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Abstract

From a given directed weighted network of knowledge links between technology fields, the paper develops a multisector dynamic model of incremental innovation and R&D activity in these fields. The model is focused on the equilibrium share distribution of these variables, which is proved to be locally stable, with reference to a simple low dimensional case. Simulation methods suggest that local, and also global, stability extend to any model dimension. It is also shown how different network structures map to different asymptotic share distributions. Using the NBER patents and patent citation data files, the analytical framework is then used to analyse some general features of the pattern of knowledge creation and transfer in the period 1975-1999. From a descriptive viewpoint, the changes in the share distribution of innovation activity predicted by the model match reasonably well the actual changes in the period.

Key Words*:* directed weighted network, knowledge spillovers, share distribution, incremental innovation and R&D dynamics, local stability, simulation, patents and patent citations.
JEL Classification*:* O30, C61

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1 Introduction

A growing body of literature in the historical and theoretical analysis of technological innovation points to the conclusion that technological search activity typically operates on 'rugged' landscapes. This takes place as a result of the relatively large number of the component ideas that define a solution in a technology field, and of the fact that the way in which performance in this field depends on the combined configuration of the components, may well differ from the influence that the same configuration exerts on performance in other fields. The potential tendency towards rising complexity resulting from innovation accumulation (Kauffman [9], Caminati [1]) is however countered by a selection for modularity. The modular design of the technological knowledge achieves and preserves the near-decomposability of problem spaces into subspaces of relatively small dimension. Notably, a specific technological lineage, such as electricity based lighting and motive power, has a specific modular pattern, or hierarchy associated with it. This hierarchy is a crucial factor explaining the ability of the given lineage to preserve its evolvability, avoiding premature 'lock-in' on a poor local optimum of the rugged landscape.

The first and major premise of this paper is that the hierarchic pattern of the ruling technology system, understood as the network topology of its constituents components, is mirrored in the organisation of the knowledge flows between its diverse technology fields. Every change in the former corresponds to a change in the latter and vice versa. Historical instances of the change in question can be associated with the advent of macroinventions (Mokyr [10, chapter10]) introducing a completely new lineage, and entailing the replacement of a pre-existing hierarchy with a new one. As in the case of electricity, the change in design pertains not only to the organisation of productive processes understood as ways of combining different pieces of hardware and other productive inputs,[1] but also to the organisation of technology understood as disciplinary knowledge (Rosenberg [14]). This implies that a relevant dimension of the change has to do with re-designing the key learning interfaces of the emerging knowledge hierarchy. The transition does not pose serious problems, and negative side effects on productivity growth can be avoided, if the new structure is smoothly integrated into the old. Problems are more serious if the replacement between hierarchies requires a structural change of a more discontinuous kind.

[1] Holding to this interpretation, Devine [4] referred to the design change imposed by the electric power drive using the catch phrase 'from shafts to wires'.

This paper is motivated by the goal of clarifying some points pertaining to the way in which the pattern of knowledge flows between technology fields affects knowledge accumulation. In this way, it will also try to shed some light on the smooth vs. discontinuous type of transition between modular hierarchies. Finally, it will consider the influence that a given knowledge hierarchy, or pattern, has on the long term distribution of R&D activity across technology fields.

Let us consider a hierarchy of modular units connected by directed links. In what follows, each modular unit is interpreted as a technology field. The directed link represents a well designed interface standard, and the associated interpretation is that information from the source module is ready for useful recombination in the target module. Directed links can be also interpreted as learning interfaces, such that target modules receive innovation opportunities, or knowledge spillovers, from the source modules. It is possible to think of the modular hierarchy as a network connecting a set of technology fields, each representing a modular unit. Most important in the network are the 'core' units of the structure. In a way strongly reminiscent of Sraffa's basic commodities [17], they are defined by the property that there is a directed sequence of links connecting every core module to every other core module and to every other module in the network. We shall refer to this defining property of the core as its 'closed-path connectivity', in the sense that there is at least one closed path of information links connecting the modules in the core and the core is itself connected to every other module of the network.

Closed-path connectivity enables the core to be the centre of cumulative, self-sustaining positive feed-back mechanisms of information generation and transmission. Modules that do not share this property may be called 'periphery' modules for convenience. Characteristically, the topological structure of network connections between core and periphery modules is asymmetric, since the former send direct or indirect information links to the latter, but the converse is not true. Since knowledge creation in the core is self sustaining and sends ideas ready for useful recombination to periphery modules, the core modules are those that collectively determine the asymptotic rate of knowledge growth of the connected network. Moreover, if the ability to achieve a self sustaining growth of innovation opportunities in the core depends on the qualitative, topological property of closed path connectivity, the aggregate measure of this connectivity depends on the intensity of links in the core, which is a property of the corresponding weighted network. Among the possibly multiple core structures in the network, the distribution of innovation activity is ultimately shaped by the 'dominant' core structure, achieving the highest measure of connectivity.

In this general setting we define an incremental dynamics in which the number, N, of technologies and their weighted network structure of connections is constant, and a structural dynamics in which the number of technologies and their topological or weighted network structure is allowed to change between every iteration of the former. The structural model, to be developed in a future paper, will consider the exogenous and endogenous factors that may be held responsible for the sequential formation/deletion, weakening/strengthening of knowledge links and of the long-term evolution in network structure.

In this paper we focus exclusively on the incremental dynamics, which is driven by the opportunities for incremental innovations and by the flows of R&D activity. We shall study how the asymptotic share distribution of the knowledge stocks and of the R&D effort across technology fields is determined by a given weighted network structure. We shall also consider, and give examples, of the way in which different forms of that structure map onto diverse asymptotic share distributions. The changes in distribution induced by a sequence of radical and network innovations, come to depend on the extent to which the mentioned global network properties are robust with respect to local perturbations induced by such innovations.

In this context, we shall outline simple, toy-model examples of continuity and discontinuity in the design of the learning interfaces. For instance, radical innovations may simply add components and links to the pre-existing dominant core structure and to its periphery. In this way, the old structure is simply embedded in the new one; the old network topology survives as part of the new hierarchy of modular units. There is wider scope for specialisation (modularity) within the larger structure and wider scope for self-sustaining positive feedback mechanisms in the expanded 'core'. In such instances, although radical innovations will imply a change in some 'key learning interface', such changes may be designed through 'on-line' adaptation by a local competence base. In these instances, the local changes of the 'key learning interfaces' may easily lead to higher rates of knowledge creation, and it may be said with Pavitt [12, p. 442] that the new pattern, or paradigm, as Pavitt puts it, simply adds to the previous knowledge structure. A case of technological discontinuity is offered by radical innovations entailing the decomposition and replacement of the pre-existing dominant core structure. The formation of a new (dominant) core around the emerging technology fields may be a lengthy and resource consuming process, if it requires the construction of learning interfaces according to new design principles that impose a global change of the knowledge hierarchy. Paul David [3] mentions the drastic change in design principles imposed by the advent of electricity, and the change of 'key learning interfaces' in the user sectors caused by the information and communication technologies.

The paper is organised as follows. Section 2 develops a simple theoretical model of incremental innovations. Section 3 contains some numerical examples that illustrate the dynamics of the model under different hypotheses about the network connections. Then, the analytical tools envisioned in the first sections of the paper are made operational in Section 4, where we try to detect from U.S.A. patent and patent-citation data, the pattern of knowledge transfer between technology fields during the last decades of the XXth century. We shall also consider how the acceleration in the information-communication revolution is reflected in this pattern and in the distribution of innovation activity across technology fields.[2] Section 5 concludes.

2 Spillover-network and the distribution of R&D: a simple model of incremental innovations

2.1 The building blocks: an heuristic presentation

We think of the technological knowledge available in the economy as subdivided into different technology fields, or simply fields. The word technology will refer in the sequel to the union of such fields. Innovations can be incremental, radical, or network. Radical innovations are understood as exogenous events affecting the number and quality of technology fields. Network innovations are changes in the matrix describing the active cross-field learning interfaces and the strength of these knowledge connections. Incremental innovations are those additions to the knowledge stock originating in a specific field that do not affect the set of available fields or the design of cross field interactions.

[2] A few words of qualification are necessary in this respect. The possibility of examining the correlation between the modes of structural evolution outlined above, with historical processes backed by empirical data, is conditioned by the operational definition of a technology field imposed by computation requirements, or embodied in the data available. If the definition of a technology field is not thin enough, we shall hardly be able to detect structural changes in the directed graph describing the topology of knowledge flows. Under a manageable definition of a technology field, it may be the case that the resulting aggregate representation of the directed graph is constant, or nearly so, in spite of the structural changes that are taking place in reality, and that only a finer representation would capture. Still, the changes in question may be signalled by quantitative changes in the flows of knowledge transfer, giving rise to a new weighted network, within a relatively constant directed network topology.

The average number of incremental innovations per unit of time in a given field depends on two main factors: in the first place, the set of innovation opportunities available in that field, in the second place the innovation effort in the same field. In this paper we assume that innovation opportunities are primarily determined by the progressive local knowledge base. This consists of the subset of ideas that are known by R&D laboratories currently operating in the given technology field and that are potentially conducive to useful recombinations and developments leading to new disciplinary knowledge. Under a recombinant interpretation of knowledge growth (Reiter [13], Weitzman [18]), the progressive knowledge base can be regarded as the repertoire of recombination possibilities from which innovations will originate. So defined, the progressive local knowledge base partly consists of ideas originated from past innovations in the same technology field, but will also partly consist of ideas originated from past innovations in other fields and that are made available to the field in question by the knowledge interfaces that are currently active across technologies. Again, what kind of interfaces are or are not active in a given situation, and the degree of their activation, is partly shaped by the specific kind of technological knowledge available, but, not less importantly, is also shaped by the nature of institutions. It follows from the above interpretation that network innovations in this context may have to do with institutions not less than with technology.

To gain a better understanding of this point, it is worth stressing that the network links envisaged in this paper are *directed* links connecting technology fields. This marks a sharp difference with respect to the network models of innovation and knowledge diffusion prevailing in the literature (see Cowan [2] for an insightful review and related bibliography). The standard network models of innovation study the way in which knowledge growth and diffusion is affected by the network topology of knowledge *interactions between economic agents*, mostly firms, or innovators. This has two immediate consequences. Since the links in the network represent interactions, they are typically undirected links. If agents a and b are connected, then the link between them elicits a potential knowledge flow from a to b, from b to a, or both, depending on circumstances. Also, it is within the power of economic agents to decide to change their knowledge interactions. The possibly changing structure of micro interactions between economic agents, and the associated emergence of distinct network topologies, interpreted as different institutional environments, is not in the focus of this paper. However, far from being irrelevant, this micro structure is an exogenous institutional factor that contributes to explaining the degree of activation, or strength, of the cross-field learning interfaces considered in the paper. In other words, the existence of a directed link from j to i, because knowledge created in field j is relevant to field i, is largely shaped by technological factors. But the extent to which field i will be in a position to exploit the knowledge created in j will also depend on the nature of the exogenously evolving institutions, among them, the prevailing network topologies of micro interactions.

The present study will proceed on the bold assumption that the qualitative influence of knowledge patterns on the distribution of incremantal innovations, which is the focus of this paper, can be approximated by a deterministic process. Moreover, innovation effort is assumed to depend on innovation opportunities. To the extent that the latter are made to depend on the progressive subset of the local knowledge base, innovation effort is likewise affected by changes in this subset. The emphasis on the available knowledge-input supply in the process of knowledge creation makes our treatment of innovation effort seemingly oblivious to Schmookler's emphasis [16] on the relation between R&D investment and demand. Although this may be partly the case, it should be added that in a characteristic Schumpeterian framework, the innovating firm creates its own demand. Demand expectations are largely shaped by innovation opportunities.

2.2 A formal outline

We consider an economy with a finite set $S = \{1, ..., n\}$ of known technology fields, to study how the network of knowledge links affects the *distribution* $\{Q_j\}$, $j = 1, ..., n$, of the R&D effort across the n fields over time. A field j is here understood as a (possibly infinite) set T_j of potential configurations, or designs. The technological state of the economy is defined by $\{G(S, L, C), A\}$. $G(S, L, C)$ is a weighted directed graph, with a set S of nodes, that are here interpreted as technology fields, a set L of directed knowledge links between these nodes, and a connection matrix C of weights, or intensity coefficients, attached to the links in question. c_{ij} is the strength of the directed link from j to i. It is a measure of the extent to which ideas developed in sector j are relevant to R&D in sector i, in the sense that A_j expands the knowledge base of the latter.[3] We can safely assume that some of the knowledge produced by past innovations in one field is always relevant to R&D activity in the same field, that is, $c_{ii} > 0$, $i = 1, ..., n$.

