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Freedom, Time Constraints and Progressive Taxation

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Abstract - In this paper I present a measure of freedom for opportunity sets which are bounded by both budget and time constraints. Then I show that, in a society in which income is distributed more unequally than leisure time, a government aiming at leaving freedom distribution unaltered should apply progressive taxation. Since incomes bind freedom only partially when time constraints bind, taxing the rich reduces his freedom proportionally less than taxing the poor reduces his. Moreover, when incomes are so high that only time constraints bind opportunity sets, income taxation can be very high without freedom being impaired.

Keywords: Freedom, opportunity sets, public goods, progressive taxation **JEL classification**: D630, D710,

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Introduction

In an objectivist approach to the analysis of freedom opportunity sets can be ranked cardinally (Pattanaik and Xu, 1990; Gravel, 1994; Gravel, Laslier and Trannoy, 1998). Then a natural measure of freedom is given by the "volume" of an opportunity set (Xu, 2004). I introduced such a measure in an essay (Screpanti, 2003) where I considered opportunity sets bounded by budgetary and institutional constraints. Now I intend to take account of another kind of constraints - the availability of leisure time - and, accordingly, to devise a new formula for measuring freedom.

An interesting result emerges when both time and budget constrains bind opportunity sets. Income is no longer a socially neutral constraint to freedom, for the limitation that it poses to the poor people's choices is stronger than that it poses to the rich people's. This occurs quite independently of preferences. It occurs because, when time constraints are taken into account, the rich's budget constraint typically bites on his choice freedom less than the poor's bites on the poor's freedom. Which seems to be a sound justification for progressive taxation. I argue that there is a degree of progressiveness which does not alter the distribution of freedom. If a conservative government is one that does want to alter this distribution, then even such a government will apply progressive taxation.

In section 1 I present a measure of freedom for an opportunity set which is only bounded by a budget constraint, whilst, in section 2, I present one that only takes account of time constraints. In section 3 a new formula is proposed that contemplates both kinds of constraints. In section 4 I show an interesting implication for taxation policy. Then the opportunity set is further expanded in section 5, where social goods are introduced.

1. Budget constraints

Let me first define an opportunity set as bounded by an income constraint alone. The bundles of goods x_1 and x_2 , with prices p_1 and p_2 , that can be chosen by an individual with income *Y* are located in the area of triangle Y/p_1 -0- Y/p_2 in figure 1. The budget constraint is $Y \ge x_1p_1 + x_2p_2$. Any point in the triangle bounded by the budget line is an opportunity. The triangle area is the quantity of freedom available to this individual.¹

Now consider *n* goods and let

$$\Gamma := \{ (x_1, ..., x_n) \mid \sum_{i=1}^n p_i x_i \le Y, x_i \ge 0 \}$$

be the opportunity set. Its volume, L, can be used as a measure of freedom:

(1)
$$L \coloneqq vol(\Gamma) = \frac{Y^n}{n! \prod_{i=1}^n p_i}$$

This formula can be interpreted in the following way: $Y^n/n!$ is the volume of a hyper-tetrahedron with *n* hedges of length *Y*; it is divided by the volume of a hyper-parallelepiped, $\prod_{i=1}^{n} p_i$, representing one choice opportunity, i.e. a bundle made up of one unit of each commodity; the division yields the number of opportunities available to our individual. This is an intuitively natural

¹ The nature of the constraint varies with the time horizon of choices. So one could refer to a monthly or a yearly income; in a long run intertemporal setting one could refer to a wealth constraint; and so on. Since there is no need to go in deep with this problem here, I will generically talk of "income".

measure of freedom as a cardinal magnitude. Of course it is determined up to a dimensionality factor, D, as it depends on the standard chosen for measuring goods.² Without loss of generality I assume D=1



Figure 1

2. Time constraints

Steedman (2001) brought to the economists' attention the significance of the constraints that time availability poses to consumption choices and perceptively investigated into the dramatic consequences that consideration of time constraints have on economic analysis. So far as I know, no one has yet taken account of the time limitations to choice freedom.

