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On Game Formats and Chu Spaces

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Abstract - It is argued that virtually all coalitional, strategic and extensive game formats as currently employed in the extant game-theoretic literature may be presented in a natural way as discrete nonfull or even-under a suitable choice of morphisms- as full subcategories of Chu(Poseṭ2).

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1 Introduction

Both games and game forms- the latter being 'games without preferences' i.e. models of pure interaction structures– come in a large variety of formats. In fact, this is clearly the case even when only the most commonly used *extensive, strategic* and *coalitional* formats are considered, and some more exotic variants are ignored. This circumstance immediately raises the following question: does there exist any available mathematical structure which happens to be general enough to accommodate –i.e. to specialize to– game forms and games of each 'basic' format as mentioned above?

Since games and game forms are typically described in common, informal set-theoretic language, the category **Set** of sets and functions is a first obvious candidate. Another, perhaps more interesting, possibility is provided by certain categories of Chu spaces. Chu spaces may be regarded as abstract representations of classifications of certain classes of 'objects' in terms of certain classes of 'types': indeed it has been explicitly suggested that the universe of games essentially amounts to the category **Chu**(**Set**,2) (see e.g. Lafont and Streicher (1991) and Pratt (1995)). Moreover, Pratt shows that virtually all known mathematical structures admit a full embedding into **Chu**(**Set**,K) for some set K. However, under the implied standard representation of games as objects of **Chu**(**Set**,2), only 2-player strictly competitive games in strategic form, an extremely specialized class of games, are covered. Therefore, it remains to be seen whether categories other than **Set** exist that include all the basic game formats as subcategories of theirs. In that connection, we focus on the categories **Chu**(**Poset**,K) for some set K, where **Poset** denotes the category of partially ordered sets and order-homomorphisms.The present note is mainly devoted to the task of showing the following two points:

i) if game-forms-as-objects (or games-as-objects) are considered while morphisms are essentially ignored, then each one of the main game-theoretic formats with their basic variants may be represented either as a (*discrete*) *non-full subcategory* of **Chu**(**Poset**,2) or as a *(discrete) concrete category* over **Chu**(**Poset**,2);

ii) moreover, for a suitable choice of morphisms the resulting category $\mathbf{Ecgf}^*$ of coalitional game forms turns out to be a *full subcategory* of **Chu**(**Poset**, 2).

Some consequences of points i) and ii) concerning the 'universality' of games as objects of mathematical discourse are also discussed.

2 Some basic categorical preliminaries

Categories provide the most natural setting for any attempt at classifying and comparing several game formats. We introduce for the sake of completeness a few basic categorical definitions and facts we shall need in the ensuing treatment of game-theoretic notions (see e.g. Adámek, Herrlich, Strecker(1990) for a thorough introduction to category theory).

A *category* is a tuple $\mathbf{C} = (\mathcal{O}, \hom, id, \circ)$ where

$\mathcal{O} = Ob(\mathbf{C})$ is a class that denotes the *objects* of $\mathbf{C}$

hom : $\mathcal{O} \times \mathcal{O} \to \mathcal{U}$ - where $\mathcal{U}$ denotes the *universe*, i.e. the class of all sets- is a function that maps each pair (A, B) of objects into a set $\hom_{\mathbf{C}}(A, B)$, the set of $\mathbf{C}$-*morphisms from* A to B

$id : \mathcal{O} \to Mor(\mathbf{C})$ - where $Mor(\mathbf{C}) = \bigcup_{A,B\in\mathcal{O}} \hom_{\mathbf{C}}(A, B)$- is a function that maps each object A into $id_A \in \hom(A, A)$, the *identity-morphism* on A

$\circ : Mor(\mathbf{C}) \times Mor(\mathbf{C}) \rightharpoondown Mor(\mathbf{C})$ is a partial binary operation on morphisms -*composition*- that maps each pair of morphisms $f \in \hom_{\mathbf{C}}(A, B), g \in \hom_{\mathbf{C}}(C, D)$ such that $B = C$ into a morphism $g \circ f \in \hom_{\mathbf{C}}(A, D)$

and the following conditions are satisfied:

Cat1) *(Associativity of $\circ$):* $h \circ (g \circ f) = (h \circ g) \circ f$ whenever both $g \circ f$ and $h \circ g$ are defined

Cat2) *(Identity):* for any $A, B \in \mathcal{O}$ and for each $f \in \hom_{\mathbf{C}}(A, B)$, $f \circ id_A = f$ and $id_B \circ f = f$

Cat3) *(Hom-Disjointness):* for any $A, B, C, D \in \mathcal{O}$, if $(A, B) \neq (C, D)$ then $\hom_{\mathbf{C}}(A, B) \cap \hom_{\mathbf{C}}(C, D) = \emptyset$.

In particular, for any class Y a category $\mathcal{C}(Y)$ induced by Y can be defined by positing $Ob(\mathcal{C}(Y)) = Y, \hom_{\mathcal{C}(Y)}(y, y) = \{y\}$ for any $y \in Y$ and $\hom_{\mathcal{C}(Y)}(x, y) = \emptyset$ for any $x, y \in Y$ such that $x \neq y, id_y = y$ for any $y \in Y$, and $y \circ y = y$ for any $y \in Y$.

A category $\mathbf{C} = (\mathcal{O}, \hom, id, \circ)$ is *discrete* if for any morphism $f \in Mor(\mathbf{C})$ there exists an object $A \in \mathcal{O}$ such that $f = id_A$. Discrete categories essentially amount to class-induced categories as defined above.

A *source* in a category $\mathbf{A}$ is a pair $\mathcal{S} =(A, (f_i)_{i\in I})$ where $A \in Ob(\mathbf{A}), I$ is a class and for all $i \in I$ there exists $A_i \in Ob(\mathbf{A})$ such that $f_i \in \hom_{\mathbf{A}}(A, A_i)$: A is also called the *domain of the source* and the family $(A_i)_{i\in I}$ its *codomain.* A *product* in a category $\mathbf{A}$ is a source $\mathcal{P} = (A, (p_i)_{i\in I})$ such that for every source $\mathcal{S} =(A, (f_i)_{i\in I})$ in $\mathbf{A}$ having the same domain as $\mathcal{P}$ there exists a unique morphism $f \in \hom_{\mathbf{A}}(A, P)$ with $f_i = p_i \circ f$ for each $i \in I$. A product $\mathcal{P}$ with codomain $(A_i)_{i\in I}$ is also called a *product of the family* $(A_i)_{i\in I}$. A product $\mathcal{P} = (A, (p_i)_{i\in I})$ is a *finite product* if I is a finite set. Similarly, products of morphisms can be defined relying on products of objects as defined above. Indeed, let $(f_i)_{i\in I}$ a family of $\mathbf{A}$-morphisms with $f_i \in \hom_{\mathbf{A}}(A_i, B_i)$ for any $i \in I$, and $(\prod_{i\in I} A_i, (\pi_i)_{i\in I}), (\prod_{i\in I} B_i, (p_i)_{i\in I})$ products of the families of objects $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$, respectively. Then, the *product of the family* $(f_i)_{i\in I}$ of $\mathbf{A}$-*morphisms* is the unique morphism $\prod_{i\in I} f_i \in \hom_{\mathbf{A}}(\prod_{i\in I} A_i, \prod_{i\in I} B_i)$ such that $(\prod_{i\in I} f_i) \circ p_j = f_j \circ \pi_j$ for each $j \in I$.

A category $\mathbf{A}$ *has (finite) products* if for every (finite) set-indexed family $(A_i)_{i\in I}$ with $A_i \in Ob(\mathbf{A})$ for each $i \in I$ there exists in $\mathbf{A}$ a product $\mathcal{P}$ -also denoted $(\prod_i A_i, (p_i)_{i\in I})$- with codomain $(A_i)_{i\in I}$.

A *functor* from category $\mathbf{A}$ to category $\mathbf{B}$ -also denoted $F : \mathbf{A} \to \mathbf{B}$- is a function

$F : Ob(\mathbf{A}) \cup Mor(\mathbf{A}) \to Ob(\mathbf{B}) \cup Mor(\mathbf{B})$

such that for any $A, B, C, D \in Ob(\mathbf{A})$ and any $f \in \hom_{\mathbf{A}}(A, B), g \in \hom_{\mathbf{A}}(C, D)$

F1)$F(A) \in Ob(\mathbf{B})$, F2)$F(f) \in \hom_{\mathbf{B}}(F(A), F(B))$, F3)$F(g \circ f) = F(g) \circ F(f)$ whenever $g \circ f$ is defined, i.e. whenever $B = C$, F4) $F(id_A) = id_{F(A)}$.

