determinants of motions. All sectoral motions are linear combinations of these motions.

Schumpeter always insisted that Walras was the greatest of all economists, not necessarily for his results, but because his methodology was best; he asked the most profound questions. The problem, essentially dating from Adam Smith is: does a freely functioning, competitive economy, achieve a given, unique, best equilibrium solution? For this linear homogeneous system, we can produce some answers. Existence and uniqueness of equilibrium are obviously guaranteed. But, only if the system is not only stable but asymptotically stable, is the equilibrium actually achieved.

Taking all adjustment coefficients as unity, to reveal the basic logic, we can say, thanks to a theorem of Frobenius, that all eigenvalues have a magnitude of less than unity, i.e. $0 < | \lambda_i | < 1$.

Therefore if all price adjustments depend only on the difference between price and cost, and if, similarly, output variations depend solely on supply less demand, then the system becomes in block matrix form:

$$\begin{Bmatrix} \dot{p} \\ \cdots \\ \dot{q} \end{Bmatrix} = \left[\begin{array}{c|c} -\lfloor 1-\lambda \rceil & 0 \\ \hline 0 & -\lfloor 1-\lambda \rceil \end{array} \right] \begin{Bmatrix} p \\ \cdots \\ q \end{Bmatrix}$$

Thus there are 2n asymptotically stable nodes: all prices and all quantities tend to the zero equilibrium state.

Equally, if not more, central to the neoclassical analysis of price-market auto-control are the twin concepts:

price variation depends on demand less supply;

output variation depends on price less cost.

The system then is

$$\begin{Bmatrix} \dot{p} \\ \cdots \\ \dot{q} \end{Bmatrix} = \left[\begin{array}{c|c} 0 & -\lfloor 1-\lambda \rceil \\ \hline +\lfloor 1-\lambda \rceil & 0 \end{array} \right] \begin{Bmatrix} p \\ \cdots \\ q \end{Bmatrix}$$

constituting 2n centers exhibiting simple harmonic motion, independent of whether the eigenvalues are real, imaginary or complex. Considering pairs of motions, there are almost certain to be one or more irrational ratios between periodicities, which means that the shape of the resultant waves will never repeat and hence will appear to have no strict periodicity. If there are many irrational ratios, the time series may appear chaotic. If we write, for either prices or quantities, in the untransformed variables:

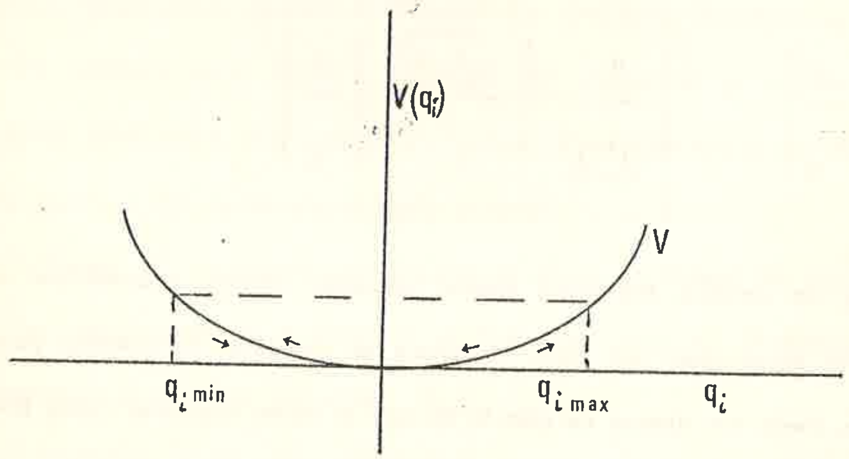
$$\langle \ddot{q} \rangle = \langle q \rangle [I - a]' [I - a] \quad ,$$

we have from the fact that

$$\big[[I - a]' [I - a] \big]' = [I - a]' [I - a]$$

the dynamical system arrives in the form of a symmetrical matrix, which means there will be only real eigenvalues, producing simple harmonic motions. Thus such dynamical tâtonnement produces stable behaviour but not asymptotic, so that equilibrium is never achieved. For each eigensector the initial condition determines completely the motion so long as the system is undisturbed.

