QUADERNI

Università degli Studi di Siena
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Probabilistic versus non-probabilistic decision making: Savage, Shackle and beyond

n. 403 – Ottobre 2003

Abstract - This paper discusses the evolution of decision theory after Savage's Foundations. Two developments are examined. First, it is presented the rationale of Shackle's proposal to abandon probabilistic decision making. Second, it is discussed the axiomatisation provided by the non-additive probability approach to account for the experimental evidence originated by the Ellsberg Paradox. An attempt is made to establish a connection between Shackle's non-probabilistic instances and non-additive probabilistic decision making. The main outgrowth of the paper is that the similarities between the non-additive approach and Shackle's theory are not limited to a number of methodological statements. In fact, it is also the formal measures used in the two contexts for representing individual preferences in uncertain environments that resemble each other.

J.E.L. Classification: B21, D81

Keywords: uncertainty, decision theory, non-additive probabilities

A previous version of this paper was presented at the Annual Meeting of the European Society for Evolutionary Political Economy, Siena, November 2001. Comments by Frank Hahn, Nicola Giocoli, Roberto Romani and Christian Schmidt are gratefully acknowledged. This paper is part of the research programme "Mathematics in the history of economics" supported by MIUR (Ministero dell'Istruzione, dell'Università e della Ricerca). Corresponding author: zappia@unisi.it

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1. Introduction

When proposed to bet in uncertain situations many decision-makers prefer to bet on unambiguous events rather than on ambiguous ones, that is they dislike ambiguity. This conclusion emerges from choice behaviour like that exhibited in the Ellsberg Paradox (Ellsberg 1961) as well as from related experimental evidence (Camerer and Weber, 1992). The fact that decision-makers dislike ambiguity contradicts the universal appropriateness of a subjective additive probability distribution representing decision-makers' beliefs. Contrary to what Savage implicitly assumed in his *Foundations of Statistics*, when decision-makers face a problem involving uncertainty they may either be unable to assign probabilities to all relevant events or express second-order probabilities. The ensuing problem is how to represent the individual agent's confidence in a probability assessment. The fact that the amount and quality of information on which a probability assessment is based matters was Keynes's main reason to introduce the "weight of evidence" in his treatment of probability (Keynes 1921).

This problem has originated new approaches, intended as drastic alternatives to probability models. These approaches aimed at making room for genuine ignorance, surprise and vagueness. Shackle (1949), in particular, developed a formal theory opposing both the frequency and the subjective probability approach. In Shackle's view (1972: 15), the standard, Bayesian meaning of probability "stands for a *language for expressing judgements* … [that] assumes, implicitly, that the hypotheses that have been enumerated, specified and presented for the assignment of weights are *the only relevant ones*. Thus the language of subjective probability is confined to the expression of a certain kind of meaning. And there are other meanings whose exclusion would be arbitrary and senseless." Shackle's proposal for dealing with uncertainty was to introduce some novel concepts like potential surprise, focus values, and so on, in place of probability distributions. These concepts were meant to capture both the mental processes and the non-repetitive, often irreversible, nature of actual economic decisions. After being the subject matter of passionate discussions in the 1950s, culminating in a monographic issue of *Metroeconomica* in 1959, Shackle's formal theory was substantially disregarded by decision theorists. However, Shackle's methodological approach, as well as his interpretation of Keynes's theory of uncertainty, still constitute a main reference point for economists working in the Keynesian and Austrian traditions (see for instance Lawson 1985 and Vaughn 1994).

More recently a huge number of alternatives to subjective expected utilities has been proposed (Kelsey and Quiggin 1992). A common feature of these attempts to provide an axiomatic foundation for decision making in uncertain environments seems to be, very much in Shackle's spirit, to dispense with the assumption that the individual has complete structural knowledge of the environment. In particular, non-additive probability theory (Schmeidler 1989 and Gilboa and Schmeidler 1994) takes into consideration decision problems in which the states of the world included in the model do not exhaust the actual ones, and argues that when an individual agent does not know how many states are omitted her beliefs can be represented by a non-necessarily-additive measure (or a set of additive probability distributions on the set of events). In this respect, non-additive probability theory calls to mind the Shackleian view that ignorance is an inherent feature of every decision concerning future events.

This paper elaborates on this resemblance. Namely, the paper focuses on the ways in which Shackle's assertion can be developed by applying the modern language of non-expected utility theory. The argument is carried out as follows. Section 2 presents the main tenets of subjective probabilistic decision making. This section, which deals with Savage's distinction between "small" and "grand" world, argues that Savage was aware that his axioms for a subjective theory of individual decision making were restrictive in crucial way. Section 3 examines the rationale of Shackle's abandonment of probabilistic decision making. Shackle, who is considered by both post-Keynesian and Austrian authors as the main representative of the so-called radical subjectivist approach to individual decisions, maintained that there is a sense in which uncertainty is intrinsically unmeasurable. Section 4 discusses the experimental evidence which is at the basis of modern criticisms of Savage. Special emphasis is placed on the notion of ambiguity introduced by Ellsberg in his seminal experiment. This section illustrates the reasons why the literature fostered by experimental evidence requires a re-examination of Shackle's contribution. Section 5 introduces the axiomatisation provided by the non-additive approach to individual decision making. An attempt is made to establish a connection between non-probabilistic instances and non-additive probabilistic decision making. Section 6 argues that the similarities between the non-additive approach and Shackle's theory are not limited to a number of methodological statements. In fact, it is also the formal measures used in the two contexts for representing individual preferences in uncertain environments that resemble each other. Section 7 provides some concluding remarks.

The main outgrowth of the paper is the following hypothesis. The reliance of standard Bayesian theory on probabilistic judgements based on point-probability estimate, a reliance which Shackle intended to oppose, is no longer a justification for dispensing with probability calculus in general. It is argued that Shackle's distinction between distributional uncertainty variables, which can be dealt with by means of probability measures, and non-distributional variables, which necessitate a non-probabilistic approach reflects an essentially non-additive feature of his theory. It is held that this feature is incorporated in the non-additive probability approach.