Let $\mathbf{A}, \mathbf{B}$ be categories. A functor $F : \mathbf{A} \to \mathbf{B}$ is an *embedding* if $F_{\upharpoonright Mor(\mathbf{A})} : Mor(\mathbf{A}) \to Mor(\mathbf{B})$ is an injective function, and is *faithful* if for any $A, A' \in Ob(\mathbf{A})$, $F_{\upharpoonright \hom_{\mathbf{A}}(A,A')} : \hom_{\mathbf{A}}(A, A') \to \hom_{\mathbf{B}}(F(A), F(A'))$ is an injective function.

Hence, F is an embedding if and only if F is faithful and $F_{\upharpoonright Ob(\mathbf{A})} : Ob(\mathbf{A}) \to Ob(\mathbf{B})$ is an injective function. Moreover, F is *full* if for any $A, A' \in Ob(\mathbf{A})$, $F_{\upharpoonright \hom_{\mathbf{A}}(A,A')} : \hom_{\mathbf{A}}(A, A') \to \hom_{\mathbf{B}}(F(A), F(A'))$ is a surjective function, and an *isomorphism* if it is *faithful, full* and $F_{\upharpoonright Ob(\mathbf{A})} : Ob(\mathbf{A}) \to Ob(\mathbf{B})$ is a *bijective* function. Two categories $\mathbf{A}, \mathbf{B}$ are *isomorphic* if there exists a functor $F : \mathbf{A} \to \mathbf{B}$ that is an isomorphism.

A category $\mathbf{B}$ is a *subcategory* of a category $\mathbf{A}$ if the following conditions are satisfied: a) $Ob(\mathbf{B}) \subseteq Ob(\mathbf{A})$;b) for any $B, B' \in Ob(\mathbf{B})$, $\hom_{\mathbf{B}}(B, B') \subseteq \hom_{\mathbf{A}}(B, B')$;c) for any $B \in Ob(\mathbf{B}), id_B(\mathbf{A}) = id_B(\mathbf{B})$; d) $\circ_{\mathbf{B}} = \circ_{\mathbf{A}\upharpoonright Mor(\mathbf{B})}$. Moreover, $\mathbf{B}$ is a *full subcategory* of $\mathbf{A}$ if it is a subcategory of $\mathbf{A}$ and $\hom_{\mathbf{B}}(B, B') = \hom_{\mathbf{A}}(B, B')$ for all $B, B' \in Ob(\mathbf{B})$.

If $\mathbf{B}$ is a subcategory of $\mathbf{A}$ then an *inclusion functor* $I : \mathbf{B} \to \mathbf{A}$ can be defined in a natural way by positing $I(B) = B$ and $I(f) = f$ for any $B \in Ob(\mathbf{B})$ and any $f \in Mor(\mathbf{B})$.Clearly enough, an inclusion functor is an *embedding.* Moreover, the inclusion functor $I : \mathbf{B} \to \mathbf{A}$ is a *full embedding* if and only if $\mathbf{B}$ is a full subcategory of $\mathbf{A}$: then, $\mathbf{B}$ is also said to be *fully embeddable* into $\mathbf{A}$ i.e. equivalently $\mathbf{B}$ is isomorphic to a subcategory of $\mathbf{A}$.

If $\mathbf{A}$ is a category, a *concrete category* over $\mathbf{A}$ -the *base* category- is a pair $(\mathbf{B}, U)$ where $\mathbf{B}$ is a category and $U : \mathbf{B} \to \mathbf{A}$ is a faithful functor, the so-called *underlying* -or *forgetful*- functor.

3 Game formats

As mentioned above, *game forms* and *games* are to be distinguished: game forms represent 'pure' interaction structures with no reference whatsoever to the players' preferences (or revealed preferences) on the outcomes, while games include information about the latter. Both game forms and games as currently used in the game-theoretic literature come in a large variety of formats. We shall focus on the following basic variants:

i) *extensive formats*: actions or moves available to players and coalition of players, and admissible *sequences* of moves are explicitly described to the effect of providing a *dynamic* structure of sorts;

ii) *strategic formats*: action plans available to players and coalitions of players as well as outcomes resulting from admissible profiles of action plans are described;

iii) *coalitional formats*: the set of outcome-subsets any player or coalition of players is able to enforce is described while the relevant actions or action plans are simply ignored.

Let us then consider the foregoing formats in more detail.

3.1 Extensive formats

In what follows, I shall implicitly focus on game forms and games *without chance moves*. As a matter of fact, their counterparts *with chance moves* could be easily accommodated within the ensuing analytical framework at the sole cost of some notational complexification: therefore, they are simply disregarded here with no essential loss of generality.

To begin with, let us introduce the following:

Definition 1 *A* multi-extensive game form with labelled moves, exits and almost perfect information (megf$_{\text{lea}}$)*is a tuple*

$\Gamma = (\ N, X, (A_i)_{i\in N}, P, \sqsubseteq, e, \xi, h)$

where

N is a set, that denotes the set of players

X is a set, that denotes the outcome *set*

$(A_i)_{i\in N}$ *is a family -or profile- of sets, where* A_i *denotes the set of* labels *of possible* moves *of player* i, *for any* $i \in N$

$P \subseteq A^{\leq\omega}$ *is a set of sequences on A such that for each* $i \in N$ *and each* $a_i^* \in A_i$ *there exist* $p \in P$,$k \in \omega$, $S \subseteq N$ *with* $p_k = (a_j)_{j\in S}$, $i \in S, a_i = a_i^*$ *(where* $A = \bigcup_{S\subseteq N}\prod_{i\in S} A_i$, A^* *denotes the set of all* lists *-or finite sequences- on A,* A^ω *denotes the set of all* streams *-or infinite sequences- on A, and* $A^{\leq\omega} = A^* \cup A^\omega$*):P denotes the set of all admissible* positions *or* move-paths

$\sqsubseteq = \sqsubseteq'_{\restriction P}$*where* $\sqsubseteq'$*is the* prefix partial order *on* $A^{\leq\omega}$ *(i.e. for any* $\pi, \pi' \in A^{\leq\omega}$, $\pi \sqsubseteq \pi'$ *iff there exists* $\alpha \in A^{\leq\omega}$ *such that* $\pi' = \pi * \alpha$ *where '$*$' denotes concatenation)*

$e = \emptyset$ *is the empty sequence on A, that denotes the* initial position

$\xi : N \rightharpoonup P\backslash((P\cap A^\omega)\cup\max_{\sqsubseteq} P)$ *is a partial function that maps players into non-terminal positions i.e. list-positions which are not* $\sqsubseteq$*-maximal, and denotes the* exit function

$h : P \rightharpoonup X$ *is a surjective partial function with domain* $dom(h) = ((P \cap A^\omega)\cup\max_{\sqsubseteq} P\ )\cup\xi(N)$ *that maps terminal positions* $P^T = ((P\cap A^\omega)\cup\max_{\sqsubseteq} P$ *i.e. stream-positions and maximal positions of* $(P, \sqsubseteq)$ *and the* exit-positions $\xi(N)$ *onto X, in such a way that* $h(P^T)\cap h(\xi(N)) = \emptyset$,*and denotes the* outcome function,

and the following 'connectedness' condition holds

(*) *P is an* order ideal *of poset* $(A^{\leq\omega}, \sqsubseteq)$ *i.e. for any* $\pi, \pi' \in A^{\leq\omega}$, $\pi \in P$ *and* $\pi' \sqsubseteq \pi$ *entail* $\pi' \in P$, *hence in particular* $e \in P$, *where e -the* initial position *of* Γ*- denotes the* empty sequence on *A, i.e. the minimum or bottom element of* $(A^{\leq\omega}, \sqsubseteq)$.

The salient features of a *megf*$_{lea}$ are i) *multiplicity* i.e. different players or coalitions may be enabled to move at the same position, ii) *move-labelling* i.e. the available moves of each player are explicitly labelled, and to a lesser extent iii) possible *early exits* as represented by the exit function and iv) *almost* perfect information i.e. players observe and recall previous moves but simultaneous -possibly agreed upon- moves are also admissible. Indeed, multiplicity can be significant and helpful when some notion of *parallel composition* of different game forms or games is to be considered. Move-labelling licences a distinction between *history-independent and history-dependent* strategies. The possibility of *early individual exits* accommodates -inter alia- *overlapping generations* modelling. By contrast, almost perfect information is to be regarded essentially as a modelling convenience.

