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# Coalitional Game Forms with Topological Closure Systems

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**Abstract -** This paper contributes to the task of classifying game forms from a structural point of view by studying properties of their concept-or Galoislattices. A characterization of those coalitional game forms that have topological closure systems is provided.It is also shown that CGFs with topological closure systems include additive effectivity functions and simple effectivity functions, but do not reduce to them, and that the resulting topologies are $T_0$ only if the underlying CGF is 'purified'.

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This paper contributes to the task of classifying game forms from a structural point of view by studying properties of their concept-or Galois- lattices. In particular, we focus on those coalitional game forms that have topological closure systems.

A *coalitional game form (CGF)* is a triple $\mathbf{G} = (N, X, E)$ where $N$ is the player set, $X$ is the outcome set, and $E : \mathcal{P}(N) \to \mathcal{P}(\mathcal{P}(X))$ is a function. A CGF $\mathbf{G} = (N, X, E)$ is said to be

*$\varnothing$-normalized* if $E(\varnothing) = \varnothing$

*never empty* if $\varnothing \notin E(S)$ for any $S \subseteq N$

*souvereign* if there exists $S \subseteq N$ such that $\mathcal{P}(X) \setminus \{\varnothing\} \subseteq E(S)$

*exhaustive* if $X \in E(S)$ for any nonempty $S \subseteq N$

*monotonic* if for any $S \subseteq T \subseteq N, A \subseteq B \subseteq X$, $A \in E(S)$ entails both $B \in E(S)$ and $A \in E(T)$

*linear* if for any $S, T \subseteq N$, $E(S) \subseteq E(T)$ or $E(T) \subseteq E(S)$

*purified* if for any $S, T \subseteq N$, and $A, B \subseteq X$, $E(S) \neq E(T)$ and $E^{-1}(A) \neq E^{-1}(B)$

*simple* if there exists $\mathcal{W} \subseteq \mathcal{P}(N)$ such that for any $S \subseteq N, A \subseteq X$, $A \in E(S)$ iff $A \neq \varnothing$ and $S \in \mathcal{W}$.

Moreover, when both $N$ and $X$ are countable sets, a CGF $\mathbf{G} = (N, X, E)$ is said to be *additive* if there exist *positive probability measures* $p : \mathcal{P}(N) \to \mathbb{R}$, $q : \mathcal{P}(X) \to \mathbb{R}$ such that for any $S \subseteq N, A \subseteq X$, $A \in E(S)$ iff $p(S) > 1 - q(A)$.

A CGF which is $\varnothing$-normalized, never empty, souvereign and exhaustive is also said to be an *effectivity function.*

The concept lattice[1] of $\mathbf{G}$ is defined as follows:

for any $\mathbf{S} \subseteq \mathcal{P}(N)$ , $\mathbf{A} \subseteq \mathcal{P}(X)$ posit

$\leftthreetimes_E(\mathbf{S}) = \{A \subseteq X : A \in E(S) \text{ for all } S \in \mathbf{S}\}$ and

$\rightthreetimes_E(\mathbf{A}) = \{S \subseteq N : A \in E(S) \text{ for all } A \in \mathbf{A}\}$ .

It is easily seen that $(\leftthreetimes_E, \rightthreetimes_E)$ is a *Galois connection* between $(\mathcal{P}(\mathcal{P}(N)), \subseteq)$ and $(\mathcal{P}(\mathcal{P}(X)), \subseteq)$ i.e. for any $\mathbf{S}, \mathbf{T} \subseteq \mathcal{P}(N)$, and $\mathbf{A}, \mathbf{B} \subseteq \mathcal{P}(X)$,

i) if $\mathbf{S} \subseteq \mathbf{T}$ then $\leftthreetimes_E(\mathbf{S}) \subseteq \leftthreetimes_E(\mathbf{T})$, and if $\mathbf{A} \subseteq \mathbf{B}$ then $\rightthreetimes_E(\mathbf{A}) \subseteq \rightthreetimes_E(\mathbf{B})$, and ii) $(\rightthreetimes_{\mathfrak{G}} \circ \leftthreetimes_{\mathfrak{G}})(\mathbf{S}) \supseteq \mathbf{S}$,$(\leftthreetimes_E \circ \rightthreetimes_E)(\mathbf{A}) \supseteq \mathbf{A}$.