fig 2



If, however, both types of error control are operative, the system becomes asymptotically stable dynamically and structurally as well.

$$\begin{pmatrix} \dot{p} \\ \cdots \\ \dot{q} \end{pmatrix} = \left[\begin{array}{c:c} -\lfloor 1-\lambda \rceil & -\lfloor 1-\lambda \rceil \\ \hdashline +\lfloor 1-\lambda \rceil & -\lfloor 1-\lambda \rceil \end{array}\right] \begin{pmatrix} p \\ \cdots \\ q \end{pmatrix}$$

In this case the Walrasian resolution is effected.

Every producer faces random short-run variations in demand and hence must maintain finished stocks. If, however, the variations in demand become biassed either up or down, his stocks become too large or too small. Producers may therefore use integral control instead of, or as well as, the proportional control already discussed. Calling stocks, s,

$$q(t) - d(t) = \Delta s$$

$$S(t) = \sum_{0}^{t} \Delta s, \quad \text{or} \int_0^t S dT$$

$$\dot{S} = \lfloor 1-\lambda \rceil q$$

Negative feedback control of output from the error in stocks gives

$$\dot{q} = -\beta (s - \bar{s}) ,$$

$\bar{s}$ being desired level of stocks, from which

$$\ddot{q} + \beta \lfloor 1-\lambda \rceil \quad q = 0$$

again producing simple harmonic motion.

This is the simplest and most comprehensible explanation of the well observed fluctuations of inventories.

The stocks cycle is a striking example of dysfunction in a control mechanism. With strong negative feedback such malfunction is surprising. The explanation is simple and highly significant; the adaptive response alters the desired goal. Each individual producer quite correctly takes his demand as given and alters his output so as to approximate his demand. His output will not, in fact, significantly affect his own demand. However, given a whole economy in which many or all producers are doing the same, then demand is strongly influenced by output. In such a case, the system may 'hunt' its equilibrium without ever finding it. Thus if pursuer and pursued each alter course in relation to the other, there need be no single, simple solution. This suggests that the conception that rational expectations would lead to an equilibrium solution is an error deriving from the analysis of simple aggregative models.

Derivative control is also likely to be found in producers' behaviour. If $q_i > d_i$, the reaction would be less if demand were increasing and much greater if it were diminishing. Therefore producers may try to equate not only q to d but also $\dot{q}$ to $\dot{d}$ (1)

$$\text{If } \dot{q}_i = -\beta_i (1 - \lambda_i) q_i - \gamma_i (\dot{q}_i - \dot{d}_i) \ ,$$

the solution is stable but, somewhat surprisingly, less stable than with simple proportional control. This makes sense for heavily disturbed mechanisms because they will be less thrown off course by the disturbances. Thus torpedoes in rough seas need both proportional and derivative control to find their target.

These error functions are linear transfer functions: given any variable input, they will transfer the consequences to the various sectors in determina-

te magnitudes. The transfer function is stable in short periods, and, being linear, cannot generate common oscillations, only transmit them. For the explanation of the origin of significant cycles, common to all sectors, one has to go to some form of non-linear relations, a large subject to be treated below.

The scale of output is arbitrary up to the limit of full employment, i.e. $< a_f > \{ q \} \leq n\ (t)$ where n (t) is the effectively available labour force. This constraint is a most potent non-linearity, available to limit any instability generated by investment.