Generally speaking, $megf_{lea}$s have been rarely studied or indeed evoked in the extant game-theoretic literature (but see Abramsky and Jagadeesan (1994) that provides a short discussion of 2-person strictly competitive games in multi-extensive form with labelled moves and perfect information; in that connection, see also Hyland (1997)). Little more study has been devoted to *unlabelled-move* counterparts of $megf_{la}$s, as described by the following

Definition 2 *A* multi-extensive game form with unlabelled moves,exits and almost perfect information ($\mathrm{megf_{uea}}$) *is a tuple*

$\Gamma = (N, X, P, \leq, p_0, \lambda, \xi, h)$

where

N *is a set, that denotes the set of* players

X *is a set, that denotes the* outcome *set*

P *is a set, that denotes the set of* positions

$\leq$ *is an order-i.e. an antisymmetric reflexive and transitive binary relation-on* P *with a minimum or bottom element, and denotes the* predecessor-relationship *between positions*

p_0 *is the minimum of poset* $(P, \leq)$*, that denotes the* initial position

$\lambda : P \backslash P^T \twoheadrightarrow\!\!\!\rightarrow \wp(N)$ *where*$P^T = \{p \in P : \#\{q \in P : q \leq p\} \geq \omega\} \bigcup \max_{\leq} P$ *is a correspondence that maps* non-terminal positions(*i.e.*non-maximal positions having a finite number of $\leq$*-predecessors) of* $(P, \leq)$ *into coalitions of players, and denotes the* move-assignment rule *that specifies -for each non-terminal position- those coalitions which are allowed to move*

$\xi : N \rightharpoondown P \setminus P^T$ *is a partial function that maps players into non-terminal positions and denotes the* exit function

$h : P \rightharpoondown X$ *is a surjective partial function with* $dom(P) = P^T \cup \xi(N)$ *that maps the set* P^T *of terminal positions of* $(P, \leq)$ *-i.e. positions having infinite* $\leq$*-predecessors or* $\leq$*-maximal positions- and the* exit-positions $\xi(N)$ *onto* X *, in such a way that* $h(P^T) \cap h(\xi(N)) = \emptyset$*, and denotes the* outcome function.

Under $megf_{uea}$ the possibility to distinguish in a natural way between history-dependent and history-independent strategies is lost. However, $megf_{uea}$ allow intersections between previously diverging sequences of positions. Hence, they accommodate as a special case so-called *cyclic* extensive game forms (see e.g. Selten,Wooders(2001)).

By contrast, the typical examples of extensive formats one meets in the extant literature are *tree-like* i.e. two diverging paths are not allowed to meet again afterwards, namely the relevant *position poset* $(P, \leq)$ is required to satisfy the following

(Tree property for a poset $(Q, \leqq)$) For any $p \in Q$ the p-generated principal order ideal $p_{\leqq} \downarrow = \{q \in P : q \leq p\}$ is a *chain* i.e. $q \leq q'$ or $q' \leq q$ for any $q, q' \in p_{\leqq} \downarrow$.

Now, it is immediately checked that while *megf* $_{uea}$ need not be tree-like, *megf* $_{lea}$ do automatically satisfy the tree property. Therefore, one may define *tree-like megf* $_{uea}$s -denoted *megf* $_{tuea}$s- by requiring that $(P, \leq)$ satisfy the tree property.

The relationship between game forms of type *megf* $_{lea}$ and *megf* $_{tuea}$ as mere objects -i.e. when no non-trivial notion of 'structure-preserving map' is singled out- is best expressed in terms of the relationships between the corresponding *discrete categories*, namely

Proposition 1 *Let* $Megf_{lea}$ *and* $Megf_{tuea}$ *denote the classes of all* megf$_{\text{lea}}$ *and all* megf$_{\text{tuea}}$ *as defined above, respectively. Then* $\mathcal{C}$ $(Megf_{uea})$ *is (isomorphic to) a* full subcategory *of* $\mathcal{C}(Megf_{lea})$.

Proof. Let $\Gamma = (N, X, P, \leq, p_0, \lambda, \xi, h)$ be a *megf* $_{tuea}$. Then, posit $A_i(\Gamma) = \{(p, q) : p \in \lambda^{-1}\{i\}$ and $q \in UC_{(P,\leq)}(p)\}$ (where $UC_{(P,\leq)}(.)$denotes the *upper cover correspondence* of $(P, \leq)$ i.e. for any $u \in P, UC_{(P,\leq)}(u) = \{q \in P : u < q$ and $u \leq r < q$ implies $u = r\})$ for each $i \in N$, $A(\Gamma) = \bigcup_i A_i(\Gamma)$ and $P(\Gamma) = \{\pi \in A(\Gamma)^{\leq\omega}$:there exists $q \in UC_{(P,\leq)}(p_0)$ such that $\pi_1 = (p_0, q)$ and for all k if $\pi_k = (p_{k-1}, p_k)$ then $p_k \in UC_{(P,\leq)}(p_{k-1})$.Therefore, it is immediately checked that the resulting tuple $\Gamma^* = (N, X, (A_i(\Gamma))_{i\in N}, P(\Gamma), \sqsubseteq (\Gamma), p_0, \xi, h)$ is indeed a *megf* $_{lea}$. Finally, observe that a *megf* $_{lea}$ Γ is implicitly endowed with a well-defined *move-assignment rule* λ_Γ (namely, for any $p \in P \setminus \max_{\sqsubseteq} P$, $\lambda_\Gamma(p) = \{S \subseteq N$:there exists $a = (a_i)_{i\in S} \in \prod_{i\in S} A_i$ such that $p * a \in P\})$, and that by definition $\lambda_{\Gamma^*} = \lambda$. ■

Extensive game forms with labelled or unlabelled moves and almost perfect information obtain from their multi-extensive counterparts by allowing *only one coalition to move* at any non-terminal position. Thus, we have the following

Definition 3 *An* extensive game form with labelled moves, exits, and almost perfect information (egf$_{\text{lea}}$) *is a* megf$_{\text{lea}}$ $\Gamma = (N, X, (A_i)_{i\in N}, P, \sqsubseteq, e, \xi, h)$ *such that*

(***) *for any* $\pi \in P$ *there exists* $S \subseteq N$ *such that for all* $a \in A$*, if* $\pi * a \in P$ *then* $a \in \prod_{i\in S} A_i$.

And similarly,

Definition 4 *An* extensive game form with unlabelled moves, exits, and almost perfect information (egf$_{\text{uea}}$) *is a* megf$_{\text{uea}}$ $\Gamma = (N, X, P, \leq, p_0, \lambda, \xi, h)$ *such that* λ *is a single-valued correspondence.*

The following proposition is immediately established

Proposition 2 *Let* Egf_{lea} *denote the class of all* egf$_{\text{lea}}$*game forms and* Egf_{uea} *the class of all* egf$_{\text{uea}}$ *game forms. Then* $\mathcal{C}(Egf_{lea})$ *is a full subcategory of* $\mathcal{C}$ $(Megf_{lea})$ *and* $\mathcal{C}(Egf_{uea})$ *is a full subcategory of* $\mathcal{C}(Megf_{uea})$.

Again, one may define *egf* $_{tuea}$s *on a tree* by requiring that $(P, \leq)$ satisfy the tree property.

Moreover, for each of the foregoing extensive formats both a *perfect information* and an *imperfect information* counterpart may be defined. *Perfect information* extensive formats obtain from their corresponding almost perfect information formats by requiring that at any position only *single* players be allowed to move.

For instance, we have the following

Definition 5 *A* multi-extensive game form with labelled moves, exits, and perfect information ($\mathrm{megf}_{\mathrm{lep}}$) *is a* $\mathrm{megf}_{\mathrm{lea}}$ $\Gamma = (N, X, (A_i)_{i \in N}, P, \sqsubseteq, e, \xi, h)$ *such that*

(***) *for any* $\pi \in P$ *and* $a \in A$*, if* $\pi * a \in P$ *then* $a \in \bigcup_{i \in N} A_i$.

and

Definition 6 *An* extensive game form on a tree with unlabelled moves, exits, and perfect information ($\mathrm{egf}_{\mathrm{tuep}}$) *is an* $\mathrm{egf}_{\mathrm{tuea}}$ $\Gamma = (N, X, P, \leq, p_0, \lambda, \xi, h)$ *such that* $\lambda(P \setminus P^T) \subseteq \{\{i\} : i \in N\}$.