Now, consider

$$\mathbb{C}(\mathbf{G}) = \{(\mathbf{S}, \mathbf{A}) \in \mathcal{P}(\mathcal{P}(N)) \times \mathcal{P}(\mathcal{P}(X)) : \mathbf{S} = \rightthreetimes_E(\mathbf{A}), \text{ and } \mathbf{A} = \leftthreetimes_E(\mathbf{S})\}.$$

In the language of formal concept analysis (see e.g. Ganter and Wille(1999)) an element $(\mathbf{S}, \mathbf{A})$ of $\mathbb{C}(\mathbf{G})$ is said to be a *concept* of the context $\mathbf{G}$, with *extent* $\mathbf{S}$ and *intent* $\mathbf{A}$ (the latter notions are amenable to straightforward dualizations).

The *concept lattice* of $\mathbf{G}$ (sometimes also referred to as its *Galois lattice*) is $\mathbb{L}(\mathbf{G}) = (\mathbb{C}(\mathbf{G}), \geqslant)$ where for any $(\mathbf{S}_1, \mathbf{A}_1), (\mathbf{S}_2, \mathbf{A}_2) \in \mathbb{C}(\mathbf{G})$

$(\mathbf{S}_1, \mathbf{A}_1) \geqslant (\mathbf{S}_2, \mathbf{A}_2)$ iff $\mathbf{A}_1 \supseteq \mathbf{A}_2$ (which is provably equivalent to $\mathbf{S}_2 \subseteq \mathbf{S}_1$), and

$$(\mathbf{S}_1, \mathbf{A}_1) \wedge (\mathbf{S}_2, \mathbf{A}_2) = (\rightthreetimes_E(\leftthreetimes_E(\mathbf{S}_1 \cup \mathbf{S}_2)), \mathbf{A}_1 \cap \mathbf{A}_2)$$
$$(\mathbf{S}_1, \mathbf{A}_1) \vee (\mathbf{S}_2, \mathbf{A}_2) = (\mathbf{S}_1 \cap \mathbf{S}_2, \leftthreetimes_E(\rightthreetimes_E(\mathbf{A}_1 \cup \mathbf{A}_2)).$$

[1] A lattice is a partially ordered set $(L, \leqslant)$ such that for any $x, y \in X$ there exist both a greatest lower bound $x \wedge y \in L$ and a least upper bound $x \vee y \in L$ of $\{x, y\}$. A lattice is *distributive* if for any $x, y, z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

It is also well-known and easily shown that both $(\curlywedge_E \circ \curlyvee_E) : \mathcal{P}(\mathcal{P}(N)) \to \mathcal{P}(\mathcal{P}(N))$ and $(\curlyvee_E \circ \curlywedge_E) : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(\mathcal{P}(X))$ are closure operators with respect to set-inclusion (recall that a *closure operator* $K$ on a preordered set $(Y, \geqslant)$ is a function $K : Y \to Y$ such that for any $y, x \in Y$ : $K(y) \geqslant y, K(y) \geqslant K(x)$ whenever $y \geqslant x$, and $K(y) \geqslant K(K(y))$ ), and *extents* and *intents* of concepts are precisely the *closed* elements -or fixed points- of $(\curlywedge_E \circ \curlyvee_E)$and $(\curlyvee_E \circ \curlywedge_E)$ respectively (i.e. $(\mathbf{S}, \mathbf{A}) \in \mathbb{C}(\mathbf{G})$ iff $\mathbf{S} = \curlywedge_E(\curlyvee_E(\mathbf{S}))$ and $\mathbf{A} = \curlyvee_E(\curlywedge_E(\mathbf{A}))$) and comprise the *(Galois) closure systems* of CGF $\mathbf{G}$.We shall also denote $(\curlywedge_E \circ \curlyvee_E)$and $(\curlyvee_E \circ \curlywedge_E)$ by $K_{\mathbf{G}}$ and $K^*_{\mathbf{G}}$, respectively.