The weighted directed graph $G(S, L, C)$ can be understood to derive from the corresponding unweighted graph $G(S, L)$ in the sense clarified by

[3] In our interpretation, $c_{ij} = 1$ if every idea developed in field j is a relevant knowledge input to R&D activity aimed at developing a new idea in field i. Since ideas are non rival, it may well be the case that $c_{ij} > 1$.

the following definition:

Definition 1 *Let $\tilde{C}$ be the $n \times n$ adjacency matrix of the directed graph $G(S,L)$. The connection matrix C of the weighted directed graph $G(S,L,C)$, associated with $G(S,L)$, is obtained multiplying every element $\tilde{c}_{ij}$ of $\tilde{C}$ by the corresponding measure $c_{ij} \geq 0$. $c_{ij} = 0$ if and only if $\tilde{c}_{ij} = 0$, so that $\tilde{c}_{ij} = 1$ implies $c_{ij} > 0$, $i = 1, ..., n$. For the reason explained above, the diagonal elements of C are assumed to be strictly positive.*

$(A_1, ..., A_n)'$ (here and throughout the paper, the symbol $'$ is the transpose operator) is the column vector of knowledge stocks and A_j, $j = 1, ..., n$, is the number of designs of j that are known in the present state. The discovery which brings j in the set S of known technologies, brings also the knowledge stock A_j to its lower bound $A_j = 1$; after that, A_j grows as a result of the cumulative flow of incremental-innovation arrivals in the technology field j, which, for the sake of simplicity is modelled as a continuous process.

Aggregate R&D effort $\chi = \sum_j Q_j$ is not explained by the model. It is assumed to grow at the exogenous exponential rate γ. Moreover, the number n of technologies and application domains is assumed constant in this section, but will be allowed to change in the examples that follow, as a result of radical innovations. A radical innovation is understood as a change in S, and a network innovation as a change in L, C, or both. The introduction of these changes yields a 'structural' model, dealing with the effects induced on the asymptotic incremental dynamics by recurrent 'structural' perturbations.

Consistent with the basic premise of the paper that the flow of useful innovations in sector i depends on the repertoire of available ideas that are the 'building blocks' of R&D in this sector, the stock A_i, $j = 1, ...n$, evolves according to the differential equation:

$$\dot{A}_i = \sigma \frac{Q_i}{A_i} \sum_j c_{ij} A_j = \sigma Q_i p_i(A) \tag{1}$$

where σ is a parameter, Q_i/A_i is effective R&D effort in field i, $p_i(A)$ is the function $\sum_j c_{ij} A_j / A_i$ with $0 \leq c_{ij}$.

Several points concerning expression (1) are worth stressing. The innovation flow depends on the effective R&D effort Q_i/A_i , rather than the absolute effort Q_i, to allow for the fact that a larger stock A_i makes innovation in i more complex, hence more R&D intensive. c_{ij} is the generic element of the $n \times n$ matrix C. Replacing every $c_{ij} > 0$ in C with 1, and leaving every zero element of C unchanged, we obtain the adjacency matrix $\tilde{C}$ of the directed graph $G(S, L)$.

The R&D effort of sector i, namely Q_i,changes according to the dynamic equation:

$$\dot{Q}_i = \left[\rho\left(p_i - \frac{1}{n}\sum_j p_j\right) + \gamma\right] Q_i \tag{2}$$

Let $a_i = A_i / \sum_j A_j$, and $r_i = Q_i/A_i$. Then

$$p_i(A) = \frac{\sum_j c_{ij} A_j}{A_i} = \frac{\sum_j c_{ij} a_j}{a_i} = f_i(a) \tag{3}$$

where $a = (a_1, ..., a_n)'$ and $f(a) : K \to R^n_+$. Notice that a is so defined that it belongs to the $n-1$ dimensional simplex K in R^n, that is, $0 \leq a_i \leq 1$, $\sum_j a_j = 1$. Indeed, since $A_i \geq 1$, a_i will approach zero if and only if $\sum_j A_j$ goes to infinity with A_i finite. From (1) and (2) we obtain:

$$\dot{a}_i = \sigma\left[r_i \sum_j c_{ij} a_j - a_i \sum_h r_h \sum_j c_{hj} a_j\right] \tag{4}$$

$$\dot{r}_i = r_i\left[\gamma + (\rho - \sigma r_i) f_i(a) - \frac{\rho}{n}\sum_h f_h(a)\right] \tag{5}$$

The following notation is now introduced: for every row n-dimensional vector of real variables $(x_1, x_2, ..., x_n)$, the corresponding label x is the column vector $(x_1, x_2, ..., x_n)'$ and X is the diagonal matrix with the elements $x_1, x_2, ..., x_n$ on its main diagonal; moreover, z is the n dimensional unit column-vector $(1, 1, ..., 1)'$. Now for the column vectors $a = (a_1, ..., a_n)'$, $r = (r_1, ..., r_n)'$ we generate the corresponding diagonal matrices A, R. From the equations above we obtain the system of non-linear differential equations:

$$\dot{a} = \sigma\left[RCa - ar'Ca\right] \tag{6}$$

$$\dot{r} = R\left[(\rho I - \sigma R) f(a) + z\left(\gamma - \frac{\rho}{n} z' f(a)\right)\right] \tag{7}$$

Our primary goal in this section is to study the system dynamics and to relate it to the topological structure of the matrix C.

Proposition 1 *Let* a^* *be the right eigenvector of* C *associated with the Perron-Frobenius eigenvalue* λ^**. Using a genericity argument, we can safely assume that* λ^* *has multiplicity* 1 *(Hirsch and Smale [6, pp. 153-157]).* (a^*, r^*) *is a stationary state of equations (6)-(7), where* r^* *is defined as follows. (i) If* $a^* > 0$, $r^* = (\gamma/\sigma\lambda^*)z$*. (ii) If* $a^* \geq 0$*, let* $n_1 < n$ *be the number of strictly positive components of* a^**,and* $n_2 = n - n_1$*.Since in this case* C *is reducible,there exists a permutation matrix* P *such that:*

$$PCP' = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix}$$

Here C_{11} *is a* $[n_1 \times n_1]$ *non negative matrix,* C_{22} *is a* $[n_2 \times n_2]$ *non negative matrix, and* $\lambda^*(C) = \lambda^*(C_{11}) > \lambda^*(C_{22})$*.On the simplifying assumption that the right eigenvector of* C_{22} *associated with* $\lambda^*(C_{22})$ *is strictly positive, we can write:*

$$r_i^* = \frac{\rho f_i(a^*) + \gamma - (\rho/n)\,(n_1\lambda^* + n_2\lambda^*(C_{22}))}{f_i(a^*)\sigma} \tag{8}$$

where, $f_i(a^*) = \lambda^*$*, if* $a_i^* > 0$*, and* $f_i(a^*) = \lambda^*(C_{22})$ *if* $a_i^* = 0$*. If the Perron-Frobenius eigenvector of* C_{22} *is not strictly positive, we can define* r_i^* *by iterating the argument above.*

Conjecture 1 *For generic initial conditions* (a, r) *such that* a *is in the relative interior of* K*, and* $r > 0$*, the dynamics of (6)-(7) converges to the fixed point* (a^*, r^*) *defined by the proposition above. On the generic assumption that* λ^* *has multiplicity* 1*,* (a^*, r^*) *is the unique stable attractor of equations (6)-(7).*

To illustrate the conjecture above we consider the following case:

Example 1

$$C = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 & c_{14} \\ c_{21} & c_{22} & 0 & 0 \\ 0 & 0 & c_{33} & c_{34} \\ 0 & 0 & c_{43} & c_{44} \end{bmatrix}$$

where $c_{11} = c_{22} = c_{33} = c_{44} = 1$, $c_{12} = c_{21} = 0.5$, $c_{34} = c_{43} = 0.2$ *and* $c_{14} = 0.4$*. Here* $\lambda^*(C) = \lambda^*(C_{11}) = 1.5 > \lambda^*(C_{22}) = 1.2$*. With* $\rho = 0.05$, $\sigma = 0.05$ *and* $\gamma = 0.1$*, the predicted attractor is* (a^*, r^*)*, where (see Fig. 1):*

$$a^* = (0.5, 0.5, 0, 0)'$$

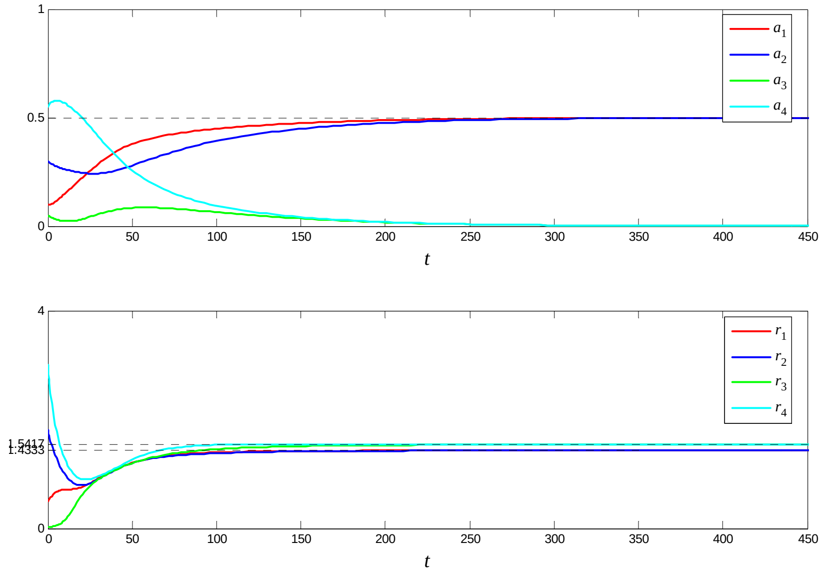

Figure 1: Convergence to the equilibrium in Example 1

and:

$$r_1^* = r_2^* = \frac{(1.5)\rho + \gamma - (\rho/2)\,(2.7)}{(1.5)\sigma} \simeq 1.4333$$

$$r_3^* = r_4^* = \frac{(1.2)\rho + \gamma - (\rho/2)\,(2.7)}{(1.2)\sigma} \simeq 1.5417$$

2.3 Local analysis

The elements of the Jacobian matrix $J(a, r)$ of system (6)-(7) evaluated at (a^*, r^*), such that $a^* > 0$, are easily computed recalling that $f_i(a^*) = \lambda^*$, $i = 1, ..., n$:

$$\left.\frac{\partial \dot{a}_i}{\partial a_i}\right|_{(a^*,r^*)} = \left(\frac{\gamma}{\lambda^*}\right)\left[c_{ii} - \lambda^* - a_i \sum_h c_{hi}\right] \qquad i = 1, ..., n \tag{9}$$

$$\left.\frac{\partial \dot{a}_i}{\partial a_j}\right|_{(a^*,r^*)} = \left(\frac{\gamma}{\lambda^*}\right)\left[c_{ij} - a_i \sum_h c_{hj}\right] \qquad i,\ j = 1, ..., n,\ j \neq i \tag{10}$$

$$\left.\frac{\partial \dot{a}_i}{\partial r_i}\right|_{(a^*,r^*)} = \sigma\lambda^* a_i (1-a_i) \qquad i=1,...,n \tag{11}$$

$$\left.\frac{\partial \dot{a}_i}{\partial r_j}\right|_{(a^*,r^*)} = -\sigma\lambda^* a_i a_j \qquad i,\ j=1,...,n \qquad j\neq i \tag{12}$$

$$\left.\frac{\partial \dot{r}_i}{\partial a_i}\right|_{(a^*,r^*)} = \left(\frac{\gamma}{\sigma\lambda^*}\right)\left[\left(\rho-\frac{\gamma}{\lambda^*}\right)\left(\frac{c_{ii}-\lambda^*}{a_i}\right)-\frac{\rho}{n}\left(\sum_h \frac{c_{hi}}{a_h}-\frac{\lambda^*}{a_i}\right)\right] \tag{13}$$

$$i = 1,...,n$$

$$\left.\frac{\partial \dot{r}_i}{\partial a_j}\right|_{(a^*,r^*)} = \left(\frac{\gamma}{\sigma\lambda^*}\right)\left[\left(\rho-\frac{\gamma}{\lambda^*}\right)\frac{c_{ij}}{a_i}-\frac{\rho}{n}\left(\sum_h \frac{c_{hj}}{a_h}-\frac{\lambda^*}{a_j}\right)\right] \tag{14}$$

$$i,\ j = 1,...,n,\ j\neq i$$

$$\left.\frac{\partial \dot{r}_i}{\partial r_i}\right|_{(a^*,r^*)} = \left[\gamma+\lambda^*\left(\rho-\frac{\gamma}{\lambda^*}\right)-\frac{\rho}{n}n\lambda^*\right]-\frac{\gamma}{\sigma\lambda^*}\sigma\lambda^* = -\gamma \tag{15}$$

$$i = 1,...,n$$

$$\left.\frac{\partial \dot{r}_i}{\partial r_j}\right|_{(a^*,r^*)} = 0 \qquad i,\ j=1,...,n \qquad j\neq i \tag{16}$$

Since a^* is strictly positive, all the elements of $J(a^*,r^*)$ are well defined. The evaluation of $J(a^*,r^*)$, for the more cumbersome case in which at least one component of a^* is zero, is still to be completed.