By contrast the scale of prices is arbitrary, though there may be temporary limits imposed by the monetary system or the foreign balance. The commonest practice is for firms to use the 'full cost' procedure, i.e. to set a mark-up on variable cost which will, at some expected average output q*, cover fixed charges, including a common rate of return on total invested capital. Since the mark-up is usually kept constant, we may absorb it into the goods and labour inputs, writing

$$\hat{p} = pa + wa_f$$

instead of

$$< pa + wa_f > \lceil 1+\pi \rceil$$

This mark-up is analogous to von Neumann's uniform, common profit rate. Actually a large proportion of producers use a fixed mark-up for setting prices, though, unlike profit or interest rate, it is different for each sector. By contrast, there is nothing corresponding to von Neumann's common growth rate; demand is not 'marked-up' by any common, fixed growth rate. The consequence is that price adjustment is far less stable than output adjustment, and, indeed, may easily be made unstable by variations in real unit labour costs as a result of wage bargaining.

Prices being normally adjusted only discontinuously, it is better to use discrete instead of continuous time. Proportional control arises from the necessity to set prices after costs (including prices) are known. Let

$P_{t+1} = = P_t a + w_t a_f$.

This shows what a serious, distributed lag is thus introduced. If wa increases, this raises prices by a smaller amount, but then the rise in prices sets a second chain of rises in motion and so on. In the limit, prices will rise proportionally to the rise in $w\ a_f$, but during the process there will have been a rise in real as well as money wages. I now define the error as price less marked-up cost, including unit labour cost. With constant unit labour cost, negative feedback control will reduce the error to zero, i.e. price = cost.

$$\Delta p = -\langle \dot{p} \rangle \left[I - a \right] - w \langle a_f \rangle \ .$$

Thus for a set of constant unit labour costs, it is a homeostatic mechanism, asymptotically stable to an equilibrium set of prices yielding a common

rate of profit.

This model illustrates forcefully how essential it is to disaggregate. There is, of course, no single wage, w, but it does represent a tendency for comparable work to be paid same wage. By contrast labour inputs, a_f, decline at strikingly different rates. The workers in the most progressive sectors will tend to press and gain at least a rate of rise of money wages equal to the rate of decline of labour input. This means that for all other sectors there will be some strong pressure to gain rises of money wage equal to those in the most progressive sector. Therefore there will be a whole set of rising unit labour costs in all other sectors. This explains why in advanced industrial countries there is an endemic tendency to some average rate of inflation. It also shows the emptiness of aggregative analysis for dealing with the difficult problem of a wages policy. Asking workers in general to accept wage increases equal to average productivity growth is no answer. Nor does it help to say wage increases may be granted to match productivity: this ignores the spin-off to other sectors. The model also illustrates the dysfunction of disaggregative decisions. Each producer knows his costs and sets his price to cover them: his price does not affect his costs. The trouble is it affects the costs of others and the price decisions of others and the price decisions of others will affect him. Yet there is no way that he could rationally calculate what to expect.

In the normal situation the cost of labour is given and determines the equilibrium level of prices. The error is the difference between price and cost; price is the variable which adjusts towards cost. But when inflation is high and persistent, this dynamical structure alters because the money illusion vanishes and wages adjust to prices as prices do to wages. This change of structure may alter the type of motion and thus constitutes a bifurcation.

Using generalized coordinates, with the money illusion,

$$\Delta p_i = -p_i\,(1 - \lambda_i) + w\,a_{fi}$$

It is helpful to consider p in deviations from the equilibrium value

$$\hat{p}_i = \frac{w\,a_{fi}}{1 - \lambda_i}$$

We have then a potential

$$V_p = 1/2 \langle p \rangle \boxed{1 - \lambda} \{p\}$$

and

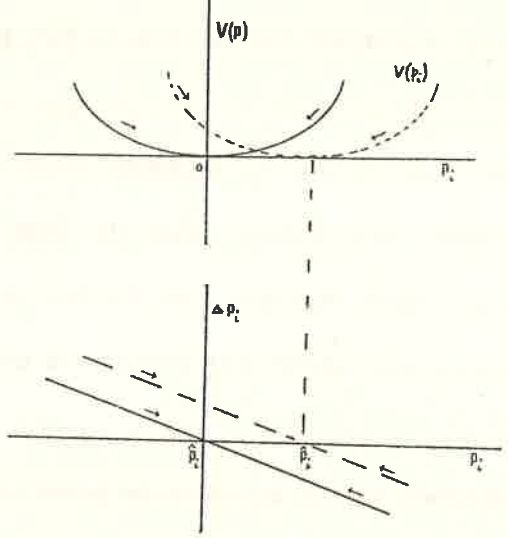
$$\Delta p_i = - \text{Grad } V_p\,(p_i)$$

yielding asymptotic stability for a given wa_f .