Since the Galois lattice of a CGF is made up of two dually isomorphic *closure systems* as induced by two corresponding *closure operators* a natural question immediately arises: *under what circumstances are those closure operators topological?*

The following definition is to be recalled here

**Definition 1** *A closure operator $K$ on a Boolean*[2] *lattice of sets $(L, \cap, \cup)$ is* topological *if it is both* i) normal *i.e. $K(\varnothing) = \varnothing$ and ii)* additive *i.e. $K(A \cup B) = K(A) \cup K(B)$.*

Indeed, whenever a CGF happens to have topological closure operators, the resulting closure systems define a topology on the player set and on the outcome sets, respectively. The underlying CGF is then said to be *topology-inducing* ( a notion that should not be confused with the more familiar notion of a topological CGF or effectivity function which simply denotes CGFs whose outcome spaces are endowed with a given topological structure (see e.g. Abdou,Keiding (1991)).

**Definition 2** *A CGF $\mathbf{G}$ is said to be* N-topology-inducing (X-topology-inducing) *whenever $K_{\mathbf{G}}$ ( $K^*_{\mathbf{G}}$) is a topological closure operator.*

It is easily checked that for any $\varnothing$-Normalized CGF $\mathbf{G}$, both $K_{\mathbf{G}}(\varnothing) = \varnothing$ and $K^*_{\mathbf{G}}(\varnothing) = \varnothing$. Thus, the topological nature of $K_{\mathbf{G}}$ and $K^*_{\mathbf{G}}$ depends in fact solely on their $\cup$-additivity (or lack of it).

This note addresses the following issue: under what circumstances are the closure operators $K_{\mathbf{G}} = (\curlywedge_{\mathbf{G}} \circ \curlyvee_{\mathbf{G}})$and $K^*_{\mathbf{G}} = (\curlyvee_{\mathbf{G}} \circ \curlywedge_{\mathbf{G}})$ topological?

To start with, a class of topology-inducing CGFs can be immediately identified. Indeed, we have the following

**Proposition 3** *Let $\mathbf{G}$ be a linear CGF. Then its closure systems are topological.*

**Proof.** Straightforward, because by definition the closure systems of a linear $\mathbf{G}$ are chains. Hence, the join of two closed sets is a closed set.e.g. for any pair $\mathbf{S}, \mathbf{T} \subseteq \mathcal{P}(N)$ it must be the case that $K_{\mathbf{G}}(\mathbf{S}) \subseteq K_{\mathbf{G}}(\mathbf{T})$ or $K_{\mathbf{G}}(\mathbf{S}) \subseteq K_{\mathbf{G}}(\mathbf{T})$.Let

[2] A lattice $(L, \wedge, \vee)$is distributive if for any $x, y, z \in L : x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ bounded if has both a greatest element (denoted 1) and a smallest element (denoted 0), and complemented if it is bounded and for any $x \in L$ there exists $y \in L$ such that $x \vee y = 1$ and $x \wedge y = 0$.

A lattice is Boolean if it is both distributive and complemented.

us suppose w.l.o.g. that $K_{\mathbf{G}}(\mathbf{S}) \subseteq K_{\mathbf{G}}(\mathbf{T})$. Therefore, $K_{\mathbf{G}}(\mathbf{S}) \cup K_{\mathbf{G}}(\mathbf{T}) = K_{\mathbf{G}}(\mathbf{T})$.Since, as it is easily checked, $K_{\mathbf{G}}(\mathbf{S}\cup\mathbf{T}) = \bigcap \{\mathbf{U} \subseteq \mathcal{P}(\mathbf{N})) : \mathbf{U} = K_{\mathbf{G}}(\mathbf{U}) \supseteq \mathbf{S} \cup \mathbf{T}\}$, and $K_{\mathbf{G}}(\mathbf{U}) \supseteq \mathbf{U}$ for any $\mathbf{U} \subseteq \mathcal{P}(\mathbf{N})$, it follows that if one posits $\mathbf{U} = K_{\mathbf{G}}(\mathbf{T})$ then $\mathbf{U} = K_{\mathbf{G}}(\mathbf{U}) = K_{\mathbf{G}}(\mathbf{T}) \supseteq \mathbf{S} \cup \mathbf{T}$.Now, let $\mathbf{U}' \subseteq \mathcal{P}(\mathbf{N})$ be such that $\mathbf{U}' = K_{\mathbf{G}}(\mathbf{U}') \supseteq \mathbf{S}\cup\mathbf{T}$: clearly $K_{\mathbf{G}}(\mathbf{U}') \supseteq K_{\mathbf{G}}(\mathbf{T})$.Hence, $K_{\mathbf{G}}(\mathbf{S}\cup\mathbf{T}) = K_{\mathbf{G}}(\mathbf{T}) = K_{\mathbf{G}}(\mathbf{S}) \cup K_{\mathbf{G}}(\mathbf{T})$, as required. A similar argument applies to $K^*_{\mathbf{G}}$.