Again, denoting $Megf_{lep}$ and $Megf_{tuep}$ the class of all *megf* $_{lep}$ and the class of all *megf* $_{tuep}$, respectively, we have the following

Proposition 3 *$\mathcal{C}$ $(Megf_{lep})$ is a full subcategory of $\mathcal{C}(Megf_{lea})$, and $\mathcal{C}(Egf_{tuep})$ is a full subcategory of $\mathcal{C}(Egf_{uea})$ (hence, modulo isomorphisms, a full subcategory of $\mathcal{C}(Megf_{lea})$ as well).*

Tokens of other extensive game formats such as *megf* $_{tup}$, *megf* $_{tlp}$, *egf* $_{lp}$, *egf* $_{up}$ are defined similarly, and the corresponding discrete categories are -modulo isomorphisms- full subcategories of $\mathcal{C}(Megf_{lea})$.

Remark 1 *The game forms introduced by Conway in his classic book (see Conway (2001)) are 2-player* megf_{tuep}*s without exits i.e. with* $\xi = \emptyset$*. They are however not characteristic of mainstream game-theoretic literature, where game forms of type* egf_{tup} *and* egf_{lp} *without exits are by far the most widely employed: see Osborne and Rubinstein(1994) where a further closure condition is imposed on the position set of an* egf_{lp} $\Gamma = (N, X, (A_i)_{i \in N}, P, \sqsubseteq, e, h)$ *to the effect that a stream whose (finite) prefixes are all admissible positions should also be an admissible position, namely*

($\cdot$) for any $p \in A^\omega$*, if* $q \in P$ *for all* $q \in A^*$ *such that* $q \sqsubseteq p$ *then* $p \in P$.

Extensive formats with *imperfect information* are meant to represent inability on the part of some players to observe or recollect previous moves. This is typically done by defining for each player a *partition* of those positions where the former is allowed to move, perhaps as a member of a certain coalition: any cell of the partition collects a maximal set of positions that the relevant player is unable to distinguish from each other (but see Thompson(1952) for a slightly different presentation of extensive game forms with imperfect information, where players are simply identified with their sets of moves). However, *non-partitional information structures* have also been occasionally considered in the extant literature: in such a case, *coverings* with possibly intersecting cells -rather than partitions-of the set of positions attached to any player are usually specified.

For each of the extensive formats presented above both an *imperfect information* (if no simultaneous moves are allowed) and a *strictly imperfect information* (if simultaneous moves are allowed) counterparts may be defined. As an example we shall consider the following

Definition 7 *A* multi-extensive game form with labelled moves, exits, and general strictly imperfect information (megf$_{\text{lesi}}$)*is a tuple*
$\Gamma^* = (N, X, (A_i)_{i\in N}, P, \sqsubseteq, e, \xi, h, (\Im_i)_{i\in N})$
where $\Gamma \equiv (N, X, (A_i)_{i\in N}, P, \sqsubseteq, e, \xi, h)$ *is a* megf$_{\text{lea}}$ *and*
for any $i \in N$, $\Im_i = \{\Im_{i_1}, .., \Im_{i_k}\}$ *is a* covering *of* $\lambda_\Gamma^{-1}(\{i\})$ *such that*
for each $\Im_{i_h}$, $h = 1, .., k$ *and all* $p, p' \in \Im_{i_h}$
$\{a \in A_i : (p * a) \in P\} = \{a \in A_i : (p * a') \in P\}$

and its more common extensive counterpart with unlabelled moves no exits and partitional imperfect information, namely

Definition 8 *An* extensive game form with unlabelled moves and partitional imperfect information (megf$_{\text{upi}}$) *is a tuple*
$\Gamma^* = (N, X, P, \leq, p_0, \lambda, h, (\Im_i)_{i\in N})$
such that $\Gamma \equiv (N, X, P, \leq, p_0, \lambda, h)$ *is an* egf$_{ua}$ *and*
for any $i \in N$, $\Im_i = \{\Im_{i_1}, .., \Im_{i_{k_i}}\}$ *is a* partition *of* $\lambda^{-1}(\{i\})$ *such that*
for each $\Im_{i_h}$, $h = 1, .., k_i$ *and all* $p, p' \in \Im_{i_h}$
$\{q \in P : q \in UC_{(P,\leq)}(p)\} = \{q \in P : q \in UC_{(P,\leq)}(p')\}\}$
where $UC_{(P,\leq)}(.)$ *denotes the* upper cover correspondence *of* $(P, \leq)$ *i.e. for any* $p \in P$,
$UC_{(P,\leq)}(p) = \{q \in P : p < q$ *and* $p \leq r < q$ *implies* $p = r\}$.

It transpires that discrete categories of imperfect information game forms may be regarded as concrete categories over discrete categories of almost perfect or perfect information game forms. In particular, the following holds:

Proposition 4 *Let* $\mathcal{C}(Meg_{\text{lesi}})$ *($\mathcal{C}(Eg)$, respectively) denote the class of all* megf$_{\text{lesi}}$ *(of all* eg$_{\text{lesi}}$*, respectively). Then i)* $\mathcal{C}(Megf_{lesi})$ *is a* concrete *category over* $\mathcal{C}(Megf_{lea})$*.Similarly, ii)*$\mathcal{C}(Egf_{lesi})$ *is a* concrete *category over* $\mathcal{C}(Egf_{lea})$.

Proof. Straightforward: the underlying functor of $\mathcal{C}(Megf_{lesi})$ is the function that sends each *megf$_{lesi}$* $\Gamma^* = (N, X, P, \leq, p_0, \lambda, h, (\Im_i)_{i\in N})$ to $\Gamma \equiv (N, X, P, \leq, p_0, \lambda, h)$ i.e. the functor that forgets the information structure $(\Im_i)_{i\in N}$. Such a functor is trivially faithful because for any $\Gamma_1, \Gamma_2 \in Megf_{lesi}$,
$\#\hom_{\mathcal{C}(Megf_{lesi})}(\Gamma_1, \Gamma_2) \leq 1$. As for $\mathcal{C}(Egf_{lesi})$ just consider the restriction of the previous function to Egf_{lesi}. ■

As mentioned above, *games* obtain from game forms by supplementing the latter with *behavioural* information concerning the evaluation of outcomes on the part of the players. More often than not, the relevant behavioural evaluation-structure amounts to a profile of binary preference relations that are taken to be (at least) *preorders*. However, more general formats for outcome-evaluations may be- and are indeed sometimes- considered, including (profiles of) *general choice functions* on the outcome set, namely *deflationary* operators on $\wp(X)$ i.e. functions $C : \wp(X) \to \wp(X)$ such that $C(A) \subseteq A$ for any $A \subseteq X$.

Again, for each type of game form as described above a corresponding class of games obtains. We provide just a pair of examples, namely the most natural common generalization of the several extensive formats for games which occasionally employed in the literature:

Definition 9 *A* multi-extensive game with labelled moves, exits, general strictly imperfect information and choice evaluation structure ($\text{meg}_{\text{lesic}}$) *is a tuple*
$\Gamma^* = (\ N, X, (A_i)_{i\in N}, P, \sqsubseteq, e, \xi, h, (\Im_i)_{i\in N}, (C_i)_{i\in N})$
where $\Gamma \equiv (\ N, X, (A_i)_{i\in N}, P, \sqsubseteq, e, \xi, h, (\Im_i)_{i\in N})$ *is a* megf_{lesi} *and*
$C_i : \wp(X) \to \wp(X)$ *denotes the* choice function *of player* i, $i = 1, .., n$

and a version of the most widely used format

Definition 10 *An* extensive game with unlabelled moves, partitional imperfect information and standard preference evaluation structure (eg_{upisp}) *is a tuple*
$\Gamma^* = (N, X, P, \leq, p_0, \lambda, h, (\Im_i)_{i\in N}, (\succcurlyeq_i)_{i\in N})$
such that $\Gamma \equiv (\ N, X, P, \leq, p_0, \lambda, h)$ *is an* egf_{up},
for any $i \in N$, $\Im_i = \{\Im_{i_1}, .., \Im_{i_{k_i}}\}$ *is a* partition *of* $\lambda^{-1}(\{i\})$ *such that*
for each $\Im_{i_h}$, $h = 1, .., k_i$ *and all* $p, p' \in \Im_{i_h}$
$\{q \in P : q \in UC_{(P,\leq)}(p)\} = \{q \in P : q \in UC_{(P,\leq)}(p')\}\}$
where $UC_{(P,\leq)}(.)$ *denotes the* upper cover correspondence *of* $(P, \leq)$ *as defined*
above, and
$\succcurlyeq_i \subseteq X \times X$ *is a preorder, that denotes the preference relation of player* i,
$i = 1, .., n$

It is easily checked that discrete categories of multi-extensive games and extensive games as defined above are also concrete categories over $\mathcal{C}(Megf_{lea})$ or some of its subcategories. In particular, we have the following

Proposition 5 *Let* Meg_{lesic} *(*Eg_{lesic}*, respectively) denote the class of all* $\text{meg}_{\text{lesic}}$($\text{egf}_{lesic}$*, respectively*). *Then, i)* $\mathcal{C}(Meg_{\text{lesic}})$ *is a* concrete *category over* $\mathcal{C}(Megf_{lea})$*.Similarly, ii)*$\mathcal{C}(Eg_{\text{lesic}})$ *is a* concrete *category over* $\mathcal{C}(Egf_{lea})$.