A change in wa_{fi} simply shifts the basin of attraction and initiates a new slide in the changed basin towards the new equilibrium.

If by contrast, money disillusion arises and $w/p_i\ a_{fi}$ becomes constant

fig 3



at a value which makes $\bar{\lambda}_i + \overline{w/p_i\,a_{fi}} > 1$, then the system becomes homogeneous with

$$-V_{p_i} = -\,1/2\,[\,(\bar{\lambda}_i + \overline{w/p_i\,a_{fi}}) - 1]\ p_i^2$$

and

$$\Delta p_i = -\,\text{Grad } V_p(p_i) = +\,[\,(\bar{\lambda}_i + \overline{w/p_i\,a_{fi}}) - 1]\ p_i\ .$$

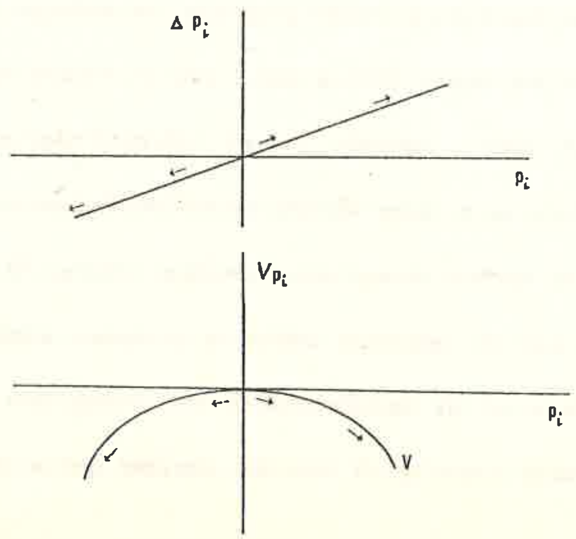


fig 4

The inflation proceeds at a constant rate as long as $w/p_i\ a_{fi}$ remains constant.

The system then becomes binary. Producers adjust prices to costs including prices and labour cost. Workers adjust the wage to the 'cost of living' as determined by prices. Not only this, but for persistent inflation of sufficient rapidity, both will use not only proportional but derivative control as well.

There are two different ways of improving the performance of a malfunctioning mechanism. The simplest is to leave its structure unaltered but add subsystems, or 'counterweights' to correct the faults. This is the familiar Keynesian method. The decision structure of producers and consumers is left untouched but parameters are changed, e.g. tax changes, government spending, interest rates, banking policy and exchange rates.

There exists, however, a second procedure -to redesign the mechanism so as to eliminate the faults. With a social system, this is no easy matter. A social revolution with a centrally planned and controlled economy is the extreme case. There is a more modest way. The failure arises from the perverse effects of certain decentralized decision making. The basic source of dysfunction is that the individual worker or producer cannot take account of the wider effects of his decisions. With the development of high speed computers with large capacity, it becomes possibly for a central planning

commission to make repeated simulations of an approximation to the interdependent effects of large numbers of decisions. The price-market mechanism is after all a very subtle analogue calculator. With the results of these simulations fed back to the individuals concerned, it is possible to show the ultimate effects of particular types of decisions, and in a modest way to alter the decisions so as to obtain better results, avoiding error, confusion, and undesired results. This is, in effect, redesigning, to some extent, the decision structure, and hence the functioning, of the economy. Only in such a context does it make sense to say that rational expectations will be realized. When I gave a lecture in Paris many years ago, I was told that the French government was trying to do something of this sort.