Direct computation with $n=2$ (see Appendix A) shows that, if a^* is strictly positive, the real part of every eigenvalue of $J(a^*,r^*)$ is negative and the fixed point of system (6)-(7) is locally stable.

2.4 More results and definitions

Remark 1 *Using the result that the Perron-Frobenius eigenvalue λ^* having multiplicity* 1 *is a generic property of the connection matrix C, in what follows we ignore the possibly more complicated non-generic cases in which the multiplicity of λ^* is larger than* 1*.*

Definition 2 *A autocatalytic set (ACS) is a subgraph of $G(S,L)$ such that each vertex in the subgraph has at least one incoming link from some vertex of the subgraph (Jain and Krishna [7]). Notice that our assumption $c_{ii}>0$, $i=1,...,n$, implies that $G(S,L)$ has n trivial ACSs. The* dominant ACS *of $G(S,L)$ is its largest subgraph with the property that the associated connection matrix C^* satisfies $\lambda^*(C)=\lambda^*(C^*)$.*

Definition 3 *The subset $S^*\subseteq S$ of the vertices corresponding to the positive* *components of a^*, together with the subset $L^* \subseteq L$ of the links between them, is the subgraph $G(S^*, L^*)$ corresponding to the attractor a^*.*

Proposition 2 *Since $\lambda^* > 0$, $G(S^*, L^*)$ is the dominant ACS of $G(S, L)$.*

Considering the case where $a_i^* > 0$, and substituting the stationary value of r_i^* into equation (1), we obtain the result that in dynamic equilibrium the variables A_i and Q_i grow at the exogenous rate γ. Still, every radical innovation changing the structure of the C matrix in a way that increases the Perron-Frobenius eigenvalue λ^* causes a persistently higher ratio between the level of R&D effort and its cumulated effects on knowledge. For the same set of initial conditions concerning R&D effort, the considered change in the C matrix would exert persistent 'level effects' on the knowledge stock.

Using the standard results on non-negative matrices, we remark that a^* is a strictly positive vector if the connection matrix C is indecomposable, but may have some zero components if C is decomposable.

3 Selected examples

3.1 Core and periphery of the dominant autocatalytic set (ACS)

Example 2 *In this example we illustrate the notion of dominant ACS, and of 'core and 'periphery' of the ACS. To fix our ideas we consider a simple case with $n = 4$, in which C admits the following block decomposition, where $\lambda^*(C) = \lambda^*(C_{11}) > \lambda^*(C_{22})$:*

$$C = \begin{bmatrix} c_{11} & c_{12} & 0 & 0 \\ c_{21} & c_{22} & 0 & 0 \\ c_{31} & 0 & c_{33} & 0 \\ 0 & 0 & 0 & c_{44} \end{bmatrix} = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix}$$

It is understood that the elements c_{ij} of C that are not explicitly fixed equal to 0 are strictly positive. The directed-graph structure corresponding to this example is shown in Fig. 2, where vertices correspond to technologies and directed links to spillovers across technologies.

Since the block C_{11}does not send links to the block C_{22}, and $\lambda^*(C) = \lambda^*(C_{11}) > \lambda^*(C_{22})$, the dominant ACS $G(S^*, L^*)$ consists of the vertices 1, 2, 3 and of the links between them. Moreover, we can distinguish within $G(S^*, L^*)$ between two structures. One is the *core* subgraph of the dominant ACS, which is formed by vertices 1 and 2, and the links between them, with

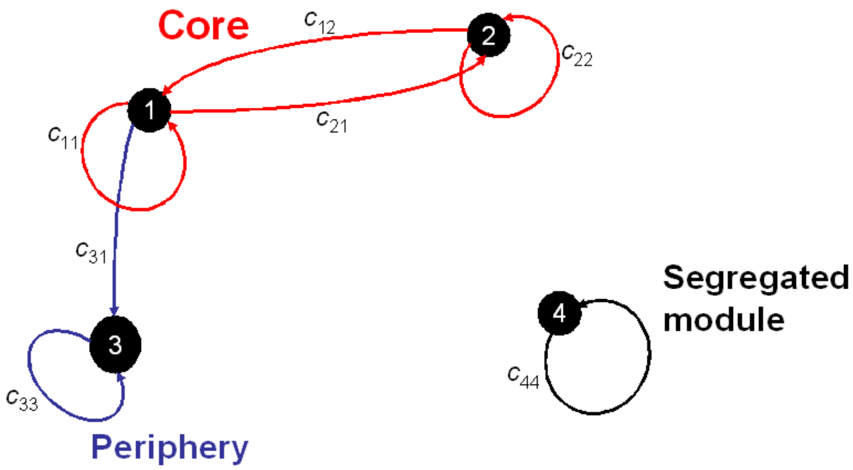

Figure 2: The directed-graph structure corresponding to Example 2

the defining property that starting from any vertex of the core, any other vertex of the autocatalytic set can be reached following a sequence of directed links. For the sake of later reference, this defining property of the core is labelled *closed path connectivity*. Vertices in the autocatalytic set that do not belong to its core, belong to its *periphery*. In our example the periphery of $G(S^*, L^*)$ consists of vertex 3, together with the link from vertex 3 to itself.

$\lambda^*(C)$ is a monotonic, strictly increasing function of the connection parameters c_{ij} attached to the links between the vertices in the core of $G(S^*, L^*)$. It is independent of the connection parameters c_{ij} attached to links outside the core, in particular, attached to the links sent from the core of $G(S^*, L^*)$ to its periphery, or from the periphery to itself (c_{31} and c_{33} in our example).

As a matter of interpretation, the directed graph and the C matrix corresponding to Example 2 can be viewed as the outcome of the coupling between a radical and a network innovation acting upon a pre-existing structure described by the C_{11} block referred to above. The radical innovation gives rise to the technology field 4 (vertex 4 in the graph of Fig. 2) and the network innovation to the connection parameter c_{44}. At the stage described in Example 2, field 4 is isolated from the rest, because the design principles embodied in its basic ideas are at variance with the design principles of the pre-existing technology. The following examples will consider two different ways in which the creation of the new technology field 4 may give rise to a

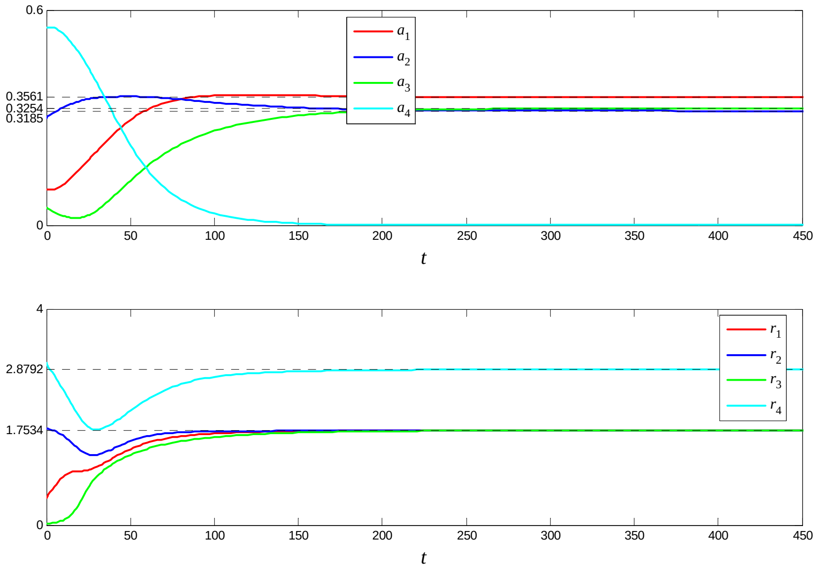

Figure 3: Convergence to the fixed point in Example 2

modified technology pattern for the economy.

3.1.1 Simulation of Example 2

With $\rho = 0.05$, $\sigma = 0.05$, $\gamma = 1$ and

$$C = \begin{bmatrix} 0.8 & 0.5 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 \\ 0.5 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

such that $\lambda^* (C) = \lambda(C_{11}) = 1.2472 > \lambda^* (C_{22}) = 0.5$, the predicted attractor is (a^*, r^*), where (see Fig. 3):

$$\begin{aligned} a^* &= (0.35613, 0.31850, 0.32538, 0)' \\ r_1^* &= \frac{(1.2472)\,\rho + \gamma - (\rho/4)\,(4.2416)}{(1.2472)\,\sigma} \simeq 1.7534 = r_2^* = r_3^* \\ r_4^* &= \frac{(0.5)\,\rho + \gamma - (\rho/4)\,(4.2416)}{(0.5)\,\sigma} \simeq 2.8792 \end{aligned}$$

3.2 Expanded dominant ACS

This is a follow-up to Example 2. There we have a technology field, 4, which does not belong to the dominant autocatalytic set $G(S^*, L^*)$ and, as such, does not participate in the growth process shaped by the structural relations between the fields 1, 2 and 3.

Example 3 *The present example refers to a smooth integration of field 4 within the pre-existing knowledge hierarchy. The integration takes place through the creation of a new interface standard that brings in the reach of field 4 knowledge inputs created in field 3, and through it, in the rest of the economy:*

$$C = \begin{bmatrix} c_{11} & c_{12} & 0 & c_{14} \\ c_{21} & c_{22} & 0 & 0 \\ c_{31} & 0 & c_{33} & 0 \\ 0 & 0 & c_{43} & c_{44} \end{bmatrix}$$

The hardware components corresponding to these knowledge inputs are labelled complementary inputs in the growth literature focused on general purpose technologies. In the example at hand, the creation of the interface standard and of the corresponding complementary inputs does not require a strong modification of the design principles previously at work. The new knowledge interface gives rise to a larger dominant autocatalytic set with an expanded core, which comes to coincide with $G(S, L)$. The relatively smooth transition to the new hierarchy is brought about by the addition to the pre-existing structure of connection parameters c_{14} and c_{43}. Now the dominant ACS $G(S^*, L^*)$ and the core in it come to coincide with $G(S, L)$. The Perron-Frobenius eigenvalue $\lambda^*(C)$ is strictly larger than in Example 2 and the attractor of the dynamics is the couple of strictly positive vectors (a^*, r^*), with r_i^* in Example 3 strictly lower than in Example 2, because structural change has made R&D effort more productive in the former, so that the ratio between Q_i and the resulting knowledge stock A_i converges to a persistently lower value.