Proof. Take as an underlying functor the function that sends each meg_{lesic} $\Gamma^* = (N, X, (A_i)_{i\in N}, P, \sqsubseteq, e, \xi, h, (\Im_i)_{i\in N}, (C_i)_{i\in N})$ to $\Gamma = (\ N, X, (A_i)_{i\in N}, P, \sqsubseteq, e, \xi, h)$ (and its restriction to Eg_{lesic}, respectively). ■

Since similar results hold for each of the relevant extensive formats considered above -including in particular those one obtains by taking $\mathcal{C}(Megf_{uea})$ and

$\mathcal{C}(Egf_{uea})$ as basic categories, one may conclude that to the extent that no non-trivial notion of morphism is considered *virtually all extensive game formats can be regarded either as full subcategories of* $\mathcal{C}(Megf_{lea})$ (or $\mathcal{C}(Megf_{uea})$) *or as concrete categories over a suitable full subcategory of* $\mathcal{C}(Megf_{lea})$ (or $\mathcal{C}(Megf_{uea})$). Moreover, if multi-extensive game forms are ignored, a similar result obtains for $\mathcal{C}(Egf_{lea})$ and $\mathcal{C}(Egf_{uea})$.

3.2 Strategic formats

In strategic formats the sequential structure of moves is ignored. Indeed, we have the following

Definition 11 *A* strategic game form (sgf) *is a tuple*
$G = (N, X, (S_i)_{i\in N}, h)$ *where*
N *is a set, that denotes the* player *set*
X *is a set, that denotes the* outcome *set*
$(S_i)_{i\in N}$ *is a family -or profile- of sets, where* S_i *denotes the* strategy set *of player* i
$h : \prod_{i\in N} S_i \rightarrow X$ *is a surjective function, that denotes the* outcome function

Remark 2 *An alternative 'contracted' presentation of a* sgf G *is the following:*
$G = (N, (S_i)_{i\in N})$
where N *and* $(S_i)_{i\in N}$ *are as in the previous definition.*
Such game forms may be regarded as a special case of sgf*s as defined above, with*
$X = \prod_{i\in N} S_i$ *and* $h = id : \prod_{i\in N} S_i \rightarrow \prod_{i\in N} S_i$.

In a strategic game form *any* possible combination of strategies is admissible. However, in some notable applications including some classic general equilibrium existence theorems the feasibility of a certain strategy is taken to be dependent on the strategies chosen by other players. This sort of situation can be represented by means of *generalized strategic game forms*, namely

Definition 12 *A* generalized strategic game form (gsgf) *is a tuple*
$G = (N, X, (S_i)_{i\in N}, (F_i)_{i\in N}, h)$
where $N, X, (S_i)_{i\in S}$ *denote the player set, the outcome set, and the strategy sets, respectively, as defined above for* sgfs*, while*
$F_i : \prod_{i\in N} S_i \rightarrow\!\!\rightarrow S_i$ *is a correspondence that denotes the* feasibility correspondence *of player* i*, for each* $i \in N$*, and*
$h : \prod_{i\in N} S_i \rightharpoondown X$ *is a surjective partial function which is defined on* $s \in \prod_{i\in N} S_i$ *if and only if* $s \in \prod_{i\in N} F_i(s)$*, and denotes the* outcome function.

It is easily checked that the following proposition holds:

Proposition 6 *Let* $Gsgf$ *denote the class of all* gsgf *and* Sgf *the class of all* sgf*. Then,* $\mathcal{C}(Sgf)$ *is (isomorphic to) a full subcategory of* $\mathcal{C}(Gsgf)$.

Proof. For all $G = (N, X, (S_i)_{i\in N}, h)$ posit $F(G) = (N, X, (S_i)_{i\in N}, (F_i^G)_{i\in N}, h)$ where for any $s \in \prod_{i\in N} S_i$ and any $i \in N$, $F_i^G(s) = S_i$ and $F(id_G) = id_{F(G)}$. ■

Of course, *strategic games* and *generalized strategic games* result from *sgf*s and *gsgf*s, respectively, by supplementing them with a suitable outcome evaluation structure i.e. either a profile of choice functions or a profile of preference relations on X. As an example, consider the following definition

Definition 13 *A* strategic game with standard preference evaluation structure (sg$_{sp}$) *is a tuple*
$G = (N, X, (S_i)_{i\in N}, h, (\succcurlyeq_i)_{i\in N})$
where $(N, X, (S_i)_{i\in N}, h)$ *is a* sgf *and* $(\succcurlyeq_i)_{i\in N}$ *is a profile of preorders on* X.

Strategic games with choice evaluation structure (sg_cs), and generalized strategic games with choice evaluation structure or with standard preference evaluation structure (gsg_{sp}s and gsg_cs) are defined in a similar way. All of them may be construed as concrete categories over $\mathcal{C}(Gsgf)$.

Remark 3 *Occasionally,* nondeterministic strategic game forms -and games- *have also been considered (see e.g. Otten, Borm, Storcken and Tijs (1995)). Indeed, a* nondeterministic strategic game form *amounts to a tuple*
$G = (N, X, (S_i)_{i\in N}, \hat{h}\ )$ *with* $N, X, (S_i)_{i\in N}$ *defined as for a* sgf *and*
$\hat{h} : \prod_{i\in N} S_i \to \wp(X)$.
Nondeterministic generalized strategic game forms *may then also be defined in the obvious way. As for* nondeterministic (generalized) strategic games *they require that the relevant outcome-evaluation structure be defined on* $\wp(X)$ *rather than on the outcome set* X.

In particular the following proposition holds

Proposition 7 *Let* Sg_{sp},Sg_c, Gsg_{sp}, Gsg_c *denote the class of all* sg$_{sp}$,*the class of all* sg$_c$,*the class of all* gsg$_{sp}$, *and the class of all* gsg$_c$, *respectively. Then, i)* Sg_{sp} *and* Sg_c *are* concrete *categories over* $\mathcal{C}(Sgf)$; *ii)* Gsg_{sp} *and* Gsg_c *are* concrete *categories over* $\mathcal{C}(Gsgf)$.

Proof. Straightforward: in each case, take as an underlying functor the functor that forgets the relevant evaluation structure. ■

Moreover, extensive game forms and games may be regarded as discrete subcategories of strategic game forms and games. In particular,

Proposition 8 $\mathcal{C}(Egf_{lea})$ *is (isomorphic to) a full subcategory of* $\mathcal{C}(Sgf)$.

Proof. Let $\Gamma = (N, X, (A_i)_{i\in N}, P, \sqsubseteq, e, \xi, h) \in Egf_{lea}$. Then posit $F(\Gamma) = (N, X, (S_i(\Gamma))_{i\in N}, h_\Gamma)$ where for any $i \in N$,

$$S_i(\Gamma) = \left\{ \begin{array}{c} \sigma_i : \ \sigma_i \ \in A^{\lambda^{-1}(\{i\})} \text{ is a function such that} \\ p * \sigma_i(p) \in P \text{ for each } p \in \lambda^{-1}(\{i\}) \end{array} \right\}, \ h_\Gamma(\sigma) = h(p(\sigma))$$

and $p(\sigma) \in P^T$ denotes the terminal position induced by strategy profile $\sigma = (\sigma_1, .., \sigma_n)$, for each $\sigma \in \prod_{i \in N} S_i(\Gamma)$. ■

The relationship between multi-extensive formats and strategic formats is more complex since a multi-extensive game form may generate indeed several strategic game forms. Thus, we shall not pursue it any further in the present paper.