**Remark 4** *It should be noticed that the foregoing Proposition entails that both* additive *CGFs and* simple *CGFs as defined above are indeed topology-inducing since -as it is easily checked- their concept lattices are chains (in particular, if* $\mathbf{G}$ *is a simple effectivity function and* $\mathbf{G}' = (N, X, E)$ *is an additive effectivity function, then* $\mathbb{L}(\mathbf{G}) = (\mathbf{4}, \leqslant)$*, and* $\mathbb{L}(\mathbf{G}') = (\mathbf{k}, \leqslant)$ *where* $\mathbf{k} = \{0, 1, .., h+1\}$ *with* $h = \# \{\mathbf{S} \subseteq \mathcal{P}(N) : \mathbf{S} = K_{\mathbf{G}'}(\mathbf{S})\}$*).*

■

In order to proceed, a few more definitions are to be introduced.

**Definition 5** *Let* $(\curlywedge, \curlyvee)$ *be a Galois connection between* $(\mathcal{P}(\mathcal{P}(N)), \subseteq)$ *and* $(\mathcal{P}(\mathcal{P}(X)), \subseteq)$ *and* $K = (\curlyvee \circ \curlywedge)$ *the closure operator on* $\mathcal{P}(N)$ *induced by* $(\curlywedge, \curlyvee)$*. Then, a* $K$*-closed set* $\mathbf{S}$ *-i.e. a fixed point of* $K$*- is*

*i)* monogenic *iff there exists* $B \subseteq X$ *such that* $\mathbf{S} = \curlyvee(\{B\})$

*ii)* prime *iff for any pair* $\mathbf{T}, \mathbf{U}$ *of* $K$*-closed sets such that* $\mathbf{T} \cap \mathbf{U} \subseteq \mathbf{S}$*, either* $\mathbf{T} \subseteq \mathbf{S}$ *or* $\mathbf{U} \subseteq \mathbf{S}$

*iii)* meet-irreducible *iff for any pair* $\mathbf{T}, \mathbf{U}$ *of* $K$*-closed sets such that* $\mathbf{T} \cap \mathbf{U} = \mathbf{S}$*, either* $\mathbf{T} = \mathbf{S}$ *or* $\mathbf{U} = \mathbf{S}$.

**Remark 6** *It is immediately checked that a prime closed set is also meet-irreducible while the converse does not necessarily hold.*

We are now ready to state the main result of this paper, namely:

**Theorem 7** *Let* $\mathbf{G} = (N, X, E)$ *be a monotonic CGF. Then* $K_{\mathbf{G}}$ *is topological iff* $\mathbf{G}$ *is* $\emptyset$-normalized *and the* monogenic $K^*_{\mathbf{G}}$-closed sets are meet-irreducible *(and dually* $K^*_{\mathbf{G}}$ *is topological iff* $\mathbf{G}$ *is* $\emptyset$-normalized *and the* monogenic $K_E$-closed sets are meet-irreducible*).*

**Proof.** The thesis follows at once from a few facts and claims as listed below.

Fact. If $L$ is a distributive lattice then for any $x \in L$, $x$ is prime if and only if it is meet-irreducible.

(This is a well-known fact about distributive lattices (see e.g. Grätzer(1998)).