3.2.1 Simulation of Example 3

With $\rho = 0.05$, $\sigma = 0.05$, $\gamma = 1$ and

$$C = \begin{bmatrix} 0.8 & 0.5 & 0 & 0.5 \\ 0.4 & 0.8 & 0 & 0 \\ 0.5 & 0 & 0.7 & 0 \\ 0 & 0 & 0.6 & 0.8 \end{bmatrix}$$

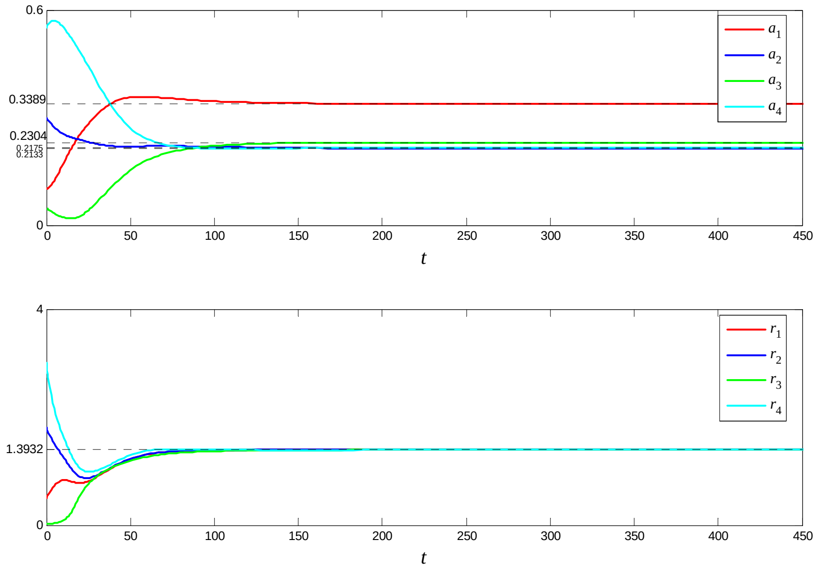

Figure 4: Convergence to the fixed point in Example 3

such that $\lambda^* (C) = 1.4356$, the predicted attractor is (a^*, r^*), where (see Fig. 4):

$$\begin{aligned} a^* &= (0.33887, 0.21328, 0.23038, 0.21748)' \\ r^* &= \frac{\gamma}{\sigma \lambda^* (C)} \simeq 1.3931 \end{aligned}$$

3.3 Deep structural change: emergence of a new dominant ACS

Example 4 *This and the following Example 5 are a different follow-up to Example 2. Taken together they attempt to capture the notion of a revolutionary, as opposed to smooth, structural change. The revolution takes place in two steps. The first step is described in Example 4, where C takes the following form:*

$$C = \begin{bmatrix} c_{11} & c_{12} & 0 & 0 \\ c_{21} & c_{22} & 0 & 0 \\ c_{31} & 0 & c_{33} & 0 \\ 0 & c_{42} & 0 & c_{44} \end{bmatrix} = \begin{bmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix}$$

This example is marked in the first place by the development of technology field 4 – corresponding to a higher coefficient c_{44} and to the new connection c_{42} – and in the second place by the saturation of the pre-existing knowledge pattern identified by the core of $G(S^*, L^*)$ in Example 2. The saturation is mirrored by the lower value of the coefficients c_{ij}, i, $j = 1, 2$, in Example 4. Two situations may now arise. If $\lambda^*(C_{11}) > \lambda^*(C_{22})$, then $G(S^*, L^*) = G(S, L)$ and $a^* > 0$. If instead $\lambda^*(C_{11}) < \lambda^*(C_{22})$, $G(S^*, L^*)$ coincides with vertex 4 and the link to itself, then the notional incremental dynamics of vector a generated by Example 4 converges to $a^* = (0, 0, 0, 1)$.

3.3.1 Simulation of Example 4

- First case

With $\rho = 0.05$, $\sigma = 0.05$, $\gamma = 1$ and

$$C = \begin{bmatrix} 0.6 & 0.5 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 \\ 0.5 & 0 & 0.4 & 0 \\ 0 & 0.5 & 0 & 0.7 \end{bmatrix}$$

such that $\lambda^*(C_{11}) = 1.0472 > \lambda^*(C_{22}) = 1$, the predicted attractor is (a^*, r^*), where (see Fig. 5):

$$\begin{aligned} a^* &= (0.25283, 0.22614, 0.19534, 0.32570)' \\ r^* &= \frac{\gamma}{\sigma\lambda^*(C)} = \frac{0.1}{0.05 \times 1.0472} \simeq 1.909 \end{aligned}$$

- Second case:

With $\rho = 0.05$, $\sigma = 0.05$, $\gamma = 1$ and

$$C = \begin{bmatrix} 0.2 & 0.1 & 0 & 0 \\ 0.1 & 0.2 & 0 & 0 \\ 0.5 & 0 & 0.2 & 0 \\ 0 & 0.5 & 0 & 0.8 \end{bmatrix}$$

such that $\lambda^*(C_{11}) = 0.3 < \lambda^*(C_{22}) = 0.8$, the predicted attractor is (a^*, r^*), where (see Fig. 6):

$$\begin{aligned} a^* &= (0, 0, 0, 1)' \\ r_1^* &= \frac{(0.8)\,\rho + \gamma - (\rho/4)\,(1.7)}{(0.8)\,\sigma} = 2.9688 = r_2^* = r_3^* \\ r_4^* &= \frac{(0.3)\,\rho + \gamma - (\rho/4)\,(1.7)}{(0.3)\,\sigma} = 6.25 \end{aligned}$$

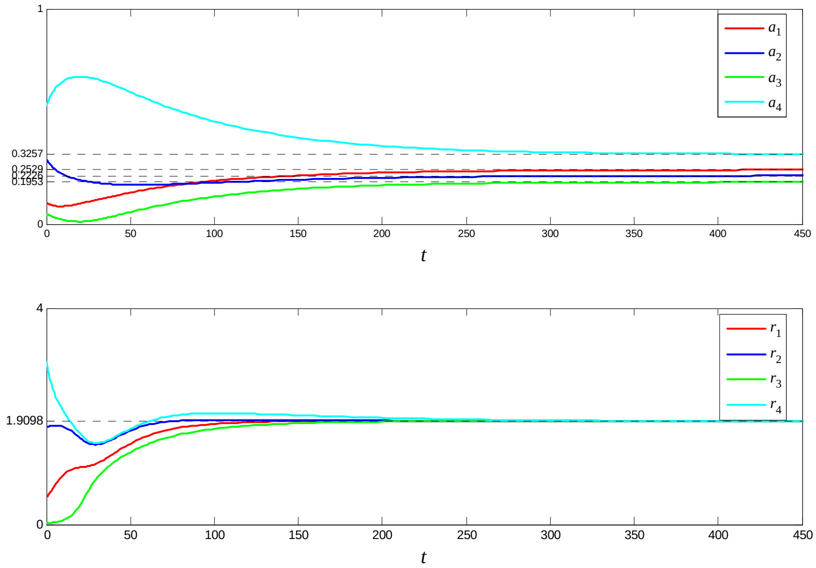

Figure 5: Convergence to the fixed point in the first case of Example 4

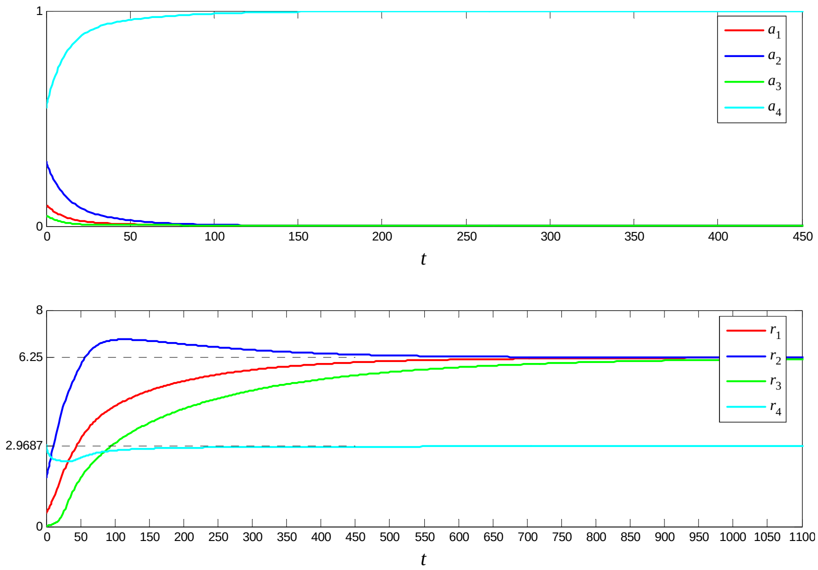

Figure 6: Convergence to the fixed point in the second case of Example 4

3.4 Structural Integration

Example 5 *The revolutionary change is completed in phase 2, when technology field 4 comes to be integrated with the rest of the network by a new emerging hierarchy brought about by the new links* $c_{14} > 0$ *and* $c_{34} > 0$*:*

$$C = \begin{bmatrix} c_{11} & 0 & 0 & c_{14} \\ 0 & c_{22} & c_{23} & 0 \\ c_{31} & 0 & c_{33} & c_{34} \\ 0 & c_{42} & 0 & c_{44} \end{bmatrix}$$

As a result, there is a new dominant autocatalytic set $G(S^*, L^*)$*, which coincides with the whole network* $G(S, L)$*. This occurs in spite of the fact that the links* c_{12} *and* c_{21} *have vanished, with the result that the way in which technologies 1 and 2 participate in the knowledge hierarchy is completely mutated with respect to Example 2. Our conjecture predicts that* a *converges to a stationary, strictly positive vector* a^**.*

3.4.1 Simulation of Example 5

With $\rho = 0.05$, $\sigma = 0.05$, $\gamma = 1$ and

$$C = \begin{bmatrix} 0.4 & 0 & 0 & 0.6 \\ 0 & 0.4 & 0.6 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.6 & 0 & 0.8 \end{bmatrix}$$

such that $\lambda^* (C) = 1.2632$, the predicted attractor is (a^*, r^*), where (see Fig. 7):

$$\begin{aligned} a^* &= (0.19428, 0.2158, 0.31042, 0.27949)' \\ r^* &= \frac{\gamma}{\sigma \lambda^* (C)} \simeq 1.5833 \end{aligned}$$

4 The pattern of knowledge flows

The second half of the 20th century witnessed the information and communication technology (ICT) revolution, most notably in the advanced countries and the USA in particular. Beside this major change, other new technology fields, such as biotechnology, experienced unprecedented dynamism. More traditional fields, such as chemicals or mechanical engineering, reached maturity. In this section, the analytical framework developed so far is set to

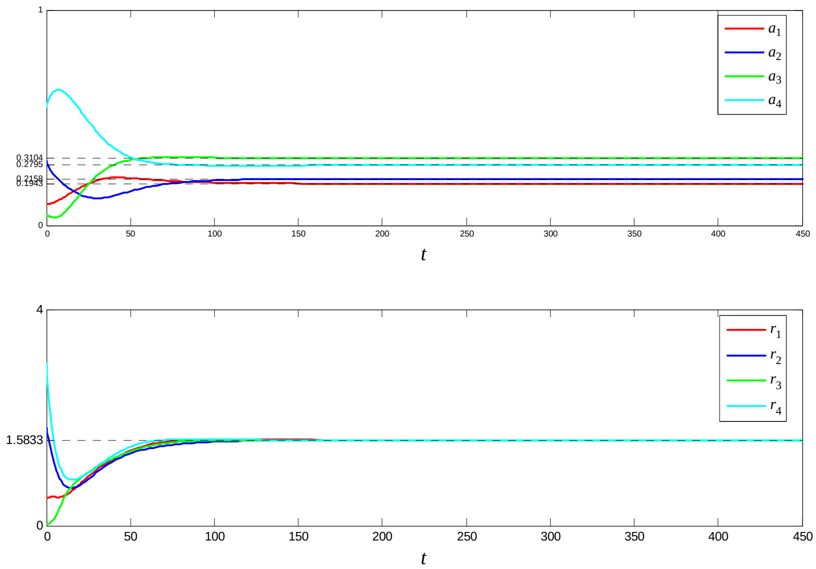

Figure 7: Convergence to the fixed point in Example 5

work with the aim of offering a synthetic representation of the knowledge pattern induced by the technological evolution taking place in the advanced economies during the middle and late 20th century. The primary focus of the analysis is on the way in which the distribution of innovation activity is influenced by the structure of knowledge links between technology fields, as primarily shaped by technological and institutional factors that are largely common across the advanced nations. More to the point, we shall consider how far the changes concerning the cumulated innovation flow distribution, that are predicted by the highly stylised model of Section 2, match the changes suggested by empirical data on knowledge flows and innovation.

Partly as a result of the level of aggregation (detailed in the sequel) at which the analysis is carried out, the notion of *core of the dominant ACS*, introduced in Section 3.1, will be replaced in this section with the looser notion of *core of the knowledge pattern*. The latter is the smallest subset S of technology fields, and the mutual connections between them, such that *(i)* the property of closed path connectivity is met; *(ii)* the subset measure of connection intensity $\lambda^*(S) \approx \lambda^*(C)$.

The data source for our exercise is the NBER Patent-Citations data file, as made available in Jaffe and Trajtenberg [8]. The main data set PAT63_99 contains all utility patents[4] granted by the U.S. Patent and Trademark Office (PTO) between January 1, 1963 and December 30, 1999. Among the variables that the PTO originally assigns to each patent, most relevant for us, in addition to the grant year, is the main U.S. patent class.[5] There were 417 patent classes in the classification in use in 1999. The 'original' variables assigned by the PTO to the various patents are enriched by the authors of the dataset with a number of 'constructed variables'. In particular, the 417 classes are aggregated by the authors into 36 technological subcategories and these further aggregated into 6 categories ('Chemical', 'Computers & Communications', 'Drugs & Medical', 'Electrical & Electronic', 'Mechanical', and 'Others'). The data set PAT63_99 can be profitably matched with a second data set, namely, CITE75_99, which contains all citations made to patents in PAT63_99 by patents issued between January 1, 1975 and December 30, 1999.

