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Sraffa's Prices

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Abstract - First we consider the existence question in Sraffa's Chapter I dismissed by counting equations and unknowns. A theorem from the theory of Markov processes, applied to distributions not now of probability but of goods to sectors, shows the general existence of non-negative prices satisfying the conditions imposed by the value equation, that value of output equals value of input. The further condition for these to be unique and positive is that the economy be irreducible, or that no independent sub-economy should exist.

Sraffa provides a precise formula determining unique prices, he barely escapes imposing too many conditions on them and certainly cannot require more. In the background and giving motive to the enquiry is the Labour Theory of Value, that goes further. It asserts that the value of anything is ultimately equal to the labour that has gone into making it; so it implies the same principle expressed by the value equation, but a further condition has been added about the nature of the unit. Since the value equation alone makes prices fully determined, there is no room for further conditions, so there is an obstacle to the application of the theory. Standing as a canonical text in a revival of interest in the Theory Of Value serving earlier thought and the later concentration of Ricardo, it offers an exercise in labour value arithmetic, where the only fruit is to find the arithmetic is impossible.

An extended interdependence, which applies to repeated production, appears as a stability condition for prices in an adjustment process, and so does the existence of what Sraffa calls a standard commodity, one depending on all others for its production.

After treating a case where there is a surplus, and joint production, the relation with Leontief and von Neumann is considered.

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SYNOPSIS

In Chapter I of Piero Sraffa's book *Production of Commodities by Means of Commodities: Prelude to a Critique of Economic Theory* (1960), entitled "Production for Subsistence", the same production is repeated every period, each commodity produced separately by a single process, and everything produced is used up in producing what is produced. (the elusive steady state sought by ecologists must be like this.) The complete intelligibility is undone when prices are introduced; it is not said what purpose they serve, as if one should know. These are not prices in the ordinary sense of when a market transaction takes place.

Being clear about a matter could spoil it for higher thought—Sraffa's book has certainly not done that. It is a canonical text in a revival of interest in the Theory of Value, whatever that is—it seems to be an inheritance from earlier thought and a later concentration of Ricardo whose significance is uncertain. Prices are not regarded as having anything to do with market transactions, competition and the equilibrium of supply and demand. They are required to be consistent with the principle that the value of anything is measured by the value that has gone into its making, so there is the *value equation*, *i.e.* value of input equals value of output. The question of the existence of such prices arises. Then there is the *interdependence*, or *irreducibility*, condition *which assures they are unique, and positive*.

An extended interdependence, which applies to repeated production, appears as a stability condition for prices in the adjustment process, and so also does the existence of what Sraffa calls a *standard commodity*, one depending on all others for its production. Inevitably, equilibrium and stability here have nothing to do with supply and demand.

In the case of joint production dealt with later by Sraffa, where the output of any process is generally several commodities instead of just one, though special conditions must permit it such as one considered, there is no general possibility for the introduction of consistent prices. However, the processes with their associated outputs and inputs, rather than being fixed, could instead belong to a system of options provided by the model of von Neumann (1938). Threshold subsistence where outputs exactly replace inputs can be modified, as by Sraffa, to replacement together with a proportion of surplus, so if the surplus is not negative, provided free disposal is permitted, subsistence is attainable. It is always possible now to choose processes which permit the introduction of consistent prices, with the new definition of consistency which, following Sraffa, extends the old definition by permitting a uniform though now possibly non-zero rate of profit to the processes, not to exceed the uniform rate of surplus on all commodities achieved by the chosen processes together.

Such a choice of processes simply for the purpose of being able to introduce such prices could appear artificial in that, at least on the surface, it seems divorced from any consideration of an overall maximality in purely real terms. The uniform rate of profit expresses a competitive selection between individual processes. Any possible process which does not achieve it is not performed, nor is any possible process which exceeds it, but there is none. The chosen processes each achieve the maximum rate of profit for all possible processes, at the prices. But still it is only a money efficiency, having regard to prices which are in fact simultaneously determined with the rate of profit. Also it is an efficiency for elements of the economy in isolation, in principle communicating nothing about the total. It reflects, at least on first appearances, no real and overall efficiency, free of the money reference and dealing with a total performance of the economy in respect to all commodities.