*Claim 1. If* $K_{\mathbf{G}}$*- or* $K^*_{\mathbf{G}}$*- is a topological closure operator then* $\mathbb{L}(\mathbf{G})$ *is a distributive lattice.*

Proof of Claim 1. First, recall that -as mentioned above- any $K_{\mathbf{G}}$-closed set $\mathbf{S}$ is an order filter of $(\mathcal{P}(\mathcal{P}(N)), \subseteq)$ i.e. for any $S, T \in \mathbf{S}$, if $S \in \mathbf{S}$ and $S \subseteq T$ then $T \in \mathbf{S}$, by monotonicity of $\mathbf{G}$: hence,the closure system $\mathcal{C}_{\mathbf{G}} = \{\mathbf{S} \subseteq \mathcal{P}(N) : \mathbf{S} = K_{\mathbf{G}}(\mathbf{S})\}$ induced by $K_{\mathbf{G}}$ is a subset of $\mathcal{F}(\mathcal{P}(N))$ the set of all order filters of $(\mathcal{P}(N), \subseteq)$.

Next, observe that -as it is easily checked- $(\mathcal{F}(\mathcal{P}(N), \subseteq)$ is a distributive lattice with inf=$\cap$ and sup=$\cup$.Since *any* closure system is $\cap$-closed, $\cup$-additivity of $K_{\mathbf{G}}$ makes the corresponding closure system asublattice of $(\mathcal{F}(\mathcal{P}(N), \subseteq)$, whence $\mathcal{C}_{\mathbf{G}}$ is a distributive lattice. The same argument applies to the dual closure system $\mathcal{C}^*_{\mathbf{G}}$.Therefore, by construction, $\mathbb{L}(\mathbf{G})$ is also a distributive lattice.

*Claim 2.* $K_{\mathbf{G}}$ is topological iff $\mathbf{G}$ is $\emptyset$*-normalized* and any *monogenic* $K^*_{\mathbf{G}}$*-closed set is prime* (and dually $K^*_{\mathbf{G}}$ is topological iff $\mathbf{G}$ is $\emptyset$*-normalized* and any *monogenic* $K_{\mathbf{G}}$*-closed set is prime*).

Proof of Claim 2. First, notice that $K_{\mathbf{G}} = (\rightthreetimes_{\mathbf{G}} \circ \leftthreetimes_{\mathbf{G}})$ is $\cup$-additive iff for any $\mathbf{S}, \mathbf{T} \subseteq \mathcal{P}(N)$ :

$(*)$ $\rightthreetimes_{\mathbf{G}}(\leftthreetimes_{\mathbf{G}}(\mathbf{S}) \cap \leftthreetimes_{\mathbf{G}}(\mathbf{T})) \subseteq (\rightthreetimes_{\mathbf{G}}(\leftthreetimes_{\mathbf{G}}(\mathbf{S})) \cup (\rightthreetimes_{\mathbf{G}}(\leftthreetimes_{\mathbf{G}}(\mathbf{T}))$.

Indeed, since $K_{\mathbf{G}}$ is inflationary i.e. $K_{\mathbf{G}}(\mathbf{S}) \supseteq \mathbf{S}$ for any $\mathbf{S} \subseteq \mathcal{P}(N)$, it follows that

$K_{\mathbf{G}}$ is $\cup$-additive iff $K_{\mathbf{G}}(\mathbf{S} \cup \mathbf{T}) \subseteq K_{\mathbf{G}}(\mathbf{S}) \cup K_{\mathbf{G}}(\mathbf{T})$ (and the same fact holds true for $K^*_{\mathbf{G}}$).

Also, observe that, by antitonicity of $\leftthreetimes_{\mathbf{G}}$and $\rightthreetimes_{\mathbf{G}}$, for any $\mathbf{S}, \mathbf{T} \subseteq \mathcal{P}(N)$, and $\mathbf{A}, \mathbf{B} \subseteq \mathcal{P}(X)$

$\leftthreetimes_{\mathbf{G}}(\mathbf{S} \cap \mathbf{T})) \supseteq \leftthreetimes_{\mathbf{G}}(\mathbf{S}) \cap \leftthreetimes_{\mathbf{G}}(\mathbf{T})$, and $\rightthreetimes_{\mathbf{G}}(\mathbf{A} \cap \mathbf{B})) \subseteq \rightthreetimes_{\mathbf{G}}(\mathbf{A}) \cap \rightthreetimes_{\mathbf{G}}(\mathbf{B})$.