Let me define "time use rates" t_1 and t_2 as the minimum quantities of time required to enjoy one unit of good x_1 or one unit of good x_2 respectively. Assuming our individual's leisure time is T, his time constraint is $T \ge x_1 t_1 + x_2 t_2$. His opportunity set, as bounded by time availability alone (provisionally ignoring budget constraints), is the area of the triangle formed by line T in figure 1. The whole area is an opportunity set, and not just the time line, for we know nothing of our individual's psychology, whether he is unable to rationally use his time or is one who just likes taking it easy.

Let

$$\Omega := \{ (x_1, ..., x_n) \mid \sum_{i=1}^n t_i x_i \le T, x_i \ge 0 \}$$

be the opportunity set as bounded by time availability alone. A measure of freedom, in this case, is the volume, Q, of opportunity set Ω .

² Let δ_i be the measuring rod for good x_i and pose $D = 1/\prod_{i=1}^n \delta_i$. Formula (1) must be multiplied by D.

(2)
$$Q \coloneqq vol(\Omega) = \frac{T^n}{n! \prod_{i=1}^n t_i}$$

3. Combining budget and time constraints

Now let me re-introduce the budget constraint. Since time use rates need not coincide with prices, the slopes of budget and time lines are typically different. And since both income and time availability may limit choices, opportunity sets can be bounded by either or both constraints. Very rich rentiers may enjoy opportunity sets that are only bounded by time availability. Very poor persons, like unemployed people, may have opportunity sets that are bounded by income alone. But both constraints bound the opportunity sets of most people. The opportunity set of an individual with leisure time *T* and income *Y* is represented by the area formed by the bold lines in figure 1. In general it is the convex polytope $\Gamma \cap \Omega$.

By defining

$$\gamma_i \coloneqq \frac{p_i T}{t_i}$$
 and $A_i \coloneqq \gamma_i \prod_{\substack{k=1\\k\neq i}}^n (\gamma_i - \gamma_k)$

the quantity of freedom, F(Y), is now measured as:³

(3)
$$F(Y) = vol(\Gamma \cap \Omega) = \frac{T^n}{n! \prod_{i=1}^n t_i} \left[1 - \sum_{j \in \mathbb{Z}} \frac{(\gamma_j - Y)^n}{A_j} \right]$$

where $Z := \{j | Y < \gamma_j\}$. All γ_j 's are assumed different.

The formula can be rewritten

(4)
$$F(Y) = Q\beta(Y) = Q[1 - \alpha(Y)]$$

with

$$\beta(Y) = 1 - \sum_{j \in \mathbb{Z}} \frac{(\gamma_j - Y)^n}{A_j}, \qquad \qquad \alpha(Y) = \sum_{j \in \mathbb{Z}} \frac{(\gamma_j - Y)^n}{A_j}$$

It holds:

 $0 < \beta(Y) \le 1, \qquad \qquad 0 \le \alpha(Y) < 1$

The geometrical meaning of $\beta(Y)$ and $\alpha(Y)$ may be grasped by observing figure 1, where $Q\beta(Y)=Q[1-\alpha(Y)]$ represents the area of the convex polytope bounded by the bold lines, whilst $Q=Q[\beta(Y) + \alpha(Y)]$ is the area of set Ω , as bounded by constraint T. Note that, if an individual's income is so higher than his time availability that it does not constrain his freedom, i.e. set Z is empty and $\alpha(Y)=0$, then F=Q. This is a case of a very rich person.

³ The formula is a re-elaboration of one worked out by Lasserre and Zeron, 2001, p. 1130.

4. A justification for progressive taxation

Take the derivative of equation (4):

(5)
$$\frac{dF}{dY} = Q\beta'(Y) = nQ\sum_{j\in Z} \frac{(\gamma_j - Y)^{n-1}}{A_j}$$

The annual taxes paid by an individual (or his income diminution due to taxes) is $dY = \rho Y d\tau$, where $\rho < 0$ is the tax-rate. Since $d\tau$ è is the fiscal year, one can write $d\tau = 1$ and $dY = \rho Y$. Substitute the latter expression into the (5) and divide by $F(Y) = Q\beta(Y)$, The result is a measure of the proportional diminution of freedom caused by taxes:

(6)
$$\frac{dF}{F} = \rho Y \frac{\beta'(Y)}{\beta(Y)}$$

Now posit that the various incomes are taxed in such a way that freedom distribution is not altered. The proportional change in freedom will be the same for all income levels, that is, c = |dF/F|. Therefore