Nevertheless the theory of the von Neumann model shows that the chosen processes together achieve the maximum uniform rate of surplus for commodities which is attainable with all possible processes, even though they have been chosen simply with the intention that they should admit consistent prices. Despite familiar linguistic devices making a connection between equilibrium and optimality, there is no counterpart of this property for supply-demand equilibrium. Dobb (1969) clarifies this and the distinctions involved, and so again does Afriat (1974). I am indebted to Christian Schmidt, of the University of Paris, for a discussion of this subject and for the suggestion that this account be produced.

For the rate of surplus introduced, the value of output is a multiple of the value of input. It is a uniform rate across sectors, suggesting a background of competition rather alien to this thought. The kind of principle intended—whether it has anything to do with real prices or is a moral formula for proper prices, or anything else—is an issue.

For Sraffa in the application to his particular model it happens to be a precise formula determining unique prices. He barely escapes imposing too many conditions on them and certainly cannot require more. In the background, and giving motive to the enquiry, is the Labour Theory of Value, a doctrine of sorts more than a theory, and that goes further. It asserts that the value of anything is ultimately equal to the labour that has gone into making it; so it implies the same principle expressed by the value equation, but if it tells us anything a further condition has been added about the nature of the unit. Since the value equation alone makes prices fully determined, there is no room for further conditions, and with production models different from Sraffa's there are too many already. There is an obstacle to the application of the theory, since the arithmetic of it is impossible.

It is an accident of his special Chapter I model that the simultaneous constraints on prices imposed by Sraffa are not inconsistent. He counts independent equations and variables and finds the numbers equal, concluding that prices with the wanted consistency property do exist. Walras did the same for prices which should clear all markets simultaneously, and Abraham Wald a hundred years later pointed out that the counting argument is ineffective, so initiating the modern theory, which goes further with the mathematics though not much further with the economics. Sraffa has linear equations for which the counting is useful. But prices which must satisfy them should also be non-negative, even positive. That might be supposed, though without knowing the significance of the prices it is impossible to know this with certainty. If this is an exercise in labour value arithmetic, the fruit is to find that the arithmetic is impossible. Sraffa's model, like Leontief's, has separate production of all goods, and if it is modified to allow joint production then the value equation alone produces an inconsistency, without any requirement about the unit. The same is true when a rate of surplus is allowed.

It might be a pity to encounter difficulties only when coming to Sraffa's arithmetic, and not before. It may fairly be asked what importance should be given to sense and logic. Here is another formula or slogan, like the 'Greatest Happiness for the Greatest Number', or the optimality of competitive equilibrium, which might lack sense but not influence. The words can be used, joined with equations where those are appreciated, and still they have effect of a stirring symbol, or flag. Joan Robinson and John Eatwell (1973, p. 3), dealing with "Metaphysics and Science", call the greatest good formula 'metaphysics', but it is certainly not that, and the same can be said of Sraffa's prices. But while those other formulae are insubstantial and give slight opportunity for an investigation, Sraffa's prices produce questions, beside whatever else, about the mathematics of his arguments. The affinity with von Neumann's economic model is well recognized, and the trinity Marx, von Neumann and Sraffa have been canonized. On such lines, in the association with von Neumann, Sraffa's thought leads to an expression of the Maximum Doctrine of Perfect Competition much better than is found in textbooks where the Walrasian system is given that duty. The maximality is now in the physical terms wanted by the Physiocrats, and behaviour concerns competition and profit. That Sraffa's ideas should find a coherence in that particular context, revealing them in a way as crypto-capitalist, is surprising.

First we consider the existence question in his Chapter I. A theorem from the theory of Markov processes—applied to distributions not now of probability but of goods to sectors—shows the general existence of non-negative prices satisfying the required conditions, imposed by the value equation. The further condition for these to be unique and positive is that the economy be *irreducible*, or that no independent sub-economy should exist.