2. Savage's *Foundations*: "small" and "grand" world

The traditional, probabilistic approach to decision making under uncertainty is structured as follows. There are two fundamental assumptions. First, a complete list of possible future states of the world is available to the individual – a list which, in an interpersonal context, is common knowledge to all individuals. The individual is endowed with subjective beliefs over the state space. These beliefs are represented by a well-defined (additive) probability function. This rests mostly on Savage's (1954) subjective expected utility theory, which made it possible to apply all rules of probability theory to a belief representation. More precisely, the subjective probability measure of the individual is derived from axioms on the preference ordering over uncertain prospects - that is, in Savage's words, over "acts" that map states of the world into contingencies - and serves as a component in the representation of that preference. As a result, in an uncertain context, individuals are supposed to be able to undertake expected-cost/expected-benefit analysis in information gathering, and hence reach an informational optimum. This is why they are termed "probabilistically sophisticated" individuals (Machina and Schmeidler 1992).

The second fundamental assumption of probabilistic decision making is that the processing of information consists in the Bayesian updating of an individual's belief (prior probability distribution), when she receives a signal on the realisation of the state. This is an outgrowth of the implicit assumption that individuals are rational in a strong sense, namely, that they can deduce every logical proposition which, in principle, can be deduced from the axioms of the theory. Of course, this second assumption is highly questionable after Simon (1982), and has been abandoned in boundedly rational and evolutionary models (see in particular Nelson and Winter 1982). But we shall not discuss computational and cognitive problems here. Rather, analysis will concentrate on the first assumption.

It hardly needs stating why Savage's representation theorem is of the utmost importance in the development of modern decision theory. Von Neumann and Morgenstern represented uncertainty by means of explicit probabilities, so that the objects of individual choices were well-defined probability distribution over outcomes. However, real-world uncertainty mostly concerns alternative events (or states), and the typical objects of choices are bets which assign outcomes to alternative events. Savage's approach derives the principle of expected utility maximisation from a number of axioms over acts: in Savage's framework, individual subjective probabilities are elicited from choices. This is in the tradition of the choice-theoretic approach to subjective probability developed by Ramsey (1926) and de Finetti (1937), an approach in which the probability measure underlies choice behaviour.

But the appropriateness of a subjective interpretation of probability rests on the claim that subjective probability satisfies the rule of additivity. De Finetti, in particular, argued that an individual's betting rates cannot violate this rule because otherwise another individual could take advantage of the situation and arrange to make money from her. This argument, known as the "Dutch book" argument, is presented as a coherence condition for the existence and uniqueness of subjective probabilities: a Dutch book is "a combination of bets devised in such a way that, profiting by an inconsistency in the odds given by the bookmaker, someone is certain to win *whatever happens*." (de Finetti 1974: 154) The whole theory of probability, de Finetti maintained, rests on a condition of consistency which "consists in *allowing no chance of a Dutch book occurring*." Yet, this way of arguing assumes that the individual should offer odds not only over certain events or proposition, but also over sets of events, and in principle over all sets covering the entire space state. The individual should also be willing to accept either side of a bet: that is, she should be prepared to be either for or against any given event at given odds. But, contrary to de Finetti, it seems more plausible that individuals are willing to take only one side of a bet; the ensuing problem is that the argument for additivity of the subjective probability distribution does not apply to one-side bets (Shafer 1986).[1]

A first and, admittedly, partial step towards a more convincing representation of uncertainty can be sketched as follows. The Bayesian assumption that, in principle, individuals are able to formulate a single subjective probability distribution in order to deal with any kind of uncertain situation is retained. What is questioned is the reliability of this distribution when individuals are aware that an unlisted event can happen in the future, or when they take decisions conditioned by a non-repetitive event. The issue of reliability of subjective probability distribution emerged, first,

1 A slightly more specific description of how the individual problem is dealt with under the assumption of a complete list of states runs as follows. The individual makes (or should make) a decision by choosing one alternative within a set, when the consequences of each action are tied to uncertain events. The individual formalises the problem by setting alternatives (acts), states of the world, and consequences on the basis of a well-defined utility function representing her preferences. This function involves an evaluation of consequences and their likelihood. The rational decision-maker's goal is to maximise her expected utility in the case in which probabilities are either objective (von Neumann and Morgenstern 1944) or subjectively held (Savage 1954). Both von Neumann-Morgenstern and Savage's theory and their mixed version (like the "horse-race/roulette wheel" theory in Anscombe and Aumann 1963) weigh consequences through a single probability measure on a set of states of the world, in such a way as to induce the linearity of the preference functional. As a consequence, the expected utility can be represented as the mathematical expectation of a real function on a set of consequences with respect to a unique probability distribution. Linearity in the probabilities is a direct consequence of two very similar axioms, namely, the "independence axiom" in von Neumann-Morgenstern's framework, and the "sure-thing principle" in Savage's theory. The independence axiom states that, given two alternatives (lotteries in technical language), with each of them composed of an action and a certain common act, preferences on them should be independent of any common consequence with identical probability (the common act). The sure-thing principle assumes that the decision-maker, when she chooses between the actions, does not take into account states in which actions yield the same consequences.

from experimental evidence which revealed systematic violations of Savage's "sure-thing principle," violations which were inconsistent with the hypothesis of expected utility maximisation. The most discussed of such violations are the Allais Paradox (Allais 1953) and the Ellsberg Paradox (Ellsberg 1961). Second, many critics have maintained that the assumption of a subjective prior does not allow a meaningful distinction between risk, or "measurable uncertainty," and proper uncertainty, or "unmeasurable uncertainty" in Knight's words.[2] Shackle, most of all, emphasised that not only knowledge is bounded, but that the bounds are necessarily imprecise. As recently recalled by Loasby (2000: 5), Shackle's attitude towards the use of probability in decision theory was dismissive: "the imposition of probability distributions, whether subjected or supposedly objective, on closed sets is a pretence of knowledge." As documented in the following sections, certain modern developments of decision theory show that the issue of reliability of probability distributions can be dealt with by relaxing the hypothesis of additivity.