(7)
$$|\rho(Y)| = c \left(Y \frac{\beta'(Y)}{\beta(Y)} \right)^{-1}$$

is the tax-rate function that does not alter the distribution of freedom. Tax-rates will be progressive if

(8)
$$\frac{d}{dY} | \rho(Y)| = -c \left(Y \frac{\beta'(Y)}{\beta(Y)} \right)^{-2} \frac{[Y\beta''(Y) + \beta'(Y)]\beta(Y) - Y(\beta'(Y))^2}{(\beta(Y))^2} > 0$$

i.e. if

(9)
$$[Y\beta''(Y) + \beta'(Y)]\beta(Y) - Y(\beta'(Y))^2 < 0$$

Thus, if taxation has to reduce freedom to the same percentage to all citizens, the rich must be taxed with a higher rate than the poor. This is certainly so when $n \le 3$, as it is proved in the appendix. It is conceivably so with an higher *n*. In fact the rationale for progressive taxation is intuitive in this approach. The income of the rich binds his opportunity set less than that of the poor, if the two citizens have the same time constraint. Thus a given tax-rate proportionally subtracts more freedom to the poor than to the rich. This is why the latter must be taxed with a higher tax-rate.

Nor is it necessary that the rich and the poor have the same time constraint, as I maintained for the sake of argument in the above reasoning. When considering a whole population of taxpayers, it would be sufficient to assume that time constraints differ less than budget constraints. And this is not difficult to justify. In fact time scarcity poses physical limits to leisure time, while there are no such limits to income accretion. In other words, income dispersion is higher than leisure time dispersion, the more so the higher income inequality. Precisely this fact justifies progressive taxation from the point of view of a government who wants to preserve the distribution of freedom.⁴ Of course the degree of tax progressiveness increases when a government aims at redistributing freedom in favour of the less free citizens.

5. Introducing social goods

In Screpanti (2003) I defined a measure of freedom by taking account of what I called "social goods". These are the goods which are offered free to all citizens. They include public goods, merit goods and, more generally, any publicly provided good or service.⁵ Many rights, for instance, can be considered social goods. Since there is no individual budget constraint to the choice of a social good, it seems that opportunity sets would become infinite. To avoid this and for the sake of simplicity, in that article I assumed that opportunity sets in the space of social goods are bounded, observing that in general they cannot be infinite due to the existence of time constraints. Now I intend to go in deep with this observation.

Assume *m* social goods exist with $p_l=0$ and $t_l>0$, (l=1,...m). Since there are *n* private goods (for which $p_i>0$ and $t_i>0$, i=1,...n), the total number of goods is v=n+m. The budget and time constraints are:

$$\sum_{i=1}^{n} p_i x_i \le Y, \qquad \sum_{i=1}^{n} t_i x_i + \sum_{l=1}^{m} t_{n+l} x_{n+l} \le T, \qquad x_s \ge 0, \qquad s = 1, \dots v$$

which can be rewritten

$$\sum_{s=1}^{\nu} p_s x_s \le Y, \quad \sum_{s=1}^{\nu} t_s x_s \le T; \quad t_s > 0, \ s = 1, \dots \nu; \ p_s > 0, \ s = 1, \dots n; \ p_s = 0, \ s = n+1, \dots \nu$$

Now define

$$\Gamma_* := \{ (x_1, ..., x_v) \mid \sum_{s=1}^v p_s x_s \le Y, x_s \ge 0 \}$$
$$\Omega_* := \{ (x_1, ..., x_v) \mid \sum_{s=1}^v t_s x_s \le T, x_s \ge 0 \}$$

Of course $Q_*=vol(\Omega_*)$ is finite. The volume of Γ_* , though, is infinite since Γ_* is not constrained in *m* dimensions. However we are interested in $F_*(Y)$, i.e. the volume of $\Gamma_* \cap \Omega_*$, and this is finite. Now, by defining

$$\gamma_s \coloneqq \frac{p_s T}{t_s}, \quad s = 1, \dots, v; \quad B_i \coloneqq \gamma_i^{m+1} \prod_{\substack{k=1\\k \neq i}}^n (\gamma_i - \gamma_k), \quad i = 1, \dots, n$$

equation (3) can be rewritten

⁴ Note that, for a very rich person, i.e. one whose freedom is only bounded by time constraints, taxation can be very high without altering the distribution of freedom, provided taxes do not cut income so dramatically to make $\beta(Y) < 1$.