3.3 Coalitional formats

Under a coalitional format actions or strategies are ignored, and the focus is on what outcome-subsets or events each coalition is able to enforce.

Let us consider the following

Definition 14 *An* effectivity-coalitional game form (ecgf) *is a tuple*
$\mathcal{E} = (N, X, E)$ *where*
N *is a set, that denotes the* player *set*
X *is a set, that denotes the* outcome *set*
$E : \wp(N) \to \wp(\wp(X))$ *is a function, the* (generalized) effectivity function

Remark 4 *Generalized effectivity functions as mentioned above correspond to* effectivity functions *as defined e.g. in Gurvich (1992). In common usage, however, effectivity functions are usually required to obey a few restrictions (see e.g. Moulin,Peleg(1982), Abdou,Keiding(1991)). Namely, given a player set N and an outcome set X, an* effectivity function (EF) *is usually defined as a function $E : P(N) \to P(P(X))$ such that EF i) $E(\emptyset) = \emptyset$; EF ii) $\emptyset \notin E(S)$ for any $S \subseteq N$; EF iii) $X \in E(S)$ for any $S \subseteq N, S \neq \emptyset$; EF iv) $E(N) = P(X) \setminus \{\emptyset\}$.*

It should also be emphasized that (generalized) effectivity functions accommodate effectivity functions over a restricted domain $D \subseteq \wp(N)$ by positing $E(S) = \emptyset$ whenever $S \in \wp(N) \setminus D$.

Remark 5 *An EF E is said to satisfy:*

X-Monotonicity *if for any $A, B \subseteq X$ and any $S \subseteq N : A \in E(S)$ and $A \subseteq B$ imply $B \in E(S)$*

N-Monotonicity *if for any $A \subseteq X$,and any $S, T \subseteq N : A \in E(S)$ and $S \subseteq T$ imply $A \in E(T)$*

Superadditivity *if for any $A, B \subseteq X, S, T \subseteq N$, $A \in E(S), B \in E(T)$ and $S \cap T = \emptyset$ imply $A \cap B \in E(S \cup T)$*

Maximality *if for any $A \subseteq X, S \subseteq N$, $A \notin E(S)$ only if there exists $B \subseteq X$ such that $B \in E(N \backslash S)$ and $A \cap B = \emptyset$*

Moreover, an EF is said to be strategically playable *if it can be regarded as the $\alpha-$effectivity function (or the $\beta-$EF of a strategic game form :see e.g. Peleg(1984), Abdou,Keiding(1991)). This requires that the EF under consideration satisfy* X-Monotonicity, N-Monotonicity and Superadditivity *(or* Maximality, *respectively).*

Example 1 *A* conditional EF *can also be defined by considering a function* $E' : P(N) \rightarrow \prod_{x\in X}[\{x\} \times P(P(X))]$ *such that for any* $S \subseteq N$ *,* $E'(S) = \prod_{x\in X}\{x\} \times E_x(S)$ *where –for each* $x \in X$*–* E_x *defines an EF (see e.g. Rosenthal (1972) for an early proposal of a special case of that notion). Clearly enough,* (conditional) characteristic function coalitional game forms *may be regarded as corestrictions of (conditional) EFs to singleton-set-values. Moreover, it should be recalled here that* simple games *may be regarded as equivalence classes of* simple EFs *namely EFs with a set* $W \subseteq P(N)$ *such that for any* $S \in P(N)\setminus\{\emptyset\}$ *:* $E(S) = P(X)\setminus\{\emptyset\}$ *if* $S \in W$ *and* $E(S) = \{X\}$ *otherwise.*

Several weakenings and extensions of EFs have also been proposed in the literature under various labels: e.g. *well-behaved functions* satisfying EF i),EF ii), EFiii), *pseudo-EFs* or *semi-well-behaved functions(* (see e.g. Ichiishi(1989)) satisfying EF i) and EF ii), or even an intermediate notion of *weak EFs* satisfying EF i), EF ii) plus EF iii') $X \in E(S)$ for any $S \subseteq N$ such that $E(S) \neq \emptyset$ and EF iv') $E(N) \supseteq E(S)$ for any $S \subseteq N$.

Remark 6 *In the last few years some'new' coalitional game formats that in fact amount to families of certain EF-like coalitional game forms have been introduced in the literature. Thus, an* effectivity structure *(as defined in Abdou and Keiding(2002)) is a tuple*

$\mathcal{E}^*[.] = (N, X, (\mathbb{E}_x)_{x\in X})$ *where for any* $x \in X$

$\mathbb{E}_x \subseteq (\wp(X))^N$ *is a set of functions* $E_x : N \rightarrow \wp(X)$ *such that there exists* $i \in N$ *with* $E_x(i) \neq \emptyset$*, and* $\mathbb{E}_x$ *is an order filter of the poset* $((\wp(X))^N, \supseteq^*)$ *where* $\supseteq^*$ *denotes the component-wise set-inclusion partial order. Clearly enough, a canonical injection* f *mappimg the foregoing functions into EFs may be defined by the following rule. For each* $x \in X$*, and each* $E_x \in \mathbb{E}_x$ *:* $f(E_x)(S) = \{A \subseteq X : A \supseteq \cap_{i\in S}E_x(i)\}$ *for any* $S \subseteq N$*. But then, it is easily checked that an effectivity structure may be regarded as a family of (generalized) EFs. Similarly,* constitutional game forms *as presented in Andjiga and Moulen (1989), and* dynamic effectivity functions *as introduced in Pauly (2001) amount to certain families of EFs.*

A quite different coalitional game format is defined as follows

Definition 15 *A* social situation form (ssf) *is a tuple*

$\mathbb{S} = (N, X, \mathcal{I})$

where

N *is a set, that denotes the player set,*

X *is a set, that denotes the outcome set*

$\mathcal{I} \subseteq \mathbf{P}(N, X) \times \wp(N) \times X \times \mathbf{P}(N, X)$

-where $\mathbf{P}(N, X) = \wp(N) \times \wp(X)$ *denotes the set of* position-forms *on* (N, X)*- is a correspondence such that*

$\mathcal{I} \subseteq \{((S, A), U, x, (T, B)) : x \in A, U \subseteq S, U \subseteq T\}$*,and denotes the* inducement correspondence *(see e.g. Greenberg(1990)).*

It can be shown that effectivity-coalitional game forms amount to a special case of social situation forms, namely

Proposition 9 *Let $Ecgf$ denote the class of all effectivity-coalitional game forms and Ssf the class of all social situation forms. Then, $\mathcal{C}(Ecgf)$ is (isomorphic to) a full subcategory of $\mathcal{C}(Ssf)$.*

Proof. For each $\mathcal{E} = (N, X, E) \in Ecgf$, posit $F(\mathcal{E}) = (N, X, \mathcal{I}(\mathcal{E}))$ where $\mathcal{I}(\mathcal{E}) = \{((T, B), S, x, (U, A)) : T = N, B = X, x \in X, S = U, A \in E(S)\}$ ■

Again, *coalitional games* in several different formats obtain by supplementing the corresponding coalitional formats as defined above with a suitable *outcome-evaluation structure* i.e. with profiles of preference relations or choice functions on the outcome set. Thus, one may define

Definition 16 *An* effectivity-coalitional game with preference evaluation structure (ecg_p) (with choice evaluation structure (ecg_c), *respectively) is a tuple*
$G = (N, X, E, (\succcurlyeq_i)_{i \in N})$
where (N, X, E) *is an* ecgf
and $(\succcurlyeq_i)_{i \in N}$ *is a profile of preorders on* X
(a tuple $G = (N, X, E, (C_i)_{i \in N})$
where (N, X, E) *is an* ecgf
and $(C_i)_{i \in N}$, $C_i : \wp(X) \to \wp(X)$, $i = 1, .., n$
is a profile of choice functions on X, respectively).