The first aim of our exercise is to obtain from the citations data just described, a computationally viable description of the knowledge flows between technology fields, and of the changes thereof. For our computation endowments, the technological classification according to the 417 3-digit classes proved far too demanding. Therefore, we had to resort to the description of technology fields according to their partition into 36 subcategories. To evaluate the intensity of knowledge spillovers across technology fields, we studied how far patenting in a subcategory xy in a time interval $[t, t+z]$ was followed by citations to xy by patents issued in every other subcategory in the time interval $[t+s, t+z]$. In this way, for each subcategory xy, we obtained a 36 dimensional vector of citations. The corresponding vector of spillover intensity from xy to the other subcategories was obtained by dividing the citations vector by the number of patents issued in xy in the period $[t, t+z]$. Proceeding in this way for each xy in the set of 36 subcategories, we arrived at a matrix of spillover intensity which is the empirical analogue of the matrix C in our model. To detect structural change, if any, in the pattern of knowledge spillovers in the period under study, we divided the latter into two sub-periods and obtained a corresponding analogue of matrix

[4] Utility patents constitute the overwhelming majority of patents, which include, in addition, design, reissue and plant patents. Cfr. Hall, Jaffe and Trajtenberg [5, p. 407, n. 4].

[5] The reason for the qualification 'main' is that each patent is assigned by the PTO to a 3-digit patent class and to a subclass, but also to any number of 'subsidiary' classes and subclasses that seem appropriate. Moreover, the system is continuously updated with new classes being added and others being reclassified or discarded. In this case, the PTO retroactively assigns patents to patent classes, according to the most recent clasification system. Cfr. Hall, Jaffe and Trajtenberg [5, p. 415.]

C for each sub-period.

The actual procedure followed was complicated by two types of consideration that have to do with those characteristics of the available data set, that are most relevant to our exercise.

The first relevant characteristic is that the number of citations in a finite time interval is affected by truncation effects related to backward and forward citation lags (Hall, Jaffe and Trajtenberg [5, pp. 421-424]). This imposed a choice of the subperiods in a way that comparisons between them were least affected by the unavoidable distortions introduced by truncation effects. In particular, the parameter s was held constant between the subperiods ($s = 12$) and differences in z were negligible ($z = 23$ in the first subperiod, $z = 24$, in the second). The corresponding choices for t were $t = 1963$ and $t = 1975$, respectively. For the sake of later reference, the intervals $[t+s, t+z] = [1975-1986]$ and $[t+s, t+z] = [1987-1999]$ are referred to below as first window ($W1$) and second window ($W2$), respectively.

The second relevant characteristic is that there is a sharp rising trend, largely common across categories, in the mean number of citations, per patent. This trend reflects, to a large extent, an increasing propensity to cite by PTO officers, as a result of the easier access to larger data sources brought about by computerisation of the PTO during the 1980's. Although the rising citations trend may not be *entirely* a pure artifact of the changed PTO practices, in the absence of a better alternative, construction of the connection matrix for the second window was carried out using discounted citations data. In particular, the number of citations made by patents issued in subcategory xy in the second window, was discounted by the xy growth rate of citations-made per patent between the first and second window.

There is a third potentially distorting characteristic in the data set, namely, the rising trend in the yearly number of patents issued since 1983. This feature is at least partly taken care of by our procedure, since according to our estimate of the connection matrix, the number of citations made by subcategory xy patents, issued in window $[t + s, t + z]$, to subcategory hk patents issued in $[t, t + z]$, is divided by the number of hk patents granted in $[t, t + z]$.

4.1 Connection matrices as knowledge flows

A visual representation of the connection matrices $C(W1)$ and $C(W2)$ for the two windows is given in Fig. 8 and 9, where colours identify different ranges of the connection coefficients. In particular, the cell corresponding to c_{ij} is coloured red if $c_{ij} \geq 1$, yellow if $1 > c_{ij} \geq 0.1$, white if $0.1 > c_{ij} \geq 0.01$, and blue if $c_{ij} < 0.01$.

***C*(*W*1)** **blue < 0.01** ≤ white < 0.1 ≤ yallow < 1 ≤ red

	11	12	13	14	15	19	31	32	33	39	21	22	23	24	41	42	43	44	45	46	49	51	52	53	54	55	59	61	62	63	64	65	66	67	68	69
11	1.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.1	0.9	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
13	0.0	0.0	1.6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	0.1	0.0	0.0	0.9	0.1	0.0	0.2	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
15	0.1	0.2	0.0	0.1	1.9	0.1	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
19	0.2	0.3	0.5	0.1	0.2	1.7	0.1	0.1	0.3	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.1	0.1	0.1	0.2	0.0	0.1	0.1	0.0	0.1	0.0	0.1	0.1	0.0	0.0	0.1	0.0	0.1	0.1	0.1	0.1
31	0.1	0.0	0.0	0.3	0.0	0.0	1.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
32	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.5	0.0	0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
33	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
39	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
21	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.7	0.3	0.2	0.1	0.1	0.0	0.1	0.1	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
22	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	2.3	0.2	0.2	0.0	0.0	0.1	0.1	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
23	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	1.7	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
41	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.1	1.2	0.0	0.0	0.0	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
42	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	1.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
43	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.1	0.1	0.0	0.0	0.0	0.0	1.3	0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
44	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	1.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
45	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.1	0.0	0.0	0.0	1.5	0.1	0.1	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0
46	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	2.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
49	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.2	0.1	0.1	0.1	0.0	0.0	0.0	0.1	1.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
51	0.0	0.1	0.0	0.0	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.2	0.1	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1
52	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.1	1.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0
53	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	1.7	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0
54	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.1	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	1.6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
55	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.2	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0
59	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.1	0.0	0.1	1.1	0.0	0.1	0.0	0.0	0.0	0.0	0.1	0.0	0.1
61	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
62	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0
63	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.2	0.0	0.0	0.0	0.0	0.0	0.0
64	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.9	0.0	0.0	0.1	0.0	0.0
65	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.2	0.0	0.0	0.0	0.0
66	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.6	0.0	0.0	0.0
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.2	0.0	0.0
68	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.4	0.0
69	0.2	0.3	0.1	0.0	0.2	0.1	0.0	0.1	0.0	0.0	0.0	0.1	0.1	0.0	0.1	0.0	0.1	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.1	0.1	0.1	0.1	0.1	0.2	0.1	1.2

Figure 8: Relative citation flows of patents issued in a column technology-field, by patents issued in the row technology-fields: 1975-1986.

In our matrix representation, the ordering of technological categories, embodied in the NBER data set, is changed through a permutation of rows and columns aimed at stressing structural properties. The permutation shifts category 3 ('Drugs & Medical', including subcategory 33, 'Biotechnology') between category 1 ('Chemical') and category 2 ('Computers & Communications'). The resulting structure of $C(W1)$ and $C(W2)$ is similar, in many respects.

4.1.1 Every subcategory is strongly connected with itself

The great majority of the cells on the main diagonal, but none of the cells outside the main diagonal, are red. The only main diagonal exceptions (3 in $W1$, 2 in $W2$) refer to connection coefficients that are in any case not far from 1. In both windows, the least self connected subcategories belong to the traditional category 'Chemical'.

$C(W2)$ blue < 0.01 ≤ white < 0.1 ≤ yallow < 1 ≤ red

	11	12	13	14	15	19	31	32	33	39	21	22	23	24	41	42	43	44	45	46	49	51	52	53	54	55	59	61	62	63	64	65	66	67	68	69
11	0.8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.1	1.0	0.0	0.0	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1
13	0.0	0.0	1.7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	0.1	0.0	0.0	0.7	0.1	0.1	0.2	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
15	0.1	0.2	0.0	0.2	1.9	0.1	0.1	0.0	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1
19	0.3	0.4	0.6	0.2	0.2	1.8	0.1	0.1	0.2	0.1	0.0	0.0	0.1	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.2	0.1	0.0	0.1	0.0	0.1	0.1	0.0	0.0	0.1	0.0	0.1	0.1	0.2	0.2
31	0.1	0.1	0.0	0.3	0.1	0.0	1.3	0.1	0.2	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
32	0.0	0.0	0.0	0.0	0.0	0.0	0.0	3.2	0.0	0.4	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.1	0.0
33	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.0	1.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
39	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	1.7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
21	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.4	0.3	0.2	0.0	0.1	0.1	0.1	0.1	0.1	0.0	0.2	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
22	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	3.3	0.5	0.5	0.1	0.0	0.2	0.1	0.1	0.1	0.3	0.0	0.0	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
23	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	2.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.0	2.4	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
41	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.1	0.0	0.0	0.0	1.6	0.0	0.0	0.0	0.1	0.1	0.1	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
42	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
43	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.0	0.0	0.1	0.1	0.0	0.0	0.1	0.0	1.5	0.2	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
44	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	1.5	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
45	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.1	0.0	0.0	0.2	0.1	0.1	0.0	1.8	0.1	0.1	0.0	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0
46	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.0	0.0	0.0	0.2	0.1	0.1	0.0	0.1	0.1	3.2	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
49	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.1	0.0	1.7	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
51	0.1	0.1	0.1	0.0	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.4	0.1	0.0	0.0	0.1	0.1	0.0	0.0	0.1	0.0	0.0	0.0	0.1	0.1	0.1
52	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.1	0.1	0.0	0.1	1.2	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.0	0.0
53	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	1.8	0.0	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.0	0.1
54	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.1	0.0	0.1	0.0	0.1	0.0	0.0	0.1	0.0	0.1	0.0	0.0	0.1	0.0	0.0	0.0	1.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
55	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.6	0.1	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0
59	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.0	0.1	1.4	0.0	0.1	0.0	0.0	0.1	0.0	0.1	0.1	0.1
61	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.6	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0
62	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.8	0.0	0.0	0.0	0.0	0.0	0.0	0.0
63	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.5	0.0	0.0	0.0	0.0	0.0	0.0
64	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.9	0.0	0.0	0.1	0.0	0.0
65	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.6	0.0	0.1	0.1	0.1
66	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.3	0.0	0.0	0.0
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.3	0.0	0.0
68	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.1	0.0	0.0	1.9	0.0
69	0.3	0.3	0.2	0.0	0.2	0.1	0.0	0.0	0.1	0.0	0.0	0.0	0.1	0.0	0.1	0.1	0.1	0.0	0.1	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.2	0.1	0.3	0.2	1.5

Figure 9: The same as in Fig. 8 for the period 1987-1999.

4.1.2 Connection matrices are nearly decomposable into blocks of categories

All the main categories tend to be more tightly connected with themselves than with the others. Most significantly, as will turn out shortly, in both periods ‘Drugs & Medical’ (3) are least connected with ‘Computers & Communications’ (2), and vice versa.

Figures 10(a) and 10(b) show the main-diagonal submatrices $C_{3,2}$ for the periods $W1$ and $W2$, respectively. Since blue numbers are negligibly different from zero, each submatrix $C_{3,2}$ has a quasi positive diagonal block structure, with blue areas in the top-right and bottom-left corners.

There are however groups of categories with relatively tight mutual connections, that give rise to near-blocks on the main diagonal of $C(W1)$ and $C(W2)$. This holds true, in particular, for the category group “‘Computers & Communications’ (2), ‘Electrical & Electronic’ (4)” and for the group “‘Chemical’ (1), ‘Drugs & Medical’ (3)”.

Figures 11(a) and 11(b) show a blown up version of the blocks $C_{2,4}(W1)$ and $C_{2,4}(W2)$.The strengthening of the mutual connections between cate-

(a)

	21	22	23	24	31	32	33	39
21	1.660	0.275	0.212	0.055	0.001	0.018	0.001	0.006
22	0.141	2.278	0.242	0.232	0.000	0.008	0.001	0.005
23	0.027	0.049	2.201	0.020	0.000	0.002	0.000	0.001
24	0.015	0.141	0.064	1.700	0.000	0.001	0.000	0.001
31	0.000	0.000	0.000	0.000	1.024	0.035	0.099	0.016
32	0.014	0.013	0.010	0.004	0.030	2.476	0.023	0.181
33	0.000	0.001	0.000	0.000	0.025	0.006	1.325	0.003
39	0.001	0.000	0.000	0.000	0.003	0.045	0.002	1.665

(b)

	21	22	23	24	31	32	33	39
21	2.365	0.266	0.184	0.042	0.001	0.029	0.002	0.007
22	0.450	3.326	0.548	0.497	0.001	0.032	0.007	0.020
23	0.069	0.125	2.405	0.032	0.000	0.006	0.000	0.011
24	0.023	0.177	0.049	2.426	0.000	0.002	0.001	0.001
31	0.001	0.001	0.000	0.000	1.279	0.068	0.177	0.067
32	0.020	0.014	0.008	0.002	0.038	3.200	0.016	0.427
33	0.002	0.004	0.001	0.001	0.096	0.020	1.539	0.018
39	0.001	0.002	0.001	0.000	0.009	0.092	0.005	1.683

Figure 10: Mutual citation flows between subcategories in 'Computers and Communications' (2) and 'Drugs and Medical' (3) for the period $1975-1986$ (a) and $1987-1999$ (b)

gories 2 and 4 in the second window is not limited to these categories, but is part of a more general trend, which is not a mere artifact of the rising patenting and citation propensity in the second period. As a result, in spite of the fact that, at the aggregation level adopted, there is considerable continuity in the structure of $C(W1)$ and $C(W2)$, the near decomposability of the connection matrix into blocks is weaker in the second window.