Joined with this condition is an elaboration quite like the *tâtonnement* of Walras for arriving at the prices, though it has nothing to do with the relation of supply and demand, which now are fixed. If in any period the prices are not exactly right, the shortages and surpluses of value for sectors which occur are compensated by price adjustments for the next period. Each price is adjusted for the right amount, as concerns income from output and regardless of the other prices being adjusted at the same time. Therefore it turns out not to be quite the right amount, and the process must be repeated endlessly, but there is a convergence. For a parallel with the Walrasian equilibrium and stability, Sraffa's prices are represented in a framework where they appear as equilibrium prices—with a global stability, moreover. After dealing with the case concerning a surplus, and joint production, the relation with Leontief and von Neumann is considered.

1 Production for subsistence

The economy produces some n goods, in the same quantities in any period. The production quantity of each can be made the unit, so the amount of any good produced in a period always equals 1. There are n sectors in the economy, each producing just one of the goods. Any good produced can be an input for the production of any other, and the total amount of a good that is used up in the production of all goods exactly equals the total amount produced. The chosen units make this 1 in each case.

Let a_{ij} be the amount of good i used up in the production of good j. Since the total amount used up exactly equals the amount produced, we have

$$a_{ij} \geq 0, \ \sum_j a_{ij} = 1 \ \text{ for all } \ i.$$

The matrix a with these elements is a distribution matrix, each of its rows being a distribution vector since the elements of it are non-negative and sum to 1. The distribution in any one row shows how the good produced by one sector is distributed to all sectors. For the matrix a we now have

$$a \geq o, \ aI = I \tag{i}$$

where I is the column vector with n elements all 1.

If the goods have prices p_i, the value of the amount a_{ij} of good i used as an input in the production of good j is $p_i a_{ij}$, and so the total value of inputs is $\sum_i p_i a_{ij}$. The output is one unit of good j, with value $p_j 1 = p_j$. Therefore on Sraffa's principle, that *the value of output equals the value of input*, it is required that

$$\sum_i p_i a_{ij} = p_j \ \text{ for all } \ j,$$

that is,

$$pa = p \tag{ii}$$

where p is the vector of the prices. Only the ratios of the prices are important for this condition. If they should be non-negative and not all zero, so that their sum is positive, then by dividing them by their sum their ratios are unaltered but their sum is made equal to 1. Then p is such that

$$p \geq o, \ pI = 1 \tag{iii}$$

and so is a distribution vector. Since prices are values of the outputs, this vector represents a distribution of value over the sectors, or an income distribution.

Sraffa suggests that, because of (i), any $n-1$ of the n equations stated by (ii) imply the remaining one, so that there are $n-1$ independent equations to determine $n-1$ unique ratios of the prices. A valid conclusion from the condition $aI = I$ in (ii) is that the equations (ii) are satisfied by some $p \neq o$. The uniqueness depends on the rank of $a - I$. A proper question is about the existence, and the uniqueness, of a solution of (ii) subject to (iii) or, possibly more suitably, to

$$p > o, \ \ pI = 1 . \tag{iii$'$}$$

That there generally exists a solution subject to (iii) is known immediately from the theory of Markov processes. Any prices which are such a solution are called *Sraffa's prices* (phrase introduced by this writer whose echo has been heard with a different commentary at symposia in Naples), or alternatively, *consistent prices*. The further issue about solutions subject to (iii'), that is, about the existence of positive Sraffa's prices, involves the irreducibility condition, put in economic terms in the last section. It is settled, again, by a theorem from the theory of Markov processes. From it, we have that *the irreducibility is necessary and sufficient for both existence and uniqueness*.