Before turning to Shackle and Ellsberg, it is worth dealing with a crucial aspect of Savage's decision theory, an aspect which is rarely considered by critics discussing the application of his theory. Savage draws a distinction between the *grand world* and the *small world*. The grand world is the complete list of states which are of concern to an individual. The small world is a construction derived from a partition of the grand world into subsets, or small-world states, which are subsets, or events, of the grand world.[3] Savage (1954: 9) maintains that an individual has to confine her attention to a relatively simple situation in almost all her decisions; this amounts to say that, in practice, the individual is concerned with a small world, which is "derived from a larger by neglecting some distinctions between states, not by ignoring some [grand-world] states outright." By considering a small world as the proper context of her decision, the individual describes roughly states of the world and consequences. It is worth noting that the individual can gradually come to consider a more refined and detailed small world, until she arrives to the grand world which takes everything into account. However, Savage's point is that it is "utterly ridiculous" to pretend that "one envisages every conceivable policy for the government of his whole life (at least from now on) in its most minute details, in the light of a vast number of unknown states of the world" (Savage 1954: 16).

As Savage (1954: 82-84) is keen to claim, subjective expected theory should be applied only to small worlds. In fact, it is only in small worlds that all possibilities can be exhaustively enumerated in advance, and all implications of all possibilities explored in detail; hence they can be

2 Knight (1921, 29) suggested that profit is the pay-off entrepreneurs get for bearing (unmeasurable) uncertainty as opposed to (measurable) risk.

3 A "small world … is determined not only by the definition of a state, but also by the definition of small-world consequences. A *small-world consequence* is a grand-world act" (Savage 1954: 85).

labelled and placed in their proper position. In his *Foundations* Savage stresses the “practical necessity of confining attention to, or isolating, relatively simple situations in almost all applications of the theory of decision developed.” Savage’s rejection of the critiques of his theory arguing that “real people frequently and flagrantly behave in disaccord with utility theory,” is mainly based on the distinction between the grand and the small world (Savage 1954: 100-101).

But Savage is forced to admit that a small world “is completely satisfactory only if it is actually a *microcosm*, that is, only if it leads to a probability measure and a utility well articulated with those of the grand world” (Savage 1954: 88). Leaving utility aside, there is no certainty that the probability of an event in the small world equals the probability of the corresponding collection of subsets in the grand world. If the probabilities are different at the two degrees of refinement, probabilities attached in the small world are right only if they equal those calculated in the grand world. Thus the individual must ultimately be able to deal with the grand world, since the condition which assures equality in probability “seems incapable of verification without taking the grand world much too seriously” (Savage 1954: 90). There is a decisive implication here: if the theory is to be consistently applied, the individual should be able to enumerate exhaustively all possibilities in advance and to explore all consequences in detail, though she works exclusively in a practical setting called the small world; thereby it is as if she had a sort of “divine” knowledge of the outside world. As a result, in situations in which outcomes and states are not clearly given in the description of the problem, it is clear neither what the normative implications of Savage’s sure-thing principle are, nor why Savage’s expected utility approach should inform actual behaviour.

3. Shackle’s endorsement of non-probabilistic decision making

The approaches intended as drastic alternatives to probability models have drawn attention to genuine ignorance, surprise and vagueness. An increasing number of economists have taken the view that the Knightian distinction concerns crucial economic problems which cannot usefully be handled by means of a probability distribution, even though subjectively derived. Most representatives of the radically subjective approach to decision making - notably Austrians, post-Keynesians, institutionalists and evolutionary economists - typically argue that the way in which mainstream economic theory deals with decisions under uncertainty is flawed since it cannot take “genuine” uncertainty into account.

Radical subjectivists’ main analytical argument was first elaborated by George L. Shackle. Shackle’s contention is that, granted that the very construction of probability calculus relies on a complete knowledge of the structure of the world, in reality individuals do not have such knowledge. Individual choices are made between alternatives which are subjective representations of alternative future sequels to action; choices are not between future sequels themselves. In Shackle's words, "choice is among imagined experiences", a view which implies that the individual is not given an exhaustive list of the alternatives between which choice is made. Most economic decisions, Shackle argued (1949b: 6), are crucial, unique experiments, namely situations where "the person concerned cannot exclude from his mind the possibility that the very act of performing the experiment may destroy forever the circumstances in which it was performed." The fact that these decisions are non-replicable precludes the possibility of applying probabilities, both numerical probabilities and subjective probabilities.[4]

Shackle's argument, as a result, is that the use of probability calculus "is inappropriate simply because the conditions appertaining to its application just do not exist in respect of decisions taken in an economic context" (Ford 1983: 21). In particular, Shackle argues, individuals are not capable of enumerating all possible contingencies, or states of the world. In a sense, the individual agent is able to take decisions only insofar as she creates her own choice set, but this necessarily implies that the choice-set thus created is non-closed.[5]

This is the main analytical point at the basis of Shackle's theory. To economists working in the above mentioned traditions of thought, this point has become an indispensable analytical reference in their effort to represent decisions under genuine uncertainty.[6] On the basis of this point, Shackle (1949 and 1961) developed a formal theory opposed to the Bayesian approach intended to capture both the mental processes and the non-repetitive, often irreversible, nature of actual economic decisions. In focusing on the subjective and idiosyncratic nature of human judgements, Shackle's aim was to emphasise the typically imprecise domain of actual decisions.[7]

4 Shackle's argument was initially intended to contrast the objective frequency-ratio interpretation of probability (1949). Later he maintained that the same argument could apply to the subjective interpretation of probability (Savage 1952: 30).

5 The necessity of dealing with non-closed choice sets bears a resemblance to the perspective endorsed by modern critics of Savage's subjectivist approach. For instance, Binmore and Brandenburger (1990: 144) contend that Savage's theory applies only to "closed universe" and define a closed universe as "one in which all the possibilities can be exhaustively enumerated in advance and all the implication of all possibilities explored in detail so that they can be neatly labeled and placed in their proper pigeonholes." Savage's small world is a closed universe.

6 In particular, this is the crucial argument upon which the distinction between "rational ignorance" and "radical ignorance" has been drawn in the Austrian tradition (Langlois 1994 and Vaughn 1994). By this distinction, the Austrians intend to distinguish Savage's approach from the Knightian tradition. On this point see Zappia (1998).

7 For a concise assessment see Currie and Steedman (1990, Ch. 9), Hamouda and Rowley (1996, Ch. 4), and Basili and Zappia (2003).