4.2 Perron-Frobenius eigenvalues and knowledge patterns

The model in Section 2 suggested that the Perron-Frobenius eigenvalue λ^* of a connection matrix, C, is an indicator of the intensity of knowledge spillovers between technology fields, and showed how it affects knowledge accumulation. Table 1 shows the λ^* values, that is the aggregate, between–and-within-category spillover intensity, in the two periods of interest. It also shows the intensity of within-category knowledge-transfer, namely, the λ^* value of the main-diagonal block C_i of $C(W1)$ and $C(W2)$ that corresponds to technological category i, for $i = 1, ..., 6$. The same table shows the λ^* value of the main-diagonal blocks $C_{2,4}$ and $C_{2,3}$ corresponding to the aggregation of categories 2,4 and 2,3, respectively. It may be worth stressing that what matters is not the absolute, but the relative, size of the eigenvalues. To emphasise this, the last two columns in the table show the ratios between each diagonal block eigenvalue $\lambda^*(C_i)$ and the corresponding matrix eigenvalue $\lambda^*(C)$. Finally, the column $W2/W1$ gives a measure of the λ^* rate of change between $W1$ and $W2$.

(*a*)

	21	22	23	24	41	42	43	44	45	46	49
21	1.66	0.27	0.21	0.06	0.11	0.04	0.07	0.06	0.04	0.04	0.10
22	0.14	2.28	0.24	0.23	0.04	0.01	0.09	0.05	0.04	0.03	0.08
23	0.03	0.05	2.20	0.02	0.00	0.03	0.00	0.01	0.00	0.01	0.04
24	0.01	0.14	0.06	1.70	0.01	0.00	0.01	0.02	0.01	0.06	0.05
41	0.08	0.05	0.02	0.05	1.18	0.04	0.03	0.01	0.07	0.15	0.08
42	0.01	0.00	0.10	0.00	0.02	1.21	0.01	0.04	0.02	0.04	0.03
43	0.07	0.10	0.03	0.01	0.05	0.02	1.26	0.16	0.04	0.01	0.04
44	0.02	0.03	0.03	0.01	0.01	0.05	0.07	1.33	0.01	0.03	0.02
45	0.04	0.06	0.02	0.03	0.11	0.04	0.04	0.04	1.51	0.10	0.06
46	0.01	0.01	0.01	0.08	0.04	0.03	0.00	0.02	0.02	2.37	0.01
49	0.08	0.10	0.19	0.12	0.08	0.05	0.03	0.04	0.05	0.05	1.31

(*b*)

	21	22	23	24	41	42	43	44	45	46	49
21	2.37	0.27	0.18	0.04	0.15	0.09	0.11	0.09	0.06	0.03	0.19
22	0.45	3.33	0.55	0.50	0.07	0.01	0.19	0.12	0.09	0.07	0.31
23	0.07	0.13	2.41	0.03	0.01	0.04	0.01	0.03	0.01	0.01	0.13
24	0.02	0.18	0.05	2.43	0.04	0.01	0.01	0.04	0.03	0.09	0.07
41	0.07	0.03	0.01	0.05	1.65	0.04	0.04	0.02	0.13	0.12	0.08
42	0.02	0.00	0.04	0.00	0.03	1.73	0.01	0.04	0.02	0.04	0.03
43	0.07	0.08	0.01	0.02	0.07	0.03	1.45	0.21	0.06	0.03	0.04
44	0.03	0.02	0.02	0.02	0.01	0.05	0.13	1.51	0.01	0.03	0.04
45	0.04	0.06	0.02	0.04	0.16	0.05	0.06	0.03	1.78	0.09	0.08
46	0.03	0.03	0.02	0.17	0.12	0.09	0.04	0.05	0.10	3.21	0.03
49	0.09	0.08	0.11	0.07	0.08	0.06	0.03	0.04	0.06	0.02	1.65

Figure 11: Mutual citation flows between subcategories in 'Computers and Communications' (2) and 'Electrical and Electronic' (4) for the period 1975–1986 (a) and 1987 – 1999 (b)

	$W1$	$W2$	$\frac{W2}{W1}$	$W1$: $\frac{\lambda^*(C_i)}{\lambda^*(C)}$	$W2$: $\frac{\lambda^*(C_i)}{\lambda^*(C)}$
$\lambda^*(C)$	2.572	3.693	1.436	1.000	1.000
$\lambda^*(C_1)$	2.023	2.092	1.034	0.787	0.567
$\lambda^*(C_2)$	2.452	3.584	1.462	0.953	0.971
$\lambda^*(C_3)$	2.487	3.227	1.298	0.967	0.874
$\lambda^*(C_4)$	2.386	3.231	1.354	0.928	0.875
$\lambda^*(C_5)$	1.677	1.897	1.131	0.652	0.514
$\lambda^*(C_6)$	1.873	1.955	1.044	0.728	0.529
$\lambda^*(\hat{C}_{2,4})$	2.523	3.669	1.454	0.998	0.994
$\lambda^*(C_{2,3})$	2.493	3.586	1.440	0.969	0.971

Table 1

The table shows a sharp difference between a group of traditional technological categories, 'Chemical' (1), 'Mechanical' (5) and 'Others' (6), and a group of progressive technologies, namely, 'Computers & Communications' (2), 'Drugs & Medical' (3), 'Electrical & Electronic' (4). The former is characterised by a lower intensity of knowledge transfer in each window [1975 – 1986] and [1987 – 1999] , and by a lower acceleration of within-category knowledge transfer between window $W1$ and $W2$. During $W1$, the category 'Drugs & Medical'. gains the top ranking in the rate of within-category knowledge transfer, followed by 'Computers & Communications'. The table clearly reveals the considerably faster relative development of

‘Computers & Communications’ during period $W2$, followed by ‘Electrical & Electronic’ (4) and ‘Drugs & Medical’ (3), in this order.

Are we to conclude that there is a change, from $W1$ to $W2$, in the subset of technology fields that crucially determine the aggregate measure of connection intensity $\lambda^*(C)$? To answer this question, let us loosely define the *near core* of the knowledge pattern to be the smallest subset of technology fields, such that the subset measure of connection intensity best approximates the aggregate measure $\lambda^*(C)$. In the light of the above definition, a closer look at Table 2 suggests the following tentative conclusion. The very weak mutual connections between categories 2 and 3, and the resulting quasi positive diagonal block structure of $C_{2,3}$ in both periods $W1$ and $W2$, have relevant implications: $\lambda^*(C_{2,3})$ is negligibly different from $\lambda^*(C_3)$ in window $W1$, and from $\lambda^*(C_2)$ in window $W2$. In both periods, neither C_3 and C_2, taken in isolation, nor their aggregation $C_{2,3}$ can be convincingly identified with the near core of the ruling pattern of knowledge transfer. In this respect, a much more convincing candidate is the quasi diagonal block $C_{2,4}$ of Fig. 11(a) and 11(b). In both periods, the block in question explains more than 99% of the aggregate measure of spillover intensity. This may be taken as clear evidence of the prominent position reached by the ICT revolution in these periods.

4.3 Perron-Frobenius eigenvectors and the distribution of knowledge accumulation

The model of Section 2 predicts that the notional dynamics of the knowledge stocks distribution, corresponding to a constant connection matrix C converges asymptotically to the right Perron-Frobenius eigenvector of C. Since the model is highly stylised and admittedly, abstracts from demand factors and other real world features that exert a non negligible influence on innovation flows, it would be highly inappropriate to expect an empirical corroboration of a point prediction of the model. Moreover, the model predictions refer to asymptotic outcomes, which may differ from transitional outcomes, not only as a result of a possibly slow convergence, but also because of the non monotonic convergence paths displayed in our simulations. At best, we may expect some degree of conformity between the *directions of change* predicted by the model and the directions of change signalled by innovation data (see Table 2).[6]

To check this hypothesis, we compared the sign of the first differences

[6] For a recent restatement of a similar point about method, see Samuelson [15].

between two couples of vectors: on the one hand, the difference $\nabla a^* = a^*(W2) - a^*(W1)$ between the model generated attractors (the normalised right Perron-Frobenius eigenvectors of $C(W1)$ and $C(W2)$), on the other, the difference $\nabla a = a(W2) - a(W1)$ between the cumulated flow distribution of patents in the two windows $W1$ and $W2$. The sign conformity is spelled out in the penultimate column of Table 2, where only 8 out of 36 subcategories have non corresponding signs.

aW1:75/86	aW2: 87/99	a'W1	a'W2	aW2–aW1	a'W2–a'W1	sign conformity	subcategories
0,0109	0,0073	0,0033	0,0012	-0,0036	-0,0022	YES	11 Agr. Food Textiles
0,0159	0,0150	0,0096	0,0049	-0,0008	-0,0047	YES	12 Coating
0,0060	0,0040	0,0057	0,0010	-0,0020	-0,0047	YES	13 Gas
0,0518	0,0279	0,0052	0,0014	-0,0239	-0,0039	YES	14 Org. Comp.
0,0390	0,0328	0,0216	0,0060	-0,0062	-0,0156	YES	15 Resins
0,1132	0,0932	0,0792	0,0294	-0,0201	-0,0498	YES	19 Misc. Chem.
0,0343	0,0524	0,0771	0,1010	0,0180	0,0239	YES	21 Communicat.
0,0176	0,0509	0,1506	0,3768	0,0334	0,2262	YES	22 Comp. Hard. Soft.
0,0052	0,0138	0,0394	0,0507	0,0086	0,0113	YES	23 Comp. Perif.
0,0126	0,0245	0,0370	0,0667	0,0119	0,0297	YES	24 Inform. Stor.
0,0288	0,0402	0,0058	0,0027	0,0114	-0,0031	NO	31 Drugs
0,0186	0,0350	0,1296	0,0342	0,0164	-0,0954	NO	32 Surg. & Med. Instrum.
0,0062	0,0182	0,0024	0,0028	0,0119	0,0003	YES	33 Biotech.
0,0052	0,0075	0,0078	0,0025	0,0023	-0,0054	NO	39 Misc. Drugs & Med.
0,0318	0,0299	0,0254	0,0198	-0,0019	-0,0056	YES	41 Electrical Dev.
0,0146	0,0165	0,0094	0,0065	0,0019	-0,0030	NO	42 Electrical Lighting
0,0293	0,0288	0,0304	0,0229	-0,0004	-0,0075	YES	43 Measuring. & Testing
0,0142	0,0157	0,0137	0,0114	0,0015	-0,0023	NO	44 Nuclear & X-rays
0,0361	0,0336	0,0327	0,0266	-0,0025	-0,0061	YES	45 Power Systems
0,0104	0,0286	0,0522	0,0788	0,0182	0,0266	YES	46 Semicond. Dev.
0,0237	0,0248	0,0373	0,0293	0,0011	-0,0080	NO	49 Misc. Elec.
0,0617	0,0448	0,0240	0,0114	-0,0169	-0,0126	YES	51 Materials Proc. & Hand
0,0353	0,0266	0,0150	0,0091	-0,0087	-0,0059	YES	52 Metal Working
0,0420	0,0337	0,0247	0,0154	-0,0083	-0,0093	YES	53 Motors, Eng. & Parts
0,0213	0,0253	0,0216	0,0180	0,0040	-0,0036	NO	54 Optics
0,0313	0,0273	0,0095	0,0077	-0,0040	-0,0019	YES	55 Transportation
0,0526	0,0462	0,0240	0,0166	-0,0064	-0,0074	YES	59 Misc. Mechanical
0,0233	0,0195	0,0055	0,0026	-0,0038	-0,0029	YES	61 Agr. Husband. Food
0,0100	0,0106	0,0047	0,0028	0,0006	-0,0020	NO	62 Amusment Dev.
0,0198	0,0144	0,0068	0,0022	-0,0054	-0,0046	YES	63 Apparel & Textile
0,0177	0,0117	0,0089	0,0030	-0,0060	-0,0059	YES	64 Earth Work. & Wells
0,0206	0,0202	0,0078	0,0031	-0,0005	-0,0047	YES	65 Furnit. House Fixtures
0,0189	0,0100	0,0102	0,0024	-0,0089	-0,0078	YES	66 Heating
0,0107	0,0075	0,0038	0,0012	-0,0032	-0,0026	YES	67 Pipes & Joints
0,0208	0,0200	0,0083	0,0035	-0,0009	-0,0048	YES	68 Receptacles
0,0884	0,0816	0,0498	0,0246	-0,0068	-0,0251	YES	69 Misc. Others