A similar analysis can be applied to the world balance of payments problem. The world has been in the grip of a slowly worsening depression with growing unemployment. Each individual country should adjust by increasing domestic demand, but cannot because this would directly worsen its balance of payments. So each country, with the unenlightened help of the I.M.F., balances its payments by cutting domestic demand and imports. This, of course, only pushes the problem onto other countries. And so the world level of output falls or fails to rise. What is needed is a proper world planning commission with the ability to persuade, or the power to force, nations to act in their common interest. As it is those with a trade deficit reduce output and employment, and those with a surplus do nothing.

From the elementary dynamics of price/cost, demand/supply behaviour, one finds strongly stable motion which can contain monotonic and oscillatory elements. This gives rise to complicated diverse motion but does not explain why a free market system shows common, albeit diverse, alternating movements up and down. I would like to outline a simple, idealized model which contains, I hope, much of the essential explanation. To make its nature clear, I shall first describe a one sector, aggregated economy, then an n sector one, partitioned into two blocks.

$$\text{demand} = d = aq + A(t) + B(t) + C(t)$$

A is real demand, independently given, though variable (eg. net government or exports).

B is real innovational investment, given by history, though in some relation to the behaviour of the economy

C is the pure accelerator, $C = \beta \dot{q}$, but is non-negative and zero for $q \leq \hat{q}$, where $\hat{q}$ is long run, desired level of output in relation to designed capacity[(2)]

β is the real capital/output ratio.

Since there is a considerable lag between investment and operational capacity, probably the condition is not only $\dot{q} > 0$ but also greater than some low positive quantity.

Considering deviations from the equilibrium determined by given A + B (with C zero), the short-term dynamics of output adjustment is highly stable:

$$\varepsilon \dot{q} = d - q$$

Thus

$$\dot{q}/q = -\frac{1-a}{\varepsilon}, \qquad 0 < a < 1, \quad 0 < \varepsilon < 1$$

giving monotonic, asymptotic stability.

By contrast, when $q > \hat{q}$,

$$(\beta - \varepsilon)\,\dot{q} = (1-a)\,q, \quad \beta > \varepsilon > 0$$

$$\text{with } \dot{q}/q = \frac{1-a}{\beta - \varepsilon} > 0,$$

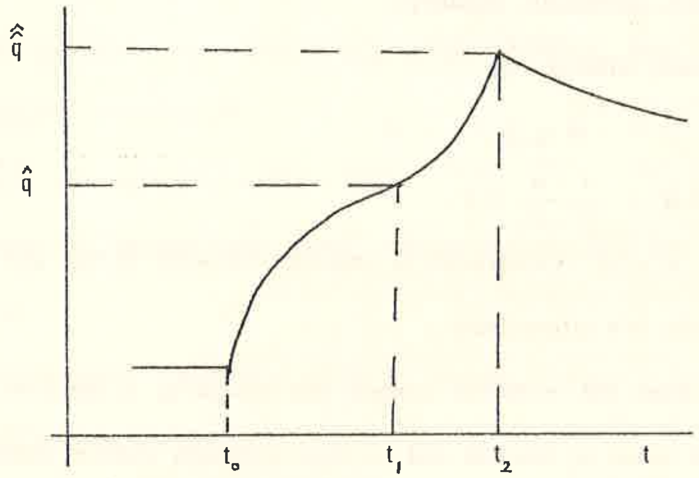
so that $q = \hat{q}$ constitutes a boundary between stable and unstable motion and hence is a bifurcation.