Similarly, one has *social situations with preference evaluation structure* and *social situations with choice evaluation structure* as defined below

Definition 17 *A* social situation with preference evaluation structure (ss_p) *is a tuple*
$(N, X, \mathcal{I}, (\succcurlyeq_i)_{i \in N})$
where $(N, X, \mathcal{I})$ *is a* ssf *and*
and $(\succcurlyeq_i)_{i \in N}$ *is a profile of preorders on* X.
Similarly, a social situation with choice evaluation structure (ss_c) *is a tuple*
$(N, X, \mathcal{I}, (C_i)_{i \in N})$
where $(N, X, \mathcal{I})$ *is a* ssf *and*
and $(C_i)_{i \in N}$ *is a profile of choice functions on* X

Again, it is immediately checked that the classes of all effectivity-coalitional games and of all social situations may be construed as concrete categories over effectivity-coalitional game forms and social situations forms, namely

Proposition 10 *i) Let Ecg_p denote the class of all* ecg_p*, and Ecg_c the class of all* ecg_c*. Then both $\mathcal{C}(Ecg_p)$ and $\mathcal{C}(Ecg_c)$ are concrete categories over $\mathcal{C}(Ecgf)$. ii) Let Ss_p denote the class of all* ss_p*, and Ss_c the class of all* ss_c*. Then both $\mathcal{C}(Ss_p)$ and $\mathcal{C}(Ss_c)$ are concrete categories over $\mathcal{C}(Ssf)$.*

Moreover, it can be shown that virtually all coalitional formats that are currently employed -or have been indeed ever proposed- in the game-theoretic literature can be regarded as versions of the foregoing formats (see e.g. Vannucci(2001) for a discussion of this point, and more examples).

4 Chu spaces

Let $\mathbf{A}$ be a category with finite products, and $K \in Ob(\mathbf{A})$. Then a *(*$\mathbf{A}$*-enriched) Chu space* over K is a tuple $\mathcal{A} = (X, Y, \dashv)$ where $X, Y \in Ob(\mathbf{A})$ and $\dashv \in \hom_{\mathbf{A}}(X \times Y, K)$.

Let $\mathcal{A} = (X, Y, \dashv)$ and $\mathcal{A}' = (X', Y', \dashv')$ be $\mathbf{A}$-enriched Chu spaces over K. A *Chu-transform* from $\mathcal{A}$ to $\mathcal{A}'$ is a pair $\varphi = (f, g)$ of $\mathbf{A}$-morphisms $f \in \hom_{\mathbf{A}}(X, X'), g \in \hom_{\mathbf{A}}(Y', Y)$ such that $\dashv' \circ (f \times id_{Y'}) = \dashv \circ (id_X \times g)$.

The following proposition is easily established (see e.g. Barr and Wells (1995) for a more general version of that result and Pratt(1995,1999) for a general discussion)

Proposition 11 *Let* $\mathbf{A}$ *be a category with finite products, and* $K \in Ob(\mathbf{A})$. *Then, consider the tuple* $\mathbf{Chu}(\mathbf{A}, K) = (\mathcal{O}, \hom, id, \circ)$ *where* $\mathcal{O}$ *is the class of all* $\mathbf{A}$*-enriched Chu spaces,* $\hom(\mathcal{A}, \mathcal{A}')$ *is the set of all Chu transforms from* $\mathcal{A}$ *to* $\mathcal{A}'$ *for any* $\mathcal{A}, \mathcal{A}' \in \mathcal{O}$, $id_{\mathcal{A}} = (id_X^{\mathbf{A}}, id_Y^{\mathbf{A}})$ *for each* $\mathcal{A} = (X, Y, \dashv) \in \mathcal{O}$, *and* $\varphi' \circ \varphi = (f' \circ_{\mathbf{A}} f, g \circ_{\mathbf{A}} g')$ *for any pair of Chu transforms* $\varphi = (f, g) \in \hom(\mathcal{A}, \mathcal{A}')$ *and* $\varphi' = (f', g') \in \hom(\mathcal{A}', \mathcal{A}'')$. *Then* $\mathbf{Chu}(\mathbf{A}, K)$ *is a category.*

In particular, the following corollary holds

Corollary 12 *Let* **Poset** *be the category having the class of all ordered sets as objects and the class of order-homomorphisms as morphisms,* K *a set and* $\widehat{K} = (K, =)$. *Then* $\mathbf{Chu}(\mathbf{Poset}, \widehat{K})$ *is a category.*

Henceforth we shall indulge to a slight abuse of notation and write $\mathbf{Chu}(\mathbf{Poset}, K)$ for $\mathbf{Chu}(\mathbf{Poset}, \widehat{K})$ since no ambiguity may arise from that usage.

As mentioned above in the Introduction, it has been observed that the category $\mathbf{Chu}(\mathbf{Set}, 2)$ is isomorphic to a category of 2-player strictly competitive strategic games (see e.g. Lafont and Streicher(1991)). Indeed, $\mathbf{Chu}(\mathbf{Set}, 2)$ is also isomorphic to a category of 2-player 2-outcome strategic game forms. However, the objects of the foregoing categories only comprise a very specialized subclass of games or game forms. What, then, about other game forms and games? Are all of them representable in a natural way within a more general category than $\mathbf{Chu}(\mathbf{Set}, 2)$?

The rest of this section is devoted to showing that in fact: i) *all the discrete categories of game forms and of games introduced in Section 3 may be either embedded in* $\mathbf{Chu}(\mathbf{Poset}, 2)$ *or construed as concrete categories over some suitable subcategory of* $\mathbf{Chu}(\mathbf{Poset}, 2)$, and ii) *for a suitable specification of morphisms the resulting categories of game forms and of games are either full subcategories of* $\mathbf{Chu}(\mathbf{Poset}, 2)$ *or concrete categories over some suitable full subcategory of* $\mathbf{Chu}(\mathbf{Poset}, 2)$.

To begin with, let us consider the following embeddings of the relevant discrete categories of game forms defined above into $\mathbf{Chu}(\mathbf{Poset}, 2)$:

Proposition 13 *i)* $\mathcal{C}(Megf_{lea})$ *is (isomorphic to) a nonfull subcategory of* $\mathbf{Chu}(\mathbf{Poset}, 2)$;
ii) $\mathcal{C}(Gsgf)$ *is (isomorphic to) a nonfull subcategory of* $\mathbf{Chu}(\mathbf{Poset}, 2)$;
iii) $\mathcal{C}(Ssf)$ *is (isomorphic to) a nonfull subcategory of* $\mathbf{Chu}(\mathbf{Poset}, 2)$.

Proof. i) For any $\Gamma = (\ N, X, (A_i)_{i\in N}, P, \sqsubseteq, e, \xi, h) \in Megf_{lea}$ take $F(\Gamma) = ((\wp(N) \times gr(h), =), (P, \sqsubseteq), \dashv)$

where $gr(h) = \{(q, x) \in P \times X : x = h(q)\}$

and for any $S \in \wp(N), (q, x) \in gr(h), p \in P$:

$\dashv (S, (q, x), p) = 1$ if either there exist $p' \in P^T$, $a = (a_i)_{i\in S} \in \prod_{i\in S} A_i$ such that $p * a \in P$, $p * a \sqsubseteq p'$ and $h(p') = x$, or there exists $i \in S$ such that $p = \xi(i)$ and $h(p) = x$, and $\dashv ((S, x), p) = 0$ otherwise.

Moreover, posit $F(id_\Gamma) = id_{F(\Gamma)}$.

Clearly -by definition, and by idempotency of the identity function- F is a well-defined functor from $\mathcal{C}(Megf_{lea})$ to $\mathbf{Chu}(\mathbf{Poset}, 2)$.Also, F is trivially faithful since $\mathcal{C}(Megf_{lea})$ is a discrete category. Moreover, F is injective on objects: if $\Gamma, \Gamma' \in Megf_{lea}$ and $F(\Gamma) = F(\Gamma')$ then, by definition of F (and by definition of a $megf_{lea}$), $\Gamma \neq \Gamma'$ only if $\xi(\Gamma) \neq \xi(\Gamma')$.Let us then assume that $\xi(\Gamma) \neq \xi(\Gamma')$. Hence there exists $(i, p) \in N \times P$ such that either $[\xi(\Gamma)(i) = p \neq \xi(\Gamma')(i)]$ or $[\xi(\Gamma')(i) = p \neq \xi(\Gamma)(i)]$. Suppose w.l.o.g.that $\xi(\Gamma)(i) = p \neq \xi(\Gamma')(i)$ holds. Then $p \in dom(h(\Gamma))$.Now, take $x = h(\Gamma)(p)$: it follows that $\dashv_{F(\Gamma')} (\{i\}, (p, x), p) = \dashv_{F(\Gamma)} (\{i\}, (p, x), p) = 1$ whence there must exist $p' \in P^T$ and $a \in A_i$ such that $p * a \in P, p * a \sqsubseteq p'$ and $h(\Gamma')(p') = x$, a contradiction since $h(\Gamma) = h(\Gamma')$ and $h(\Gamma')(P^T) \cap h(\Gamma')(\xi(N)) = \emptyset$.