Moreover, for any $\mathbf{S}, \mathbf{T} \subseteq \mathcal{P}(N)$, and $\mathbf{A}, \mathbf{B} \subseteq \mathcal{P}(X)$,

$\leftthreetimes_{\mathbf{G}}(\mathbf{S} \cup \mathbf{T}) = \leftthreetimes_{\mathbf{G}}(\mathbf{S}) \cap \leftthreetimes_{\mathbf{G}}(\mathbf{T})$and $\rightthreetimes_{\mathbf{G}}(\mathbf{A} \cup \mathbf{B})) = \rightthreetimes_{\mathbf{G}}(\mathbf{A}) \cap \rightthreetimes_{\mathbf{G}}(\mathbf{B})$,

because $\leftthreetimes_{\mathbf{G}}(\mathbf{S} \cup \mathbf{T}) \subseteq \leftthreetimes_{\mathbf{G}}(\mathbf{S}) \cap \leftthreetimes_{\mathbf{G}}(\mathbf{T})$and $\rightthreetimes_{\mathbf{G}}(\mathbf{A} \cup \mathbf{B})) \subseteq \rightthreetimes_{\mathbf{G}}(\mathbf{A}) \cup \rightthreetimes_{\mathbf{G}}(\mathbf{B})$

by antitonicity of $\leftthreetimes_{\mathbf{G}}$and $\rightthreetimes_{\mathbf{G}}$respectively, while for any $A \subseteq X, A \in \leftthreetimes_{\mathbf{G}}(\mathbf{S}) \cap \leftthreetimes_{\mathbf{G}}(\mathbf{T})$ entails $A \in E(S)$ for any $S \in \mathbf{S}$ and $A \in E(T)$ for any $T \in \mathbf{T}$, whence $A \in E(U)$ for any $U \in \mathbf{S} \cup \mathbf{T}$ (a similar argument holds for $\rightthreetimes_{\mathbf{G}}$).

Hence, $(*)$ follows, (and dually $\leftthreetimes_{\mathbf{G}}(\mathbf{B}, \mathbf{C}) \subseteq (\leftthreetimes_{\mathbf{G}}(\rightthreetimes_{\mathbf{G}}(\mathbf{A})) \cup (\leftthreetimes_{\mathbf{G}}(\rightthreetimes_{\mathbf{G}}(\mathbf{B})))$.

Now, let us suppose that $K_{\mathbf{G}}$ is $\cup$-additive and that $\mathbf{A} \subseteq \mathcal{P}(X)$ is a monogenic $K^*_{\mathbf{G}}$-closed set i.e. there exists $S \in \mathcal{P}(N)$ such that $K^*_{\mathbf{G}}(\mathbf{A}) = \mathbf{A} = \leftthreetimes_{\mathbf{G}}(\{S\})$.

We have to show that $\mathbf{A}$ is prime. In order to prove that, consider any pair $\mathbf{B}, \mathbf{C}$ of $K^*_{\mathbf{G}}$-closed sets such that

$\mathbf{A} \supseteq \mathbf{B} \cap \mathbf{C}$.Then,

$\rightthreetimes_{\mathbf{G}}(\mathbf{A}) \subseteq \rightthreetimes_{\mathbf{G}}(\mathbf{B} \cap \mathbf{C})$, by antitonicity, or equivalently,

$$\rightthreetimes_{\mathbf{G}}(\mathbf{A}) \subseteq \rightthreetimes_{\mathbf{G}}(\leftthreetimes_{\mathbf{G}}(\rightthreetimes_{\mathbf{G}}(\mathbf{B})) \cap \leftthreetimes_{\mathbf{G}}(\rightthreetimes_{\mathbf{G}}(\mathbf{C}))).$$