If and when the economy crosses this boundary, it becomes unstable upward (in the sense of Harrod) and so must gradually reduce unemployment towards zero. This means that the economy is growing at a pace which cannot continue: the economy must decelerate which brings gradually a reduction in the level of C, furthering the deceleration, and thus cancelling the accelerator. This is a second bifurcation, annulling the first and returning the economy to monotonic and/or oscillatory stability, q proceeding to its equilibrium,

$$q \rightarrow \frac{A(t) + B(t)}{1 - a'}$$

Starting from an underemployment equilibrium, if there occurs an innovation large enough to carry the economy to desired capacity output $\hat{q}$, the accelerator then sweeps it eventually to full employment of labour, $\hat{\hat{q}}$. From here it relaxes to a new, generally higher, level of underemployment.

Schematically



We may think of three regions:

I Unemployed capacity and labour;

II The capacity is no longer in excess, but labour remains in excess;

III Unemployed labour disappears as well.

In II real wages and profits both rise because output is growing. In III output becomes constrained by scarce labour; real wages rise more than productivity; the tendency for the share of labour to rise is resisted by infla-

tion.

When the system relaxes, it does not return to its previous position. To initiate an expansion, a major innovation (a new good or new process) is sufficient. The consequence of this is that both relative prices and relative quantities will alter, so that the position to which the economy returns is not the same as the one from which it started. The productive structure in many sectors will have altered, always in one direction, greater outputs per units of inputs. It is morphogenesis in the sense of René Thom and further than that it is evolutionary selection of a new species of productive structure in the sense of Darwin. There is technological unemployment, combined with conjunctural unemployment. This means that when the economy next expands, it will rise well above the previous highest level, thus generating economic growth. For simplicity of analysis, I assume that the expansion occurs with a substantially unaltered productive structure and that the new structure only becomes effective in the downswing.

A qualitative analysis for an n sector model can be simplified by assuming all sectors are producing either durable goods or non-durables, never both. I partition the matrix into section a, sectors 1 to r, producing non-durables, and section b, sectors s to n producing durables. Also I assume that accumulation of capacity in b is ad hoc and any investment is included in $A_b(t)$ or $B_b(t)$.

$$\begin{bmatrix} \varepsilon_a & 0 \\ 0 & \varepsilon_b \end{bmatrix} \begin{Bmatrix} \dot{q}_a \\ \dot{q}_b \end{Bmatrix} = \begin{Bmatrix} d_a \\ d_b \end{Bmatrix} - \begin{Bmatrix} q_a \\ q_b \end{Bmatrix} =$$

$$= \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} q_a \\ q_b \end{Bmatrix} + \begin{Bmatrix} A_a \\ A_b \end{Bmatrix} + \begin{Bmatrix} B_a \\ B_b \end{Bmatrix} + \begin{Bmatrix} C_a \\ C_b \end{Bmatrix} - \begin{Bmatrix} q_a \\ q_b \end{Bmatrix}$$

Setting A, B and C equal to zero

$$\left[\varepsilon_a\right] \dot{q}_a = -\left[I - a\right] q_a + \left[b\right] q_b$$

but q_b will be zero, so that we have the asymptotically stable dynamical transfer function which will carry the economy to any new equilibrium determined by, not too large, variations in A and B, C remaining zero, if all sectors a continue to have excess capacity. Given a major innovation or series of innovations, requiring large investment, δB_a, and inducing similar investments in other industries, the rising outputs will carry one sector after another into the region of accelerational investments. Consider the situation if all sectors a are undertaking accelerational investment

$$\left[\varepsilon_b\right] \dot{q}_b = \left[\beta \ \ 0\right] \begin{Bmatrix} \dot{q}_a \\ \dot{q}_b \end{Bmatrix} - \begin{Bmatrix} q_b \end{Bmatrix}$$

or

$$\left[b\right]\left\{q_b\right\} = \left[b\right]\left[\beta\right]\left\{\dot{q}_a\right\} - \left[b\right]\left[\varepsilon_b\right] \dot{q}_b$$

so that the dynamical transfer function becomes

$$\left[\varepsilon_a \dot{q}_a\right] = -\left[I - a\right] q_a + \left[b\right]\left[\beta\right]\left\{\dot{q}_a\right\} - \left[b\right]\left[\varepsilon_b\right] \dot{q}_b$$

$$\left[b\right]\left[\beta\right]\left\{\dot{q}_a\right\} - \left[\varepsilon_a\right]\left\{\dot{q}_a\right\} - \left[b\right]\left[\varepsilon_b\right] \dot{q}_b - \left[I - a\right] q_a = 0$$