To dispense with probability, Shackle put forward the concept of potential surprise. First, he distinguished between distributional uncertainty variables, which can be used if "the list [of suggested answers to a question] is *complete without a residual hypothesis*," and non-distributional variables, which must be used when "the list in order to attain formal completeness must be rounded off with a residual hypothesis." (Shackle 1961: 49-50). This distinction shows that his theory is essentially non-additive. Shackle was aware of this characteristic from the very beginning of his effort. In fact, in response to some critics of his 1949 volume *Expectation in Economics*, he made explicit that his system was non-additive. In order to describe "mental states of uncertainty", what was needed was "a measure of acceptance, of a hypothesis proposed in answer to some question, that shall be independent of the degrees of acceptance simultaneously accorded to rival hypothesis;" that is, Shackle's framework required "a measure of acceptance by which the individual can give to new rival hypotheses, which did not at first occur to him, some degree, and even the highest degree, of acceptance without reducing the degrees of acceptance accorded to any of those already present in his mind" (Shackle 1949-50: 70). This measure of acceptance, called "potential surprise," amounts to a substitute for probability distributions.

Shackle's next step was to apply this (non-additive) measure. He analyses a decision-maker, typically an entrepreneur, who has to choose among alternative sequels to actions on the basis of two elements: an index of the gains and losses embedded in a sequel, called face-value, and a valuation of the "possibility" of the hypothetical sequel, called potential surprise. The latter element should be considered as a degree of disbelief, or implausibility of the hypothesis that supports the sequel; it can be normalized from 0 (absence of disbelief) to 1 (absolute disbelief).[8] When the decision maker chooses among alternative sequels, she re-considers the face-value of each sequel by its degree of potential surprise. This criterion amounts to a rule of thumb by which the decision maker takes into account both "the best possible and the worst possible outcome of each course of action and makes these pairs of outcomes the basis of his decision" (Shackle 1953: 43), respectively called "focus-gain" and "focus-loss."

Next, Shackle defines a function $\phi \in [0,1]$, called "degree of stimulus" (or "ascendancy function") whose arguments are the face-value and the potential surprise implicit in a sequel. Given

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8 The aim of measuring the degree of belief in a certain event by means of its opposite, the degree of disbelief, or potential surprise, is instrumental to the construction of a non-additive index. The emergence of a new unanticipated event does not necessarily reduce the degree of disbelief previously assigned to other events as necessarily is if (the opposite of) this degree is measure by an probability. "By disbelief I do not now mean the absence of perfect certainty, but the positive recognition of some disabling circumstances … and there is, in general, no limit to the number of mutually exclusive hypotheses to all of which simultaneously a person can, without logical contradiction, attach zero potential surprise" (Shackle 1952: 30-31).

a degree of stimulus, it is possible to determine a prospect of the possible outcomes of a sequel weighed by degrees of reliability. On this ranking of sequels, Shackle superimposes “the particular potential surprise curve which [the decision maker] assigns to some particular project.” In this way it is possible to determine “the highest bids which the particular project in question can make for the decision maker’s attention and interest. One of these bids is the most powerful suggestion of success and the other the most powerful suggestion of disaster that the conception of project conveys” (Shackle 1953: 46). These extreme values represent the limits of all possible outcomes of any feasible sequel. This development can be phrased in the language of modern decision theory. Shackle’s decision maker orders prospect revenues of an act (sequel) on the basis of both their value and their reliability, and then she re-evaluates them by her specific (that is, linked to the project at hand) attitude towards the uncertain environment. At this point, she takes into account the best and the worst outcomes involved in the feasible act, and considers them as the limits (best/worst) of the possible values of the act, instead of calculating an expected value of the act.[9]

But the procedure of reducing the expectational elements involved in the decision to two (monetary) values was criticized because of a lack of “axiomatic” justification from the early critics. For instance, Arrow (1951) singled out Shackle’s theory as one of the few approaches alternative to von Neumann and Morgenstern axiomatisation, but regarded the process of eliciting the focus values as arbitrary. As a result, the this lack of justification was one of the main reason why Shackle’s formal theory, after being widely discussed in the 1950s, sunk to oblivion.[10] The final section of this paper focuses on the fact that Shackle’s procedure, with its emphasis on the notion of possibility rather than probability, has a counterpart in modern decision theory. On the basis of this recognition, it will be argued that the reliance of standard Bayesian theory on probabilistic judgements based on point-probability estimates, a reliance that Shackle intended to oppose, is no longer a justification for dispensing with probability calculus once the non-additive probability approach is endorsed. The next step, however, is a brief analysis of the experimental evidence against Savage’s theory.

9 Shackle (1953: 47) summarises: “because the project is a non-divisible non-seriable experiment, his [the entrepreneur] various hypotheses as to its outcome are mutually exclusive and … therefore there is here no logical basis for the additive procedure by which a ‘mathematical expectation’ is assigned to a divisible experiment.”

10 After reviewing Shackle’s theory, Arrow (1951: 38) commented: “This theory is not based on consideration of rational behavior, which Shackle specifically rejects, but on an alleged inability of the mind to consider simultaneously mutually exclusive events.”

4. Ellsberg and the experimental evidence

The challenge posed to expected utility theory by the Ellsberg Paradox can be considered a crucial counter-example concerning the descriptive validity of the theory. The paradox focuses on the belief side of the decision problem, and involves considerations of ambiguity and degree of confidence. The following experiment was put forward by Ellsberg (1961). An individual faces an urn that contains 30 red balls and 60 balls in some combination of black and yellow; there is no information whatsoever about the number of black and yellow balls in the urn (unknown proportion). A ball will be drawn from the urn. There are two pairs of acts, X and Y, and X' and Y.' Acts have consequences of 1 or 0 as follows: choosing X one gets 1 if the ball is red and 0 if it is black or yellow; choosing Y one gets 0 if red or yellow and 1 if black, choosing X' one gets 1 if red or yellow and 0 if not; choosing Y' one gets 0 if red and 1 if black or yellow. Ellsberg reported that, among those asked, most people choose X instead of Y, and Y' instead of X,' thus revealing a remarkable preference for betting on known probabilities of winning. That is, it appeared that confidence in estimates of subjective probabilities is taken into account by individuals when they make a choice. This type of decisions is inconsistent with Savage's sure-thing principle. In fact, both pairs of acts have different consequences only when the yellow state occurs, and these consequences are the same both for X and Y (the individual gets 0) and for X' and Y' (the individual gets 1).