It follows- from $(*)$- that

$$\rightthreetimes_{\mathbf{G}}(\mathbf{A}) \subseteq \rightthreetimes_{\mathbf{G}}(\leftthreetimes_{\mathbf{G}}(\rightthreetimes_{\mathbf{G}}(\mathbf{B})) \cup \rightthreetimes_{\mathbf{G}}(\leftthreetimes_{\mathbf{G}}(\rightthreetimes_{\mathbf{G}}(\mathbf{C}))).$$

But $\rightthreetimes_{\mathbf{G}} = \rightthreetimes_{\mathbf{G}} \circ \leftthreetimes_{\mathbf{G}} \circ \rightthreetimes_{\mathbf{G}}$(and $\leftthreetimes_{\mathbf{G}} = \leftthreetimes_{\mathbf{G}} \circ \rightthreetimes_{\mathbf{G}} \circ \leftthreetimes_{\mathbf{G}}$) by an elementary property of Galois connections.

Hence, $\rightthreetimes_{\mathbf{G}}(\mathbf{A}) \subseteq \rightthreetimes_{\mathbf{G}}(\mathbf{B}) \cup \rightthreetimes_{\mathbf{G}}(\mathbf{C})$. Since, by definition of $\mathbf{A}$, $\rightthreetimes_{\mathbf{G}}(\mathbf{A}) = \rightthreetimes_{\mathbf{G}}(\leftthreetimes_{\mathbf{G}} \{S\}) = K_{\mathbf{G}}(\{S\}) \supseteq \{S\}$,

it follows that $S \in \rightthreetimes_{\mathbf{G}}(\mathbf{B}) \cup \rightthreetimes_{\mathbf{G}}(\mathbf{C})$. Let us assume w.l.o.g. that $S \in \rightthreetimes_{\mathbf{G}}(\mathbf{B})$, that is $\{S\} \subseteq \rightthreetimes_{\mathbf{G}}(\mathbf{B})$.

Then, by antitonicity of $\leftthreetimes_{\mathbf{G}}$, $\mathbf{B} = \leftthreetimes_{\mathbf{G}} \rightthreetimes_{\mathbf{G}}((\mathbf{B})) \subseteq \leftthreetimes_{\mathbf{G}}(\{S\}) = \mathbf{A}$. Therefore, $\mathbf{A}$ is indeed prime.

Conversely, let us suppose that any monogenic $K^*_{\mathbf{G}}$-closed set is prime.

Consider now two $K^*_{\mathbf{G}}$-closed sets $\mathbf{A}, \mathbf{B}$, and let $S \in \curlywedge_{\mathbf{G}}(\mathbf{A} \cap \mathbf{B})$. Posit $\mathbf{C} = \curlyvee_{\mathbf{G}}(\{S\})$.

Since $\curlyvee_{\mathbf{G}} = \curlyvee_{\mathbf{G}} \circ \curlywedge_{\mathbf{G}} \circ \curlyvee_{\mathbf{G}}$it follows that

$K^*_{\mathbf{G}}(\mathbf{C}) = \curlyvee_{\mathbf{G}}(\curlywedge_{\mathbf{G}}(\mathbf{C})) = \curlyvee_{\mathbf{G}}(\curlywedge_{\mathbf{G}}(\curlyvee_{\mathbf{G}}(\{S\}))) = \curlyvee_{\mathbf{G}}(\{S\}) = \mathbf{C}$

hence $\mathbf{C}$ is a monogenic $K^*_{\mathbf{G}}$-closed set.

Moreover, by antitonicity of $\curlyvee_{\mathbf{G}}$, $\mathbf{C} \supseteq \curlywedge_{\mathbf{G}}(\curlyvee_{\mathbf{G}}(\mathbf{A} \cap \mathbf{B})) = \mathbf{A} \cap \mathbf{B}$ (recall that the meet of two closed set must

be a closed set). Thus, $\mathbf{C} \supseteq \mathbf{A}$ or $\mathbf{C} \supseteq \mathbf{B}$ since $\mathbf{C}$ is prime, by hypothesis. Let us assume w.l.o.g. that $\mathbf{C} \supseteq \mathbf{A}$.