If $q_b = \beta \dot{q}_a$ (i.e. output = demand),

$$b \beta \dot{q}_a - \varepsilon_a \dot{q}_a - \left[I - a\right] q_a = 0$$

$$\left[b\left[\beta\right] - \left[\varepsilon\right] \right] \dot{q}_a - \left[I - a\right] q_a = 0 \ .$$

If some, most, or all $\left[\left[b\right]\left[\beta\right] - \left[\varepsilon_a\right]\left\{1\right\}\right] > 0$

then some, most, or all motions are unstable: the whole system is unstable.

If, however, the durable goods sector also adjust dynamically to demand, then the stronger condition for instability is

$$\left\{ b\beta - \left[\varepsilon_a + b\left[\varepsilon_b\right]\right] \right\}\left\{1\right\} > 0$$

or

$$\left\{\left[b\beta - b\left[\varepsilon_b\right]\right] - \left[\varepsilon_a\right]\right\}\left\{1\right\} > 0 \ .$$

This condition in also likely to be fulfilled.

The purpose of this model is to show

a) that the economy is basically very stable dynamically, so that

b) we may use gradient dynamics to study the impact of given changes in the exogenous elements

c) there are resource limits, especially labour, which necessarily nullify the instability at some level, in such a way that

d) the system again becomes dynamically stable

e) that the observed large fluctuations are not true oscillations but rather the combined result of the reaction of the system and the given impulses of economic history in the form of technical progress.

As such it can be a better explanation than various forms of cycle theory. This derives from the basic observed fact that capitalism is dominated by the unending search for profit. In the depressed phase profits are low or negative because of excess capacity. This gives a strong impulse to seek for new goods or new processes which will prove successful commercially. The investment required, unlike that of the accelerator, is independent of excess capacity. Consequently, sooner or later, the economy is lifted off the bottom by such investment. This aspect is much strengthened by the fact that major innovations undergo long periods of improvement and adaptation to other industries. This means that whilst such an investment tends to be inhibited by the collapse of the boom, it is soon resumed when the economy ceases to decelerate. This means prolonged expansions and short, sharp recessions. In this fashion lies, I suspect, the best possibility of explaining the so-called Kondratief long waves.

Once the system has relapsed to a stable equilibrium solution we await a major innovation. It is impossible to give any precise analysis of the subsequent process because of the large and heterogeneous lags involved, but a schematic, or idealized description is revealing. First come the exogenous investment outlays to creste the new productive enterprises. (These will come in successive bursts). The new process constitutes a change of a number of parameters, which will yield a lower cost structure. In whatever complicated fashion it may happen, we may assume a gradual tendency for the price to fall towards cost. A new equilibrium state has been created, so that even if the system was previously in equilibrium, it is no longer. As the price of the innovational good falls, so also do the prices of the goods of which it is an input. The consequence of this is that we must enlarge the system to include the choice of known techniques.

$$V(p,x) = \langle p \rangle \{x\} - \langle p,w \rangle \begin{bmatrix} a & a_c \\ a_f & a_{ff} \end{bmatrix} \begin{Bmatrix} x \\ f \end{Bmatrix} - \langle B \rangle \{x\} - \langle p \rangle \{A\}$$

$$\text{Grad } V_x = \langle p \rangle - \langle p; w \rangle [a \; ; \; a_c] - \langle B \rangle$$

where x represent intensitis, with p repeated as necessary.

For any change in the set of p; w, the Best Technique is defined as the minimal of the inner product for each good plus the estimated unit fixed costs. This chooses the Best Technique but it may or may not be promptly initiated, depending on comparison of total unit cost and variable unit cost.