Moreover, the beliefs of the individual exhibiting such preferences cannot be represented through an additive probability distribution. Suppose p(r), p(b) and p(y) are the subjective probabilities of each possible draw. Setting $U(0)=0$, Savage's subjective expected utility implies that X is to be preferred to Y if and only if $p(r)U(1)> p(b)U(1)$ or $p(r)>p(b)$. Likewise Y' is preferred to X' if and only if $p(b\cup y)>p(r\cup y)$. This contradicts the assumption that probabilities are additive: in fact, given $p(b\cap y)=0$, if $p(b\cup y)=p(b)+p(y)$, then to prefer Y' to X' implies $p(b)>p(r)$, which conflicts with what is implied by preferring X to Y, that is, $p(b)<p(r)$.

As a result, these preferences contradict not only the expected utility theory, but also every other theory of rational behaviour under uncertainty that assumes a unique additive probability measure underlying choices. In Ellsberg's (1961: 654) words, "it is impossible, on the basis of such choices, to infer even qualitative probabilities for [the] events in question … [it is impossible] to find probability numbers in terms of which these choices could be described – even roughly or approximately – as maximising the mathematical expectation of utility." The violations of both the complete ordering of actions and the sure-thing principle pointed out by Ellsberg in his hypothetical experiments have been confirmed by many other experiments replicated in recent years (Camerer

2000). These results suggest that most agents prefer making choices in unambiguous environments rather than in ambiguous ones, and that individual choices can be affected by the nature of one's information concerning the relative likelihood of events. What is at issue, Ellsberg (1961: 657) clarified, "might be called the *ambiguity* of this information, a quality depending on the amount, type, reliability and unanimity of information, and giving rise to one's degree of confidence in an estimate of relative likelihood."

It is of course true that the traditional formulation of the problem of decision under uncertainty is still dominant. The main reason for this dominance is the central role that traditionalists ascribe to Bayesian individuals maximising expected utility. But, mainly thanks to Ellsberg's contribution, an increasing number of papers in leading mainstream journals have been devoted to the study of alternative ways of formalising uncertainty.[11] Making Knight's distinction operational, in other words, is no longer the business of alternative traditions of thought only. The next section discusses one of these line of research that provides a formal solution to the problem of subjective probability, a solution that resembles Shackle's solution.

5. Probabilistic decision making: the non-additive approach

A major problem emerging from our reconstruction is how to represent the individual agent's confidence in a probability assessment. In situations featuring ambiguous and unambiguous events, like that exemplified in the Ellsberg Paradox, decision-makers underweigh the probabilities attached to ambiguous events and prefer to bet on unambiguous events.[12] Therefore, it is affirmed, the weights of the priors depend on the decision-makers' attitude towards their probability assessments; that is to say that weights depend on the agents' confidence in their specification of states of the world. This section of the paper points out that it is not necessary to abandon probabilistic reasoning even if uncertainty is taken to refer to the intuitive concepts of ambiguity and vagueness. Through a more accurate description of both the world and individual beliefs, a precise notion of uncertainty (as different from risk) will be formulated.

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[11] As Hamouda and Rowley (1997: *xx*) put it in introducing their collection of influential articles on probability and decision theory, "while many textbooks retain and stress the notions of probability as established by the beginning of the 1970s, two decades of active innovation with vague and imprecise alternatives has undermined earlier myopia and complacency, widened the conventional structure of policy analyses involving uncertainty, produced some means of translating common forms of imprecision into useful ingredients for modelling frameworks, and thus generated a less hostile audience for unconventional views of uncertainty and their application to real phenomena."

Consider a decision problem in which the states of the world included in the model do not exhaust the actual ones. A description of the world can be considered as a mis-specified model if certain states are not explicitly included. If an individual agent does not know how many states have been omitted, her beliefs can be represented either by a non-necessarily-additive measure or by a set of additive probability distributions on the set of events. Gilboa and Schmeidler (1994) contend that individuals maximise the expected value of their utility functions with respect to a non-additive belief. In other words, they represent preferences by a Choquet integral of expected utility with respect to a (convex) capacity, that is, to a non-necessarily-additive probability distribution.

A convex capacity, μ, is a monotone measure which, like probability, is normalised to 1 on the full set and 0 on the null set. But, unlike probability, the sum of the capacities of two subsets may be different from the capacity of the union of the same sets.[13] If the sum is strictly less than the capacity of the union, the capacity is said to be convex.[14] The convexity of the capacity is a property suggested by the Ellsberg Paradox. As shown in the previous section, the Ellsberg Paradox demonstrates that belief in the unambiguous event of drawing a black or a yellow ball strictly exceeds the sum of beliefs in the ambiguous events that a black ball is drawn or a yellow ball is drawn. Since μ is a non-additive measure, the integration of a real-valued function with respect to μ is impossible in the Lebesgue sense. It has been shown that the proper integral for a capacity is the Choquet integral. The Choquet integral with respect to a capacity is a generalisation of the Lebesgue integral, which requires that states of the world have been ranked from the most favourable to the least favourable, or vice versa, with respect to their consequences.[15] As a result, the Choquet integral is a generalisation of the mathematical expectation usually used in expected utility models with respect to a capacity.

Basically, two non-expected utility theories have been proposed to encompass both uncertainty attitude (versus risk attitude) and the expected utility maximisation. Gilboa (1987) and Schmeidler (1989) axiomatise a generalisation of expected utility, which provides a derivation of both utilities and non-necessarily-additive probability by means of the Choquet integral. Gilboa and

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12 For instance, in the three-color example reported in the previous section, individuals prefer acts with a known probability of winning.

13 In a more formal way, let $\Omega=\{w_1,\dots,w_n\}$ be a non empty set of states of the world and let $S=2^\Omega$ be the set of all events. A function $\mu:S\rightarrow R_+$ is a non-necessarily-additive probability measure, or capacity, if $\mu(\emptyset)=0$ and $\mu(\Omega)=1$, and if for all $s_1,s_2\in S$ such that $s_1\supset s_2$, $\mu(s_1)\geq\mu(s_2)$.