Table 2

As a further descriptive check on our hypothesis, for each subcategory i, we computed the growth rates:

$$g_i(a^*) = \frac{\nabla a_i^*}{a_i^*}, \; g_i(a) = \frac{\nabla a_i}{a_i}$$

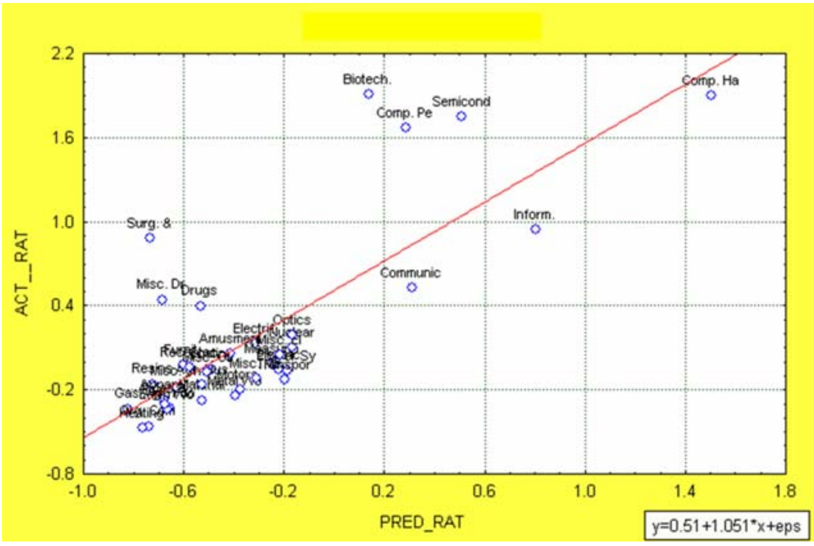

Figure 12: Least-squares regression line between model-generated (pred.) growth rates and actual growth rates.

In this way, we obtained $n = 36$ pairs of measurements $g_i(a^*)$, $g_i(a)$, and regressed the actual growth rates $g_i(a)$ on the model-generated growth rates $g_i(a^*)$. The resulting regression equation, that bears only a descriptive statistical interpretation,[7] is $g_i(a) = 0.51 + (1.0507)g_i(a) + \varepsilon_i$, with an $R^2 = 0.598$. As Fig. 12 illustrates, from a graphical perspective, the model fits the data reasonably well.

5 Conclusions

The last part of the paper considers patent activity and citations in the period 1975-1999, to compare the directions of change predicted by the theoretical model developed in Section 2, with the actual directions of change in the distribution of innovation accumulation between technology fields. Great caution in the interpretation of our results is suggested not only by the highly stylised model structure, which fully abstracts, for instance, from demand effects on innovation opportunities, but also by the fact that we are taking patent citations between technology fields as indicators of knowl-

[7]The reason is that our computation procedure makes the model generated growth rates non independent of the actual growth rates.

edge links between these fields. Further caution, and moderate predictive success, is also suggested by the fact that the model generated changes are obtained by comparing the *asymptotic* attractors of each incremental dynamics corresponding to the average network structure in the sub-period concerned. For this reason, the incremental dynamics admits only a *notional* interpretation. Moreover, our simulations show that convergence paths to the asymptotic-equilibrium distributions may not be monotonic. In spite of these limitations and qualifications, we conclude that the model generated changes fit the actual changes reasonably well.

Our paper was not concerned with the detection of radical or network innovation and morphogenesis in the real world patterns of discovery, or with the factors explaining the formation of their structural properties. These objectives will be addressed in our future work.

A Appendix

In the simplified case with only two sectors and two known technologies ($n = 2$), the dynamical system (6)-(7) reduces to:

$$\begin{aligned} \dot{a}_1 &= \sigma \{ r_1 (c_{11} a_1 + c_{12} a_2) \\ &\quad - a_1 [r_1 (c_{11} a_1 + c_{12} a_2) + r_2 (c_{21} a_1 + c_{22} a_2)] \} \end{aligned} \tag{17}$$

$$\begin{aligned} \dot{a}_2 &= \sigma \{ r_2 (c_{21} a_1 + c_{22} a_2) \\ &\quad - a_2 [r_1 (c_{11} a_1 + c_{12} a_2) + r_2 (c_{21} a_1 + c_{22} a_2)] \} \end{aligned} \tag{18}$$

$$\begin{aligned} \dot{r}_1 &= r_1 \left[\gamma + \frac{(\rho - \sigma r_1)(c_{11} a_1 + c_{12} a_2)}{a_1} \right. \\ &\quad \left. - \frac{\rho}{2} \left(\frac{c_{11} a_1 + c_{12} a_2}{a_1} + \frac{c_{21} a_1 + c_{22} a_2}{a_2} \right) \right] \end{aligned} \tag{19}$$

$$\begin{aligned} \dot{r}_2 &= r_2 \left[\gamma + \frac{(\rho - \sigma r_2)(c_{21} a_1 + c_{22} a_2)}{a_2} \right. \\ &\quad \left. - \frac{\rho}{2} \left(\frac{c_{11} a_1 + c_{12} a_2}{a_1} + \frac{c_{21} a_1 + c_{22} a_2}{a_2} \right) \right] \end{aligned} \tag{20}$$

The strictly positive fixed points of the system are found by imposing $\dot{a}_1 = \dot{a}_2 = \dot{r}_1 = \dot{r}_2 = 0$ in the system of equations (17)-(20), i.e. by solving:

$$r_1^* (c_{11} a_1^* + c_{12} a_2^*) - a_1^* [r_1^* (c_{11} a_1^* + c_{12} a_2^*) + r_2^* (c_{21} a_1^* + c_{22} a_2^*)] = 0 \tag{21}$$

$$r_2^* (c_{21} a_1^* + c_{22} a_2^*) - a_2^* [r_1^* (c_{11} a_1^* + c_{12} a_2^*) + r_2^* (c_{21} a_1^* + c_{22} a_2^*)] = 0 \tag{22}$$

$$\gamma + \frac{(\rho - \sigma r_1^*)(c_{11}a_1^* + c_{12}a_2^*)}{a_1^*} - \frac{\rho}{2}\left(\frac{c_{11}a_1^* + c_{12}a_2^*}{a_1^*} + \frac{c_{21}a_1^* + c_{22}a_2^*}{a_2^*}\right) = 0 \tag{23}$$

$$\gamma + \frac{(\rho - \sigma r_2^*)(c_{21}a_1^* + c_{22}a_2^*)}{a_2^*} - \frac{\rho}{2}\left(\frac{c_{11}a_1^* + c_{12}a_2^*}{a_1^*} + \frac{c_{21}a_1^* + c_{22}a_2^*}{a_2^*}\right) = 0 \tag{24}$$

Notice that from (23) and (24) we obtain:

$$r_1^* = \frac{\rho}{\sigma}\left[\left(\frac{a_1^*}{c_{11}a_1^* + c_{12}a_2^*}\right)\left(\frac{\gamma}{\rho} - \frac{c_{21}a_1^* + c_{22}a_2^*}{2a_2^*}\right) + \frac{1}{2}\right] \tag{25}$$

$$r_2^* = \frac{\rho}{\sigma}\left[\left(\frac{a_2^*}{c_{21}a_1^* + c_{22}a_2^*}\right)\left(\frac{\gamma}{\rho} - \frac{c_{11}a_1^* + c_{12}a_2^*}{2a_1^*}\right) + \frac{1}{2}\right] \tag{26}$$

Then, inserting in (21) and (22):

$$\left[a_1^*\left(\frac{\gamma}{\rho} - \frac{c_{21}a_1^* + c_{22}a_2^*}{2a_2^*}\right) + \frac{(c_{11}a_1^* + c_{12}a_2^*)}{2}\right](1 - a_1^*) - \left[a_2^*\left(\gamma - \frac{c_{11}a_1^* + c_{12}a_2^*}{2a_1^*}\right) + \frac{(c_{21}a_1^* + c_{22}a_2^*)}{2}\right]a_1^* = 0$$

$$\left[a_2^*\left(\frac{\gamma}{\rho} - \frac{c_{11}a_1^* + c_{12}a_2^*}{2a_1^*}\right) + \frac{c_{21}a_1^* + c_{22}a_2^*}{2}\right](1 - a_2^*) - \left[a_1^*\left(\frac{\gamma}{\rho} - \frac{c_{21}a_1^* + c_{22}a_2^*}{2a_2^*}\right) + \frac{(c_{11}a_1^* + c_{12}a_2^*)}{2}\right]a_2^* = 0$$

from which it follows that in equilibrium we must have:

$$\frac{a_1^*}{c_{11}a_1^* + c_{12}a_2^*} = \frac{a_2^*}{c_{21}a_1^* + c_{22}a_2^*} \tag{27}$$

Then, inserting in (25) and (26), we find:

$$r_1^* = \frac{\gamma}{\sigma}\left(\frac{a_2^*}{c_{21}a_1^* + c_{22}a_2^*}\right) = \frac{\gamma}{\sigma}\left(\frac{a_1^*}{c_{11}a_1^* + c_{12}a_2^*}\right) = r_2^* = r^* \tag{28}$$

Summarising, the fixed points of the dynamical system of our model are given by:

$$(a_1^*, a_2^*, r_1^*, r_2^*) = (a_1^*, 1 - a_1^*, r^*, r^*)$$

where a_1^* and a_2^* must satisfy condition (27).

Expressing the same condition in terms of a_1^* only, we obtain:

$$(c_{12} + c_{22} - c_{11} - c_{21})\, a_1^{*2} - (2c_{12} + c_{22} - c_{11})\, a_1^* + c_{12} = 0$$

We must distinguish different cases, according to whether $c_{12} + c_{22} \neq c_{11} + c_{21}$ or $c_{12} + c_{22} = c_{11} + c_{21}$. When $c_{12} + c_{22} \neq c_{11} + c_{21}$, we find:[8]

$$a_1^* = \frac{2c_{12} + c_{22} - c_{11} - \sqrt{\Delta}}{2\left(c_{12} + c_{22} - c_{11} - c_{21}\right)} \tag{29}$$

$$a_2^* = 1 - a_1^* = \frac{-2c_{21} + c_{22} - c_{11} + \sqrt{\Delta}}{2\left(c_{12} + c_{22} - c_{11} - c_{21}\right)} \tag{30}$$

where $\Delta = \left(c_{22} - c_{11}\right)^2 + 4c_{12}c_{21}$, whereas, when $c_{12} + c_{22} = c_{11} + c_{21}$:

$$a_1^* = \frac{c_{12}}{c_{12} + c_{21}} \tag{31}$$

$$a_2^* = \frac{c_{21}}{c_{12} + c_{21}} \tag{32}$$

These results imply that, in the special case such that $c_{12} = c_{21}$, namely, in the case of perfect reciprocity of knowledge spillover across technology fields, we have:

$$0 \leq a_1^* = \frac{2c_{12} + c_{22} - c_{11} - \sqrt{\Delta}}{2\left(c_{22} - c_{11}\right)} \leq 1$$

$$0 \leq a_2^* = \frac{-2c_{12} + c_{22} - c_{11} + \sqrt{\Delta}}{2\left(c_{22} - c_{11}\right)} \leq 1$$

when $c_{11} \neq c_{22}$, whereas:

$$a_1^* = a_2^* = \frac{1}{2}$$

when $c_{11} = c_{22}$.