The Sraffa matrix a is a Quesnay *tableau économique*, and because of the choice of units making outputs all 1 it is also a Leontief input-output matrix. The special feature of the Sraffa subsistence economy is that the outputs equal the inputs so that net outputs are all zero. In terms of the input-output theory, this may not be a productive economy, or even a semi-productive one, since no goods are produced finally. In any case, we cannot freely think that Sraffa's economy is a Leontief economy, where there is a choice of activity on the linear model, even though, for that matter, Leontief did take such a liberty with similar data.

A subgroup E of sectors is an *independent sub-economy* if

$$a_{ij} = 0 \ \text{ for } \ i \in \bar{E}, j \in E.$$

That is, sectors which are in E use no inputs produced by sectors which are not. Such a sub-economy would, out of self-interest, were such a thing understandable here, break away from the others and possibly become a better-than-subsistence economy on its own. Irreducibility means the non-existence of such a sub-economy. It. guarantees the existence of positive Sraffa's prices, and is implied by their existence. It is also equivalent to Sraffa's prices being unique.

Commodity i is necessary for the production of commodity j, or *j depends* on i, if $a_{ij} > 0$. A group of commodities are *independent* if they are independent of all others outside the group. Thus if commodities *1,…, r* are independent the distribution matrix has the form

$$a = \begin{pmatrix} a_{00} & a_{01} \\ o & a_{11} \end{pmatrix}$$

where a_{00} is $r \times r$. Commodities are *interdependent* if no independent proper subgroup of them exists, in other words, the system is irreducible. There is the theorem that *in just this case consistent prices are unique and all positive.*

Commodity i is necessary for the production of commodity j t periods later if there exists a chain of t commodities following i and ending in j each of which is necessary for its successor. An equivalent condition is $a_{ij}^t > 0$. If $a_{ij}^t = 0$ for all t then j is *completely independent* of i. A group is completely independent if its members are so in respect to all others. If no such proper subgroup exists, the commodities are *completely interdependent*. Because it is a strengthening of immediate interdependence, this condition also gives the conclusion that consistent prices are unique and positive.

If $a^t > o$ for some t then the same holds for all larger t, so this implies complete interdependence. The converse, which is not immediate, is also true. Thus these conditions are equivalent, and they are also equivalent to the convergence $a^t \to \bar{a}$ $(t \to \infty)$, the limit being a positive distribution matrix whose rows are all identical, and equal to the unique distribution vector $\bar{p}$ such that $\bar{p}a = \bar{p}$, that is to the unique vector of consistent prices. This shows that, with any initial distribution vector p_0, and $p_t = p_0 a^t$,

$$p_t \to \bar{p} \ (t \to \infty).$$

The same conclusion is obtained if a, or some power of a, has a positive column, that is if there exists some commodity which, immediately or after several periods, is dependent on all commodities for its production. Such immediate dependence corresponds to a *standard commodity* of Sraffa.

The algebraical theorems which give the general existence of consistent prices, and then that they be unique and positive in the case of interdependence, and approached by successive approximations in the case of complete interdependence or the existence of a Sraffa standard product, are familiar from probability theory in connection with Markov processes.

2 Interdependence and stability

With the Sraffa distribution matrix a, and any prices p,

$$v = pa - p$$

is the vector of value losses to sectors, and $-v$ the gains, or profits. The algebraical sum of the losses, or the gains, is zero. For with $aI = I$ we have

$$vI = paI - pI = pI - pI = 0.$$

There is a loss to sector i if $v_i > 0$ and a gain or profit if $v_i < 0$, and the total of losses equals the total of gains, as in a zero-sum game, so the winners take away from the losers. With Sraffa's prices we have $v = 0$, and so no such imbalance, but equilibrium. Whenever $v \neq o$ there is inequity, exploitation; forces are present—if not for revolution, then for a change of prices. The price p_i which determines the value of the product of sector i can be adjusted to compensate the current loss v_i by making it $p_i^+ = p_i + v_i$ in the next period. The prices therefore become