14 That is, a capacity is convex if for all $s_1,s_2\in S$ such that $s_1\cup s_2$, $s_1\cap s_2\in S$, $\mu(s_1\cup s_2)\geq\mu(s_1)+\mu(s_2)-\mu(s_1\cap s_2)$.

15 See Choquet (1954). The Choquet integral of f with respect to μ is

$$\int fd\mu=\int_0^{\infty}\mu(\{w|f(w)\geq t\})dt+\int_{-\infty}^{0}[\mu(\{w|f(w)\geq t\})-\mu(\Omega)]dt$$

Schmeidler (1988) extend the range of classical expected utility by representing preferences through a utility function and a set of additive probabilities, in place of a single additive function on the set of events. But it is worth stressing that the two approaches not only coincide with respect to the issue we are concerned with but, under certain technical conditions, also have the same formal representation.

At first it might appear that the Ellsberg Paradox and Shackle's problem refer to different situations, so that the solution of the Ellsberg Paradox is not a solution to Shackle's problem.[16] In the former, the question is one of ambiguous probabilities with a complete list of all possible events.[17] In the latter, the question is one of providing an exhaustive list of possible events. But in recent literature on non-additive probability it has been argued, first, that non-additive beliefs may arise naturally from imperfect knowledge about the relationship between states of the world and pay-off relevant events. Second, scholars have maintained that the omission of possibly relevant details in constructing a model of this relationship may give rise to this kind of imperfect knowledge.

To be more specific, Mukerji (1997) has shown that a decision model with a non-additive measure on the state space can be embedded in a decision model with an additive measure (probability). In this model, the enlarged state space includes all possible missing states (namely, the "objectively" given states in the sense of Savage's "divine" space states). As a result, it is possible to relax the non-additivity of a measure at the expense of the dimension of the decision model. In the representation of uncertainty through a non-additive measure on the space state, a relationship between the epistemic status of the individual (that is, her awareness of both incomplete knowledge and the limited reliability of likelihood assessments) and her choice is implicitly assumed. Mukerji (1997: 25) clarifies this relationship by means of a "two-tiered" state space model which embeds "a space on which the individual assigns primitive beliefs and a space of payoff relevant states, i.e. states on which the available acts are directly defined."[18] At first the individual assigns her beliefs (priors) on her state space perceived as simpler (primitive), and then she infers beliefs about the events to which the outcomes of acts are directly related. The inferred space is called derivative world. It is straightforward to interpret the primitive and derivative worlds as Savage's small and grand worlds, respectively.

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16 This point has been emphasized by Runde (2000).

17 In Ellsberg's three-colour example, ambiguity emerges because of the impossibility of knowing in advance the combination of blue and yellow balls in the second urn, but the range of this combination is known with certainty.

18 See Gilboa and Schmeidler (1994) for the formal counterpart of this analysis. Mukerji's two-tiered state space is mathematically isomorphic to the enlarged space of Gilboa and Schmeidler.

The primitive state space (the small world) is a set of objects of which the individual has direct experience, clear intuition and empirical knowledge. Belief assessments on this state space express this confidence. Likelihood assessments on the derivative world (the grand world) are deduced from an "implication mapping" which represents the individual's knowledge of the association between the two worlds. As a result, the decision-maker's knowledge about the likelihood of an event in the derivative frame is constituted by the sum of the beliefs assigned to those elements of the primitive frame whose implications are sub-events of the event in question (Mukerji 1997: 33).

Depending on the epistemic condition of the individual, the beliefs on the derivative frame may have a non-additive representation. In fact, if the individual transfers a likelihood assigned to an event in the small world to an event in the grand world, the implication is that she is unable to distribute beliefs across the elements of the grand world. It follows that the non-additivity of subjective probability measures becomes an expression of the limits of the decision-maker's understanding of the possibilities of the world, as well as of her awareness of these limits. Hence it is legitimate to assume that an individual with a perception of the grand world as fuzzy, incomplete, or vague behaves as if she had a set of priors or a non-additive measure rather than a well-defined probability.

By means of this representation we attain a formal solution of the problem of the relationship between the small and the grand world. Savage tackled this problem by referring to the small world as a microcosm (as shown in section 2 above), but eventually failed to solve it. Additionally, this representation leads to a definition of uncertainty which makes the Knightian distinction operational. The argument can be posited as follows. A decision-maker faces Knightian, radical uncertainty when either she has a mis-specified description of the states of the world or is unable to assign a reliable probability distribution to states of the world. Furthermore, a decision-maker expresses Knightian uncertainty aversion if she assigns larger probabilities to states in which consequences are unfavourable than to states in which consequences are favourable. This attitude entails that her non-additive measure is convex (Dow and Werlang 1992). Hence, the convexity of the capacity captures the decision-maker's Knightian uncertainty aversion and substantiates the conservative statement that the decision-maker acts "as though the worst were somewhat more likely than his best estimates of likelihood would indicate [and] he distorted his best estimates of likelihood, in the direction of increased emphasis on the less favourable outcomes and to a degree depending on his best estimate" (Ellsberg 1961: 661).

Alternatively, Knightian uncertainty can be represented by a set of possible priors. That is, the individual knows enough about the problem at hand to rule out a number of possible distributions. In this case the agent has multiple additive probability measures P on $\Omega=\{w_1,...,w_n\}$, and her preferences are compatible with either the maxmin or the maxmax expected utility decision rule, where the maxmin (maxmax) expected utility postulates that an individual with multiple priors considers the least (most) value of expected utility for any act and chooses that act for which this least (most) value is greatest. Gilboa and Schmeidler (1988) and Chateauneuf (1991) have demonstrated that when an arbitrary (closed and convex) set of possible priors P is given, and one of these defines either a non-additive probability measure υ (convex) or ν (concave) on Ω, such that all additive probability measures in P majorise υ or minorise ν, the non-additive expected utility theory coincides with either the maxmin or the maxmax decision rule, respectively.[19]

Our analysis can now be summarised. For both theoretical and empirical reasons economists working in decision theory have sought to generalise the expected utility model. At the basis of this developments there is the distinction between risk and uncertainty usually attributed to Knight. Although this distinction is deemed unimportant to scholars working in a Bayesian perspective (for textbook evidence see Hirshleifer and Riley 1992), we have briefly expounded an axiomatic development which incorporates such a distinction. The model discusses an individual maximising her expected utility with a non-additive probability; and, in order to reflect the individual's attitude to uncertainty, the model includes the possibility that the subjective probability attributed, for instance, to two mutually exclusive events does not necessarily add up to 1. What is more, the axiomatisation provides a base for dealing with situations in which the uncertainty of the individual involves the existence of a third (or more) event, whose occurrence had no probability attached at the outset.[20]

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19 For a detailed review of these developments see Basili (2001).