⁵ If some goods, call them "spurious social goods", are publicly provided at positive but low prices or tickets, they are likened to private goods in the analysis below. Their public provision rises freedom, *ceteris paribus*, to the extent that tickets are lower than market prices.

(3')
$$F_*(Y) = vol(\Gamma_* \cap \Omega_*) = \frac{T^v}{v! \prod_{i=1}^v t_i} \left[1 - \sum_{j \in \mathbb{Z}} \frac{(\gamma_j - Y)^v}{B_j} \right]$$

Note that the last *m* of the γ_s coefficients are zero. Also note that it still is $Z := \{j \mid Y < \gamma_j\}$, but now we know that the last *m* indices do not belong to *Z*.

By defining

$$\beta_*(Y) = 1 - \sum_{j \in \mathbb{Z}} \frac{(\gamma_j - Y)^{\nu}}{B_j}, \qquad \alpha_*(Y) = \sum_{j \in \mathbb{Z}} \frac{(\gamma_j - Y)^{\nu}}{B_j}$$

equation (4) can be rewritten

(4')
$$F_*(Y) = Q_*\beta_*(Y) = Q_*[1 - \alpha_*(Y)]$$

The analysis of section 4 applies. The result is that income taxation must be progressive if

(8')
$$[Y\beta_*'(Y) + \beta_*'(Y)]\beta_*(Y) - Y(\beta_*'(Y))^2 < 0$$

Conclusion

A new formula for measuring choice freedom is devised which takes into account both budget and time constraints. It makes cardinal ranking of opportunity sets possible and seems to be a natural measure of freedom. Even more important, it allows playing with exercises in interpersonal comparisons of freedom. On the ground of one such exercise, an engaging policy result is appended to those I reached in Screpanti (2003). Not only an egalitarian and a liberal government, but even a conservative one has good reasons to: 1) apply progressive taxation; 2) tax heavily the very rich.

Appendix

For $n \le 3$ it is possible to prove that

$$[Y\beta''(Y) + \beta'(Y)]\beta(Y) - Y(\beta'(Y))^{2} < 0$$

or

$$\left[-Yn(n-1)\sum_{j\in\mathbb{Z}}\frac{(\gamma_{j}-Y)^{n-2}}{A_{j}}+n\sum_{j\in\mathbb{Z}}\frac{(\gamma_{j}-Y)^{n-1}}{A_{j}}\right]\left(1-\sum_{j\in\mathbb{Z}}\frac{(\gamma_{j}-Y)^{n}}{A_{j}}\right)-Yn^{2}\left(\sum_{j\in\mathbb{Z}}\frac{(\gamma_{j}-Y)^{n-1}}{A_{j}}\right)^{2}<0$$

Put the γ_i indices in increasing order, so that $\gamma_1 < \gamma_2 < ... < \gamma_n$, and consider separately the two cases with n=2 and n=3.

The case with *n*=2.

 $\gamma_1 < Y < \gamma_2$ is the only interesting case. It holds:

$$\beta(Y) = 1 - \frac{(\gamma_2 - Y)^2}{A_2}, \quad \beta'(Y) = 2\frac{(\gamma_2 - Y)}{A_2}, \quad \beta''(Y) = -2\frac{1}{A_2}, \quad A_2 = \gamma_2(\gamma_2 - \gamma_1)$$

Thus

$$[Y\beta''(Y) + \beta'(Y)]\beta(Y) - Y(\beta'(Y))^2 = -2\frac{\gamma_2}{A_2^2}[(Y - \gamma_1)^2 + \gamma_1(\gamma_2 - \gamma_1)] < 0$$

The case with n=3.

Two situations may arise: $\gamma_1 < Y < \gamma_2 < \gamma_3$ and $\gamma_1 < \gamma_2 < Y < \gamma_3$.