Then, $\curlywedge_{\mathbf{G}}(\mathbf{C}) \subseteq \curlywedge_{\mathbf{G}}(\mathbf{A})$ by antitonicity of $\curlywedge_{\mathbf{G}}$. Therefore, $\{S\} \subseteq K_{\mathbf{G}}(\{S\}) = \curlywedge_{\mathbf{G}}(\curlyvee_{\mathbf{G}}(\{S\})) = \curlywedge_{\mathbf{G}}(\mathbf{C}) \subseteq \curlywedge_{\mathbf{G}}(\mathbf{A})$.

It follows that

$$(**) \qquad \curlywedge_{\mathbf{G}} (\mathbf{A} \cap \mathbf{B}) \subseteq \curlywedge_{\mathbf{G}}(\mathbf{A}) \cup \subseteq \curlywedge_{\mathbf{G}}(\mathbf{B}).$$

Notice, however, that $(**)$ entails $(*)$ above since $\curlyvee_{\mathbf{G}} = \curlyvee_{\mathbf{G}} \circ \curlywedge_{\mathbf{G}} \circ \curlyvee_{\mathbf{G}}$implies that $\curlyvee_{\mathbf{G}}$-images are in fact $K^*_{\mathbf{G}}$-closed sets. As a result, $K_{\mathbf{G}}$ turns out to be $\cup$-additive, as required. ∎

The class of topology-inducing CGFs as characterized above include, but does not reduce to, the class of linear CGFs. To see this, consider the following example of a CGF with a topological closure system, which is not a chain.

**Example 8** *Let* $\mathbf{G} = (N, X, E)$ *with*

$N = \{1, 2, 3, 4, 5, 6\}, X = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6\}$,

*and* $E$ *defined as follows. For any* $S \subseteq N$,*and* $A \subseteq X$, $A \in E(S)$ *iff one of the following clauses is satisfied:*

*i) either* $S \supseteq \{1, 3\}$ *or* $S \supseteq \{1, 5\}$, *and* ($B \supseteq X \setminus \{x_1, x_3\}$,
*or* $B \supseteq X \setminus \{x_1, x_5\}$, *or* $B \supseteq X \setminus \{x_0, x_1\}$, *or* $B \supseteq X \setminus \{x_0, x_2\}$.

*ii) either* $S \supseteq \{2, 4\}$ *or* $S \supseteq \{2, 6\}$, *and* ($B \supseteq X \setminus \{x_2, x_4\}$,
*or* $B \supseteq X \setminus \{x_2, x_6\}$, *or* $B \supseteq X \setminus \{x_0, x_1\}$, *or* $B \supseteq X \setminus \{x_0, x_2\}$.

*iii)* $S = N$ *and* $B \neq \varnothing$.

*iv)* $S \neq \varnothing$ *and* $B = X$.

*It is easily checked that* $\mathbb{L}(\mathbf{G}) = \mathbf{2} \oplus \mathbf{2}^2 \oplus \mathbf{2}$, *hence definitely not a chain. However the only meet-reducible* $K_{\mathbf{G}}$*-closed set is*

$\mathbf{S} = \{S \subseteq N : S \supseteq \{1, 3\}, \text{ or } S \supseteq \{1, 5\}, \text{ or } S \supseteq \{2, 4\}, \text{ or } S \supseteq \{2, 6\}\}$, *which is not monogenic. Therefore,* $\mathbb{L}(\mathbf{G})$ *amounts to a topological closure system.*

**Remark 9** *It should be noticed that, generally speaking, closure systems of topology-inducing CGFs do* not *satisfy even* $T_0$, *the weakest of classical separation axioms. However, when a CGF is* purified *then the resulting topological closure systems is* $T_0$, *e.g. for any pair* $S, T$ *of distinct subsets of* $N$ *there exists a* $K_{\mathbf{G}}$*-closed set* $\mathbf{U}$ *such that* $S \in \mathbf{U}, T \notin \mathbf{U}$. *This is so, because when* $\mathbf{G} = (N, X, E)$ *is purified, for any distinct* $S, T \subseteq N$, $\curlyvee_{\mathbf{G}}(\{S\}) \neq \curlyvee_{\mathbf{G}}(\{T\})$ .

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