20 A number of important applications have been proposed in the fields of financial markets (Epstein and Wang 1994, and Mukerji and Tallon 2001), incomplete contracts (Mukerji 1998), environmental problems concerning irreversibility (Basili 1998), and in game-theoretical contexts (Eichberger and Kelsey 2000). It is worth stressing that the representation of beliefs through real-valued set functions which do not necessarily satisfy additivity is not new. In particular, "belief functions" were introduced by Dempster (1967) and Shafer (1976). Although these theories were not directly related to decision under uncertainty, it turned out that "beliefs functions" were a special case of non-additive measures (or capacities) (Gilboa and Schmeidler 1994). Likewise, Zadeh's (1978) theory of fuzzy sets has been shown to be compatible with the non-additive probability approach (Wakker 1990).

6. Shackle and modern decision theory

This section of the paper argues that there is a precise analogy between Shackle's propositions and certain aspects of the non-expected utility theory based on capacity.[21] In particular, Shackle's rule can be regarded as the counterpart of the maxmin and maxmax rules. These are appropriate in the cases in which the decision maker faces absolute disbelief and absolute belief, respectively, with respect to an act carried out in contexts marked by ambiguity.

In Shackle's view the second independent variable assigned to a sequel of action (act) is the potential surprise. This is a function measuring the reliability of the face-value. It depends on the influence of the non-excluded hypotheses, that is, on the assessment of possible uncertain states of the world. In the words of modern decision theory, potential surprise depends on ambiguity and it is embedded into a capacity. A capacity can be considered as a distortion of the individual's subjective probability, a distortion that depends on the reliability of the decision maker's description of states of the world.[22] According to Choquet Expected Utility (CEU), the decision maker expresses her ambiguity aversion (pessimism) or ambiguity loving (optimism) by a convex or a concave capacity. The Choquet integral modifies the decision maker's priors to encompass her attitude towards ambiguity. To be more specific, the Choquet integral replaces prior probabilities with a new measure (based on marginal probabilities) which depends on the induced order of consequences defined by the decision maker's attitude towards ambiguity. With respect to her priors, a pessimistic (optimistic) decision maker attaches a larger (smaller) probability to unfavorable consequences than to favorable ones.

Decision theorists agree that if the decision-maker's prior is vague and she feels subjectively either confident or sceptical about her assessment, a reasonable solution of a given decision problem is achieved either by Wald's maxmin rule (if she is confident) or by its counterpart, the maxmax rule (if she is sceptical).[23] As recalled above, Gilboa and Schmeidler (1989) and Chateauneuf (1991) have pointed out that CEU relating to a convex or concave capacity is equal to either the minimum or the maximum expected utility with respect to the additive probability measures in the core of that capacity. In the context of Shackle's theory, the maxmin and maxmax solutions correspond to the focus-gain and focus-loss values.

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21 This section draws and elaborates on Basili and Zappia (2003).

22 Under stochastic dominance, given a probability P on the set of states of the world and a non-decreasing normalized function γ: $[0,1]\to[0,1]$ a capacity ν is a distorted probability if $\nu: =\gamma\circ P$ (see Wakker 1990b).

23 See Wald (1950).

But the analogy between Shackle's decision rule and modern measures of uncertainty can be further developed. Two different measures have been recently proposed: the α-Maxmin Expected Utility (α-MEU) and the Ellsberg capacity (E-capacity). Both theories assume that the existence of ambiguity is expressed by a set of priors (additive probability distributions), that is, both theories belong to the class of multiple priors models. The starting point of these theories is an acknowledgement of the inadequacy of CEU to characterize information. That is, they contend that, although the CEU functional represents a solution of the Ellsberg paradox, it imposes *homogeneity of ambiguity* and *extreme ambiguity attitude*. Roughly speaking, the standard CEU solution of the Ellsberg paradox rests on the extreme (worst) case, regardless of any consideration relative to the individual's degree of confidence in her probability assessment.

This limitation is overcome by a generalized version of maxmin expected utility called α-Maxmin Expected Utility (α-MEU),[24] as well as by the CEU with respect to an E-capacity.[25] In both theories a crucial role is played by the parameter $\alpha \in [0,1]$, which expresses, respectively, the decision maker's ambiguity attitude and degree of confidence about her assessment. The α-MEU points to the decision maker's degree of ambiguity perception, and the E-capacity Expected Utility refers to the degree of reliability that the decision maker attaches to her best additive assessment. Both these non-expected utility theory involve a valuation of the quality of the decision maker's information.

The α-MEU preference model recently axiomatised by Girardato, Maccheroni and Marinacci (2003) establishes a connection between the opposite valuations (the best and the worst) of each act, given a set of relevant probability distributions (priors). In the α-MEU model the functional shows a *sandwiching property*, because it is placed between the worst and the best scenario evaluation of the decision maker. Straightforwardly, the assumed constant function α is the ambiguity aversion index: the higher (lower) it is, the bigger (smaller) the pessimism in the decision maker's evaluation is. If α is equal to one, then a standard CEU functional is obtained.[26]

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24 The α-MEU may be thought of as a generalization of Hurwicz's α-maxmin decision rule.

25 In discussing the paradoxical results his example generated, Ellsberg proposed a representation of preferences over uncertain acts, which resolves the paradox. Though intuitively appealing**,** this representation lacks an axiomatic foundation. Eichberger and Kelsey (1999) introduce a particular class of capacities, called E-capacities, and show that the Choquet integral of an E-capacity is the Ellsberg representation. Ellsberg capacities are a "parameterized version of a capacity based on an additive probability distribution that makes it possible to include known probabilities for a partition of unambiguous events" (Eichberger and Kelsey 1999, p. 133).