To derive the conditions of local asymptotic stability of the fixed point, we consider the Jacobian matrix of the system evaluated at $(a_1^*, a_2^*, r_1^*, r_2^*)$:

$$\mathbf{J} = \begin{bmatrix} j_{11} & j_{12} & j_{13} & j_{14} \\ j_{21} & j_{22} & j_{23} & j_{24} \\ j_{31} & j_{32} & j_{33} & 0 \\ j_{41} & j_{42} & 0 & j_{44} \end{bmatrix}$$

[8] The solution $(a_1^*, a_2^*) = (a_1^*, 1 - a_1^*)$, where:

$$a_1^* = \frac{2c_{12} + c_{22} - c_{11} + \sqrt{\Delta}}{2\left(c_{12} + c_{22} - c_{11} - c_{21}\right)}$$

must be disregarded because, as it easily proved, such solution always gives values of a_1^* greater than 1 when $c_{12} + c_{22} > c_{11} + c_{21}$ or negative when $c_{12} + c_{22} < c_{11} + c_{21}$.

where, given (9)-(16), we know that:

$$\begin{aligned}
j_{11} &= \left.\frac{\partial \dot{a}_1}{\partial a_1}\right|_{\left(a_1^*, a_2^*, r^*, r^*\right)} = \left(\frac{\gamma}{\lambda^*}\right)\left[c_{11} - \lambda^* - a_1^*\left(c_{11} + c_{21}\right)\right] \\
&= -\left(\frac{\gamma}{\lambda^*}\right)\left[\frac{c_{12}a_2^*}{a_1^*} + a_1^*\left(c_{11} + c_{21}\right)\right] < 0 \\
j_{12} &= \left.\frac{\partial \dot{a}_1}{\partial a_2}\right|_{\left(a_1^*, a_2^*, r^*, r^*\right)} = \left(\frac{\gamma}{\lambda^*}\right)\left[c_{12} - a_1^*\left(c_{12} + c_{22}\right)\right] = \left(\frac{\gamma}{\lambda^*}\right)\left(c_{12}a_2^* - c_{22}a_1^*\right) \\
j_{13} &= \left.\frac{\partial \dot{a}_1}{\partial r_1}\right|_{\left(a_1^*, a_2^*, r^*, r^*\right)} = \sigma\lambda^* a_1^* a_2^* > 0 \\
j_{14} &= \left.\frac{\partial \dot{a}_1}{\partial r_2}\right|_{\left(a_1^*, a_2^*, r^*, r^*\right)} = -\sigma\lambda^* a_1^* a_2^* = -j_{13} < 0 \\
j_{21} &= \left.\frac{\partial \dot{a}_2}{\partial a_1}\right|_{\left(a_1^*, a_2^*, r^*, r^*\right)} = \left(\frac{\gamma}{\lambda^*}\right)\left[c_{21} - a_2^*\left(c_{11} + c_{21}\right)\right] = -j_{11} - \gamma \\
j_{22} &= \left.\frac{\partial \dot{a}_2}{\partial a_2}\right|_{\left(a_1^*, a_2^*, r^*, r^*\right)} = \left(\frac{\gamma}{\lambda^*}\right)\left[c_{22} - \lambda^* - a_2^*\left(c_{12} + c_{22}\right)\right] = -j_{12} - \lambda \\
j_{23} &= \left.\frac{\partial \dot{a}_2}{\partial r_1}\right|_{\left(a_1^*, a_2^*, r^*, r^*\right)} = -\sigma\lambda^* a_2^* a_1^* = j_{14} = -j_{13} < 0 \\
j_{24} &= \left.\frac{\partial \dot{a}_2}{\partial r_2}\right|_{\left(a_1^*, a_2^*, r^*, r^*\right)} = \sigma\lambda^* a_2^* a_1^* = j_{13} > 0 \\
j_{31} &= \left.\frac{\partial \dot{r}_1}{\partial a_1}\right|_{\left(a_1^*, a_2^*, r^*, r^*\right)} = \left(\frac{\gamma}{\sigma\lambda^*}\right)\left[\left(\rho - \frac{\gamma}{\lambda^*}\right)\left(\frac{c_{11} - \lambda^*}{a_1^*}\right) - \frac{\rho}{2}\left(\frac{c_{11}}{a_1^*} + \frac{c_{21}}{a_2^*} - \frac{\lambda^*}{a_1^*}\right)\right] \\
&= r^*\left[-\frac{\rho}{2}\left(\frac{c_{21}}{a_2^*} + \frac{c_{12}a_2^*}{a_1^{*2}}\right) + \frac{\sigma c_{12} a_2^* r^*}{a_1^{*2}}\right] \\
j_{32} &= \left.\frac{\partial \dot{r}_1}{\partial a_2}\right|_{\left(a_1^*, a_2^*, r^*, r^*\right)} = \left(\frac{\gamma}{\sigma\lambda^*}\right)\left[\left(\rho - \frac{\gamma}{\lambda^*}\right)\frac{c_{12}}{a_1^*} - \frac{\rho}{2}\left(\frac{c_{12}}{a_1^*} + \frac{c_{22}}{a_2^*} - \frac{\lambda^*}{a_2}\right)\right] \\
&= r^*\left[\frac{\rho}{2}\left(\frac{c_{12}}{a_1^*} + \frac{c_{21}a_1^*}{a_2^{*2}}\right) - \frac{\sigma c_{12} r^*}{a_1^*}\right] = -\left(\frac{a_1^*}{a_2^*}\right) j_{31} \\
j_{33} &= \left.\frac{\partial \dot{r}_1}{\partial r_1}\right|_{\left(a_1^*, a_2^*, r^*, r^*\right)} = -\gamma < 0 \\
j_{41} &= \left.\frac{\partial \dot{r}_2}{\partial a_1}\right|_{\left(a_1^*, a_2^*, r^*, r^*\right)} = \left(\frac{\gamma}{\sigma\lambda^*}\right)\left[\left(\rho - \frac{\gamma}{\lambda^*}\right)\frac{c_{21}}{a_2^*} - \frac{\rho}{2}\left(\frac{c_{11}}{a_1^*} + \frac{c_{21}}{a_2^*} - \frac{\lambda^*}{a_1}\right)\right] \\
&= r^*\left[\frac{\rho}{2}\left(\frac{c_{21}}{a_2^*} + \frac{c_{12}a_2^*}{a_1^{*2}}\right) - \frac{\sigma c_{21} r^*}{a_2^*}\right]
\end{aligned}$$

$$
\begin{aligned}
j_{42} &= \left.\frac{\partial \dot{r}_2}{\partial a_2}\right|_{\left(a_1^*,a_2^*,r^*,r^*\right)} = \left(\frac{\gamma}{\sigma\lambda^*}\right)\left[\left(\rho-\frac{\gamma}{\lambda^*}\right)\left(\frac{c_{22}-\lambda^*}{a_2^*}\right)-\frac{\rho}{2}\left(\frac{c_{12}}{a_1^*}+\frac{c_{22}}{a_2^*}-\frac{\lambda^*}{a_2^*}\right)\right] \\
&= r^*\left[-\frac{\rho}{2}\left(\frac{c_{21}a_1^*}{a_2^{*2}}+\frac{c_{12}}{a_1^*}\right)+\sigma\frac{c_{21}a_1^*}{a_2^{*2}}r^*\right] \\
&= -\left(\frac{a_1^*}{a_2^*}\right)j_{41} \\
j_{44} &= \left.\frac{\partial \dot{r}_2}{\partial r_2}\right|_{\left(a_1^*,a_2^*,r^*,r^*\right)} = -\gamma < 0
\end{aligned}
$$

Thus, the characteristic equation of the system is given by:

$$
\begin{vmatrix}
j_{11}-\lambda & j_{12} & j_{13} & -j_{13} \\
-j_{11}-\gamma & -j_{12}-\gamma-\lambda & -j_{13} & j_{13} \\
j_{31} & -\left(a_1^*/a_2^*\right)j_{31} & -\gamma-\lambda & 0 \\
j_{41} & -\left(a_1^*/a_2^*\right)j_{41} & 0 & -\gamma-\lambda
\end{vmatrix}
$$
$$
= -\left(\gamma+\lambda\right)^2\left[-\lambda^2-\left(\gamma+j_{12}-j_{11}\right)\lambda-\gamma\left(j_{12}-j_{11}\right)-\left(1/a_2^*\right)\left(j_{41}j_{13}-j_{31}j_{13}\right)\right]=0
$$

which implies that two eigenvalues are equal to γ:

$$\lambda_1 = \lambda_2 = -\gamma < 0$$

while the remaining two (λ_3 and λ_4) are the roots of:

$$\lambda^2 + \left(\gamma + j_{12} - j_{11}\right)\lambda + \gamma\left(j_{12}-j_{11}\right) + \frac{j_{13}\left(j_{41}-j_{31}\right)}{a_2^*} = 0$$

The following two conditions guarantee that λ_3 and λ_4 are negative if real and have negative real parts if complex and therefore that the fixed point of the system is locally stable:

$$\gamma + j_{12} - j_{11} > 0 \tag{33}$$
$$\gamma\left(j_{12}-j_{11}\right) + \frac{j_{13}\left(j_{41}-j_{31}\right)}{a_2^*} > 0 \tag{34}$$

In terms of the parameters of the model, condition (33) can be written as:

$$\gamma+j_{12}-j_{11} = \left(\frac{\gamma}{\lambda^*}\right)\left[\frac{c_{21}a_1^*+c_{22}a_2^*\left(1-a_1^*\right)}{a_2^*}+c_{12}a_2^*+\frac{c_{12}a_2^*}{a_1^*}+a_1^*\left(c_{11}+c_{21}\right)\right] > 0$$

which is always true.

Moreover, in (34), we have:

$$\gamma\left(j_{12}-j_{11}\right)=\left(\frac{\gamma^2}{\lambda^*}\right)\left[c_{12}a_2^*-c_{22}a_1^*+\frac{c_{12}a_2^*}{a_1^*}+a_1^*\left(c_{11}+c_{21}\right)\right]$$
$$=\gamma\left[\sigma r^*\left(c_{12}a_2^*-c_{22}a_1^*\right)-\sigma r^*\left(c_{11}a_2^*-c_{21}a_1^*\right)+\gamma\right]=\gamma^2\left[\frac{c_{11}a_1^*+2c_{12}a_2^*-c_{22}a_1^*}{c_{11}a_1^*+c_{12}a_2^*}\right]$$

and

$$\frac{j_{13}\left(j_{41}-j_{31}\right)}{a_2^*}=\sigma\lambda^*a_1^*\left(\frac{\gamma}{\sigma\lambda^*}\right)\left[\left(\rho-\frac{\gamma}{\lambda^*}\right)\frac{c_{21}}{a_2^*}-\frac{\rho}{2}\left(\frac{c_{11}}{a_1^*}+\frac{c_{21}}{a_2^*}-\frac{\lambda^*}{a_1}\right)\right.$$
$$\left.-\left(\rho-\frac{\gamma}{\lambda^*}\right)\left(\frac{c_{11}-\lambda^*}{a_1^*}\right)+\frac{\rho}{2}\left(\frac{c_{11}}{a_1^*}+\frac{c_{21}}{a_2^*}-\frac{\lambda^*}{a_1^*}\right)\right]=\gamma\left(\rho-\sigma r^*\right)\left(\frac{c_{21}a_1^*}{a_2^*}+\frac{c_{12}a_2^*}{a_1^*}\right)$$

Thus, in terms of the parameters of the model, the second condition for local stability can be expressed as:

$$\gamma\left(\frac{c_{11}a_1^*+2c_{12}a_2^*-c_{22}a_1^*}{c_{11}a_1^*+c_{12}a_2^*}\right)+\left(\rho-\sigma r^*\right)\left(\frac{c_{21}a_1^*}{a_2^*}+\frac{c_{12}a_2^*}{a_1^*}\right)$$
$$=\gamma\left(\frac{c_{11}a_1^*+2c_{12}a_2^*-c_{22}a_1^*}{c_{11}a_1^*+c_{12}a_2^*}\right)+\rho\left(\frac{c_{21}a_1^*}{a_2^*}+\frac{c_{12}a_2^*}{a_1^*}\right)$$
$$-\gamma\left(\frac{a_2^*}{c_{21}a_1^*+c_{22}a_2^*}\right)\frac{c_{21}a_1^*}{a_2^*}-\gamma\left(\frac{a_1^*}{c_{11}a_1^*+c_{12}a_2^*}\right)\frac{c_{12}a_2^*}{a_1^*}$$
$$=\gamma c_{22}\left[\frac{a_2^*\left(c_{11}a_1^*+a_2^*c_{12}a_2^*\right)-a_1^*\left(c_{21}a_1^*+a_2^*c_{22}\right)}{c_{11}a_1^*+c_{12}a_2^*\left(c_{21}a_1^*+c_{22}a_2^*\right)}\right]$$
$$+\rho\left(\frac{c_{21}a_1^*}{a_2^*}+\frac{c_{12}a_2^*}{a_1^*}\right)=\rho\left(\frac{c_{21}a_1^*}{a_2^*}+\frac{c_{12}a_2^*}{a_1^*}\right)>0$$

which is also always true.

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