Finally, F is obviously nonfull: to check this, consider any pair $(X, \leq), (Y, \leq')$ of posets with $X \neq Y$ and any Chu transform (f, g) from Chu space $\mathcal{A} = ((X, \leq), (X, \leq), \dashv)$ to Chu space $\mathcal{A}' = ((Y, \leq'), (Y, \leq'), \dashv')$. Clearly, there is no $\Gamma \in Megf_{lea}$ such that $(f, g) = F(id_\Gamma)$.

ii) For any $G = (N, X, (S_i)_{i\in N}, (F_i)_{i\in N}, h) \in Gsgf$ take

$F(G) = ((N \times \prod_{i\in N} S_i, =), (X, =), \dashv)$ where for any $(i, s, x) \in N \times \prod_{i\in N} S_i \times X$

$\dashv ((i, s), x) = 1$ if $h(s) = x$ and $\dashv ((i, s), x) = 0$ otherwise, and posit

$F(id_G) = id_{F(G)}$.

Again, it is immediately checked that F is a well-defined faithful functor from $\mathcal{C}(Gsgf)$ to $\mathbf{Chu}(\mathbf{Poset}, 2)$.To see that F is also injective on objects, hence an embedding of categories, consider any pair $G, G' \in Gsgf$ such that $F(G) = F(G')$. If $G \neq G'$ then it must be the case that $h(G) \neq h(G')$ i.e. there exists $(s, x) \in \prod_{i\in N} S_i \times X$ such that either $[h(G)(s) = x \neq h(G')(s)]$ or $[h(G)(s) \neq x = h(G')(s)]$. Let us suppose w.l.o.g. that $[h(G)(s) = x \neq h(G')(s)]$ holds.

But then, for any $i \in N$

$\dashv_{F(G)} ((i, s), x) = 1$ while $\dashv_{F(G')} ((i, s), x) = 0$ i.e. $\dashv_{F(G)} \neq \dashv_{F(G')}$, a contradiction since $F(G) = F(G')$.

Nonfullness of F follows from the same argument introduced under i).

iii) For any $\mathbb{S} = (N, X, \mathcal{I}) \in Ssf$ posit

$F(G) = ((\wp(N) \times \wp(X) \times X \times \wp(N), =), (\wp(N) \times \wp(X), \Supset), \dashv)$

where for any $(S, A), (T, B) \in \wp(N) \times \wp(X)$,

$(S,A) \Supset (T,B)$ iff $[S \supseteq T$ and $A \supseteq B]$, while

$\dashv ((S,A,x,U),(T,B)) = 1$ if $((S,A),x,U,(T,B)) \in \mathcal{I}$ and $\dashv ((S,A),(T,B)) =$ 0 otherwise. Moreover, take

$F(id_{\mathbb{S}}) = id_{F(\mathbb{S})}$.

Now, F is by definition a well-defined functor, and is faithful by discreteness of $\mathcal{C}(Ssf)$. To see that it is also injective on objects, consider any pair $\mathbb{S}, \mathbb{S}' \in Ssf$ such that $F(\mathbb{S}) = F(\mathbb{S}')$ and $\mathbb{S} \neq \mathbb{S}'$.Then there exists a tuple $((S,A),U,x,(T,B)) \in (\mathcal{I}(\mathbb{S}) \setminus \mathcal{I}(\mathbb{S}')) \cup (\mathcal{I}(\mathbb{S}') \setminus \mathcal{I}(\mathbb{S}))$.Let us assume w.l.o.g. that $((S,A),U,x,(T,B)) \in \mathcal{I}(\mathbb{S}) \setminus \mathcal{I}(\mathbb{S}')$.It follows that $\dashv_{F(\mathbb{S})} ((S,A,x,U),(T,B)) = 1$ while $\dashv_{F(\mathbb{S}')} ((S,A,x,U),(T,B)) = 0$, a contradiction.

Finally, F is nonfull by the same argument presented above under i). ■

Therefore, one may indeed conclude that-at least when only identity morphisms are considered- virtually all game forms or games may be regarded either as subcategories of **Chu**(**Poset**, 2)or as concrete categories over subcategories of **Chu**(**Poset**, 2).

Of course, one should like to know if for some reasonable choice of morphisms of game forms the latter may be represented as *full* subcategories of **Chu**(**Poset**, 2). The following proposition provides a (partially) positive answer to that question, namely

Proposition 14 *Let* $\mathbf{Ecgf}^* = (Ecgf, \hom_{\mathbf{Ecgf}^*}, id, \circ)$ *where for any* $\mathcal{E} = (N,X,E), \mathcal{E}' = (M,Y,E') \in Ecgf$,

$$\hom_{\mathbf{Ecgf}^*}(\mathcal{E},\mathcal{E}') = \{(f,g) : f \in \hom_{\mathbf{Poset}}((\wp(N),\supseteq),(\wp(M),\supseteq))$$
$$g \in \hom_{\mathbf{Poset}}((\wp(Y),\supseteq),(\wp(X),\supseteq)) \text{ such that}$$
$$\text{for each } S \subseteq N \text{ and each } B \subseteq Y$$
$$B \in \mathcal{E}'(f(S)) \text{ iff } g(B) \in \mathcal{E}(S)\}.$$

Then, **Ecgf*** *is a* full subcategory *of* **Chu**(**Poset**, 2).

Proof. It is immediately checked that **Ecgf***is a well-defined category.

Next, for any $\mathcal{E} = (N,X,E), \mathcal{E}' = (M,Y,E') \in Ecgf$, and any $\phi = (f,g) \in \hom_{\mathbf{Ecgf}^*}(\mathcal{E},\mathcal{E}')$ posit $F(\mathcal{E}) = ((\wp(X),\supseteq),(\wp(Y),\supseteq),\dashv_{\mathcal{E}})$, where for all $S \subseteq N, A \subseteq X$, $\dashv_{\mathcal{E}} (S,A) = 1$ if $A \in \mathcal{E}(S)$ and $\dashv_{\mathcal{E}} (S,A) = 0$ otherwise, and $F(\phi) = \phi$.

Clearly, F is -by definition- a functor from **Ecgf***to **Chu**(**Poset**, 2).

Moreover, F is trivially faithful and full, by definition. To see that F is also injective on objects, hence an embedding, notice that for any $\mathcal{E} = (N,X,E), \mathcal{E}' = (M,Y,E') \in Ecgf$, if $F(\mathcal{E}) = F(\mathcal{E}')$ and $\mathcal{E} \neq \mathcal{E}'$ then necessarily $N = M$, $X = Y$ and $E \neq E'$ i.e. there exist $S \subseteq N, A \subseteq X$ such that either $[A \in E(S) \setminus E'(S)]$ or $[A \in E'(S) \setminus E(S)]$. In any case, it follows that $\dashv_{\mathcal{E}} \neq \dashv_{\mathcal{E}'}$, a contradiction. ■

It should be noticed that the foregoing result might be extended to all classes of game forms whose corresponding discrete categories are subcategories of $\mathcal{C}(Ecgf)$, by defining morphisms in such a way that they preserve the relevant effectivity-coalitional structure. The details however, will not be pursued here. I also conjecture that a similar result could be established for social situation forms, but I leave this issue as a topic for further research.

5 Concluding remarks

The present paper has been to a large extent a tour de force among game formats. By establishing that virtually all of them may be represented as subcategories (or even full subcategories) of the category $\mathbf{Chu}(\mathbf{Poset}, 2)$ of **Poset**-enriched Chu spaces over 2, the results provided above help make three main points, namely:

i) given the remarkable variety of extant game formats, reducing them to special versions of an underlying common structure is only possible if the latter is indeed a *very general* one;

ii) the category $\mathbf{Chu}(\mathbf{Poset}, 2)$ seems to be a better candidate for that role than $\mathbf{Chu}(\mathbf{Set}, 2)$,which has also been proposed in the literature as a good representation of (certain) games and game forms;

iii) as a result, the family of $\mathbf{Chu}(\mathbf{Poset}, K)$ categories as indexed by the class of all sets are arguably a better candidate as a tool for representing and classifying the objects of mathematical discourse than the corresponding family of $\mathbf{Chu}(\mathbf{Set}, K)$ categories as proposed in Pratt(1995). This observation is reinforced by the fact that for any K, $\mathbf{Chu}(\mathbf{Set}, K)$ may be construed as a full subcategory of $\mathbf{Chu}(\mathbf{Poset}, K)$. It should also be remarked that the results of the present paper suggest that suitably fuzzified versions of game formats inhabit the categories $\mathbf{Chu}(\mathbf{Poset}, K)$.

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