26 Formally the α-MEU functional of an act is:

$$V(f) = \alpha \max_{\pi \in \Pi} E_{\pi}(u \circ f) + (1-\alpha) \min_{\pi \in \Pi} E_{\pi}(u \circ f) .$$

The CEU preference model with respect to an Ellsberg capacity has been axiomatised by Eichberger and Kelsey (1999). It starts with Ellsberg's own solution of the problem of explaining the observed behavior in the three-color example. E-capacities are a representation of the beliefs of an individual, a representation which takes into account both her probability assessments of events and the reliability of her probability assessments ("degree of confidence" in Ellsberg's (1961: 664) words). The degree of confidence is considered a measure of reliability of the decision-maker's probability distribution. In a convex combination, the CEU preference model with respect to an Ellsberg capacity links an additive probability distribution (the most reliable probability distribution in the set of the relevant ones) with a capacity, or "a term which is easily recognized as the minimum expected utility over all information consistent probability distributions" (Eichberger and Kelsey 1999: 123). The decision maker links her best estimation with the set of *information-consistent* probabilities, which are included in the set of the decision maker's priors. The set of information-consistent probabilities is defined on *unambiguous* events.

Consider Ellsberg's three colors example once more. There are 30 red and 60 black or yellow balls in the urn, with probabilities 1/3 and 2/3, respectively. This is a partition of the set of states of the world; events in this partition are called unambiguous. The Choquet integral of an Ellsberg capacity "is the weighted average of the expected utility with regard to an additive probability distribution and the worst expected outcome obtained in the unambiguous events" (Eichberger and Kelsey 1999: 133). The weight in the average is derived from the degree of confidence in the best assessment: the larger (shorter) α is, the bigger (smaller) the weight of optimism over the best assessment is. If α is equal to one, then a standard SEU functional is obtained.[27]

[27] Let $\Omega=\{w_1,...,w_n\}$ be a non-empty finite set of states of the world and let $S=2^{\Omega}$ be the set of all events. Let g be an act, such that $g{:}\,\Omega\rightarrow C$, and let C be the set of finite consequences. Let $\{E_1,....,E_n\}$ be a partition of Ω with probabilities $p(E_i)$, such that $\sum_{i=1}^{n} p(E_i)=1$, that is a partition of *unambiguous events*. Given an additive probability distribution π on Ω, let $\Pi(p)$ be the set of *information consistent* additive probabilities, such that $\Pi(p):=\{\pi\in\Delta(\Omega)\big|\sum_{\omega\in E_i}\pi(\omega)=p(E_i)\}$ with $i=1,2,...,n$ and for all $A\in\Omega$ let $\beta_i(A):\Omega\rightarrow\{0,1\}$, such that $\beta_i(A)\rightarrow\{1 \text{ if } E_i\subseteq A;\ 0 \text{ otherwise}\}$ be the function characterizing events including at least one unambiguous event (Eichberger-Kelsey 1999, p. 118).
Due to ambiguity aversion, the decision-maker considers all the sets of conditional probability distributions compatible with her incomplete information on the basis of her degree of confidence $\alpha\in[0,1]$. Consequently, the *E-capacity* $\upsilon(\pi,\alpha)$ is $\upsilon(A|\pi,\alpha)=\sum_{i=1}^{n}[\alpha\pi(A\cap E_i)+(1-\alpha)p(E_i)\beta_i(A)\ \ \forall A\in\Omega.$

Although Shackle does not explicitly set the functional of his rule, it is straightforward to note that it overlaps the α-MEU functional. Furthermore, Shackle's rule comes very close to the intuition of the functional obtained by means of an Ellsberg capacity.

7. Concluding remarks

Ellsberg's seminal paper presented compelling examples in which decision-makers prefer to bet on known rather than on unknown probabilities. Related experimental evidence has demonstrated that many decision-makers do not assign probabilities which are consistent in the sense of de Finetti and Savage. In "ambiguous" situations, decision-makers may not assign probabilities to all possible events, though the likelihood of "unambiguous" events is represented in the standard probabilistic way. Furthermore, decision theorists have increasingly recognised that "the fundamental assumption that the state space (either objective or subjective) is known to the individual … is problematic." As a result, a theory of decision making under uncertainty is made to rest on the assumption that "the individual knows the set of available actions and he knows that payoffs occur to each action in each period. But he has no further knowledge of the decision problem he is facing. In particular, states are not a part of his view of the world. He does not necessarily have knowledge of the objective or even some subjective state space" (Easley and Rustichini, 1999: 1158).

Interestingly, current attempts to provide an axiomatic foundation for decision making in complex environments focuses on how to overcome the patent inadequacy of the assumption about the structural knowledge of the environment held by the individual. Earlier on, we have pointed to the difficulty of representing the individual agent's confidence in a probability assessment. As Shackle's case illustrates, this problem was the focus of approaches aiming at constituting drastic alternatives to probability models. Characteristically, alternative approaches took genuine ignorance, surprise, and vagueness into account.

This paper has suggested the potential of a, relatively straightforward, modification of the axiomatic system of the traditional Bayesian approach to decision making under uncertainty. This modification has made it possible both to solve many emerging descriptive paradoxes and to tackle problems related to genuine uncertainty. One of these problems is the inherent inability of the individual agent to know in advance the domain of his uncertainty, an inability which we have labelled radical ignorance. We regard the axiomatisation reviewed in section 5 as a consistent attempt to give concrete, operational meaning to the dichotomy between radical and rational ignorance in decision making.

The inability of pure economic theory to deal with proper uncertainty should no longer be taken for granted. We do not view the kind of formal representation of decision making under uncertainty one finds in recent developments in decision theory, namely, the non-additive approach to decision making, as some representatives of the radical subjectivist approach do. The non-additive approach is not intended to describe agents "striving to formulate the correct vision of the future as if the future were something already implicit in the data and one's only problem is to guess correctly what the future will be" (Vaughn 1994: 147). On the contrary, it recognises as a starting point for research the view that ignorance is an inherent feature of every decision regarding future events. In this crucial respect, the non-additive approach resembles the Shackleian conception that the future is the unpredictable consequence of creative choices made by individual agents.

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