Consider the first. It holds:

$$\beta(Y) = 1 - \frac{(\gamma_2 - Y)^3}{A_2} - \frac{(\gamma_3 - Y)^3}{A_3}, \qquad \beta'(Y) = 3\frac{(\gamma_2 - Y)^2}{A_2} + 3\frac{(\gamma_3 - Y)^2}{A_3},$$

$$\beta''(Y) = -6\frac{(\gamma_2 - Y)}{A_2} - 6\frac{(\gamma_3 - Y)}{A_3}$$

with $A_2 = \gamma_2(\gamma_2 - \gamma_1)(\gamma_2 - \gamma_3)$ and $A_3 = \gamma_3(\gamma_3 - \gamma_1)(\gamma_3 - \gamma_2)$. Calculation yields:

$$[Y\beta''(Y) + \beta'(Y)]\beta(Y) - Y(\beta'(Y))^{2} = 3\frac{\gamma_{2}\gamma_{3}}{A_{2}^{2}A_{3}^{2}}(\gamma_{3} - \gamma_{2})^{4}(Y - \gamma_{1})[-(\gamma_{2} + \gamma_{3} - \gamma_{1})Y^{3} + 3\gamma_{1}(\gamma_{2} + \gamma_{3} - \gamma_{1})Y^{2} - 3\gamma_{1}\gamma_{2}\gamma_{3}Y + \gamma_{1}^{2}\gamma_{2}\gamma_{3}]$$

The sign is given by the polynomial of third degree

$$C(Y) \coloneqq -(\gamma_2 + \gamma_3 - \gamma_1)Y^3 + 3\gamma_1(\gamma_2 + \gamma_3 - \gamma_1)Y^2 - 3\gamma_1\gamma_2\gamma_3Y + \gamma_1^2\gamma_2\gamma_3$$

which is decreasing, since, by simple computation,

$$C'(Y) = -3[(\gamma_2 + \gamma_3 - \gamma_1)Y^2 - 2\gamma_1(\gamma_2 + \gamma_3 - \gamma_1)Y + \gamma_1\gamma_2\gamma_3] < 0$$

and negative in γ_l

$$C(\gamma_{1}) = -2\gamma_{1}^{2}(\gamma_{3} - \gamma_{1})(\gamma_{2} - \gamma_{1}) < 0$$

Thus C(Y) < 0 for $\gamma_1 < Y < \gamma_2$.

Now consider the case $\gamma_1 < \gamma_2 < Y < \gamma_3$. It holds:

$$\beta(Y) = 1 - \frac{(\gamma_3 - Y)^3}{A_3}, \quad \beta'(Y) = 3\frac{(\gamma_3 - Y)^2}{A_3}, \quad \beta''(Y) = -6\frac{(\gamma_3 - Y)}{A_3}$$

Calculation yields:

$$[Y\beta''(Y) + \beta'(Y)]\beta(Y) - Y(\beta'(Y))^{2} = 3\frac{\gamma_{3}}{A_{3}^{2}}(\gamma_{3} - Y)[Y^{3} - 3\gamma_{3}Y^{2} + 3(\gamma_{1}\gamma_{3} + \gamma_{2}\gamma_{3} - \gamma_{1}\gamma_{2})Y + \gamma_{1}\gamma_{2}\gamma_{3} - \gamma_{1}\gamma_{3}^{2} - \gamma_{2}\gamma_{3}^{2}]$$

The sign is given by the polynomial of third degree

$$C(Y) := Y^{3} - 3\gamma_{3}Y^{2} + 3(\gamma_{1}\gamma_{3} + \gamma_{2}\gamma_{3} - \gamma_{1}\gamma_{2})Y + \gamma_{1}\gamma_{2}\gamma_{3} - \gamma_{1}\gamma_{3}^{2} - \gamma_{2}\gamma_{3}^{2}$$

which is decreasing in the interval $[\gamma_2, \gamma_3]$, since, by simple computation,

$$C'(Y) = 3[Y^{2} - 2\gamma_{3}Y + \gamma_{1}\gamma_{3} + \gamma_{2}\gamma_{3} - \gamma_{1}\gamma_{2}] < 0$$

and negative in γ_2

$$C(\gamma_{2}) = -(\gamma_{3} - \gamma_{2})[\gamma_{1}(\gamma_{3} - \gamma_{2}) + \gamma_{2}(\gamma_{3} - \gamma_{1}) + \gamma_{2}(\gamma_{2} - \gamma_{1})] < 0$$

Thus C(Y) < 0 for $\gamma_2 < Y < \gamma_3$.

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