As new, induced changes in processes become operational, a secondary set of price changes is put in motion, with the same sorts of consequences as the original one. These parametric changes do not bring a bifurcational change of type of motion, unless and until full capacity is achieved. This is what growth and development is all about. We do not have a given mechanism to analyze, but rather the slow evolution of an everchanging mechanism. However, in the short run, the productive structure can alter little so that it is permissable to study the stable behaviour in response to changes in investment, government outlay, and international trade.

Appendix

To develop the analysis in the tradition of classical mechanics, I take what I have called 'cross field' dynamics with $\dot{q}$ a function of value and $\dot{p}$ a function of output. Define a Hamiltonian Function, of the 2n variables

$$H(\dot{q}, q) = \sum_{i=1}^{n} \dot{q}_i^2 / 2 + V_q(q_i), \quad i=1, \dots, n.$$

Then

$$DH = \sum_i \frac{\delta H}{\delta \dot{q}_i} d\dot{q}_i + \sum_i \frac{\delta H}{\delta q_i} d q_i = \phi(\dot{q}, q)$$

The Hamiltonian vector field will be

$$\left(\frac{\delta H}{\delta \dot{q}_1}, \dots, \frac{\delta H}{\delta \dot{q}_n}; \frac{\delta H}{\delta q_1}, \dots, \frac{\delta H}{\delta q_n} \right) = \phi^{-1} DH$$

$$\frac{\delta H}{\delta \dot{q}_i} = \dot{q}_i$$

$$(\dot{q}_i) = \ddot{q}_i = -\frac{\delta H}{\delta q_i} = -\frac{\delta V_q}{\delta q_i}$$

We have the quadratic, generalized potential (page)

$$V_q = 1/2 \langle q \rangle \lfloor 1 - \lambda \rfloor \{ q \} =$$

$$= 1/2\,(1-\lambda_1)^2\, q_1^2 + \dots + (1-\lambda_n)^2\, q_n^2$$

so that

$$\frac{\delta V}{\delta q_i} = (1-\lambda_i)^2\, q_i \,, \quad i = 1, \dots, n$$

and

$$\ddot{q}_i = -(1-\lambda_i)^2\, q_i$$

Therefore we have n simple harmonic motions as the dynamic constituents of the system. It is easy to show that for solution curves

$$\frac{d}{dt}\left[H(\dot{q}(t), q(t))\right] = 0.$$

Thus the system moves on a level Hamiltonian surface, which level being determined by the initial conditions. The solutions are smooth curves and, in this conservative case, closed ellipses.

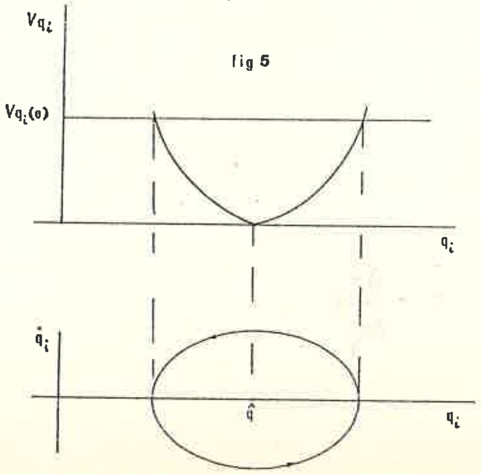


fig 5

Notes

(1) It should be noted that none of these control patterns imply that these are decisions made in eigensectors; it is merely a representation of a type of decision made in actual sectors.

(2) Calling actual capacity k, if desired capacity is $\hat{k} = (1+\gamma)q$, with $1>\gamma>0$, and $\hat{q} = k/(1+\gamma)$, then, when $q > \hat{q}$, $C = \dot{q}\beta$ and when $q \le \hat{q}$, $C = 0$.

Of course, the causation is the other way round. Desired capacity is some constant multiple, greater than unity, of actual output. Investment for future capacity is undertaken only when desired capacity is greater than existing capacity.

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