$$\begin{aligned} p^+ &= p + v \\ &= p + (pa - p) \\ &= pa \end{aligned}$$

The new losses and gains generally will not be zero and the process must be repeated, indefinitely, producing a series of prices pa^t $(t = 0,1,2,\ldots)$. Sraffa's prices p^* always exist. But under a certain condition, which also assures they are unique and positive, we have

$$pa^t \to p^* \left(t \to \infty\right),$$

so the series is always convergent, to a limit which is independent of the initial prices p and equal to the Sraffa price vector p^*, so we have $v \to o.$ The required condition is more than the irreducibility of the Sraffa distribution matrix a, only to exclude the periodic case. In that special case there can be chains of dependence which close into cycles involving a subgroup of sectors, leaving others outside the circle. It is reflected by some power of the the matrix a being reducible, even if a is not. If there is such a power at all it will occur before the nth. In that case the prices would tend to run through a cycle of values, and so to oscillate indefinitely instead of converging, even though the various values on the cycle converge. Contrivance is needed to produce such a case, and if it is excluded then irreducibility is the required convergence condition. One way of excluding it is to require all powers up to the nth to be irreducible. The condition has a direct economic sense which extends to other models, beside Sraffa's and Leontief's. The concern of it is interdependence between sectors, so it is relational rather than quantitative, and it is also readable directly from Quesnay's *tableau économique*.

The standard of value regulating prices is a total commodity standard, having reference to all that is produced and giving this unit value. The stabilizing Standard Product could simply be Government, which taxes part of output and supplies part of the input of every process, creating interdependence. Sraffa's model suggests economic arrangements where what is produced and distributed is constrained to at least meet specific needs or agreements, but is permitted tolerances and is subject to hazards so accounts will never quite balance with any prevailing prices. But prices can be continually readjusted to offset losses and gains in account balances. Then they will not have destabilizing movements but will tend to settle down, except that the movements would become more pronounced and then again attenuated following an alteration in the production and distribution pattern. Processes being subject to hazards, on the sides of inputs, which might not be delivered, and outputs, which might not reach expectations, Government, in having command over a portion of all commodities, beside creating a stabilizing interdependence would be able to function as a kind of Input-Output Insurance Company, redistributing the impacts of such hazards.

This treatment of Afriat (1975) has, so I understand, been followed by Hahn (1982).

3 Production with a surplus

A Quesnay table has the form $T\ Y\ X$ where T is the transaction table, X the gross product vector and Y the net product, or surplus after the factors of production have been replaced. All entries are taken to be non-negative and there is the accounting identity

$$TI + Y = X.$$

We have chosen the units to make $X = I.$ Then the Leontief coefficients are

$$\begin{aligned} a_{ij} &= T_{ij} / X_j \\ &= T_{ij} \end{aligned}$$

so $a = T$ and the transaction matrix T already is the Leontief matrix. Thus we have

$$aI + Y = I,$$

the matrix a being ambiguously the Sraffa, Leontief and Quesnay matrices simultaneously. In the subsistence case there is no surplus so that $Y = O$ and hence $aI = I$. In any case, $aI \leq I$, since it is understood that $Y \geq O$. The surplus or net product is

$$Y = I - aI = (1 - a)I$$

and some goods are produced with a surplus if $aI \lesssim I$, and all are if $aI < I$.

For the case of production with a surplus, Sraffa introduces a rate of profit r simultaneously with prices p by means of the condition

$$(1 + r)\, pa = p$$

which makes the value of output in any sector the profit factor $e = 1 + r$ times the value of the inputs.

Sraffa argues that here are n independent equations to uniquely determine n unknowns, the profit factor e and $n - 1$ independent ratios of the prices p. These are not linear equations in all the variables, so the existence question is not so straightforward, and is even less so if prices are taken to be semi-positive, as expressed by

$$p \geq o, \ \ pI = 1.$$

The Perron-Frobenius theorem on non-negative matrices shows that his conclusion is correct as concerns existence provided a is irreducible. Also, under this condition, if r is given the smallest possible value for any solution, then the corresponding p is unique and positive.

For another view, consider an interest factor i across a production period when the prices are $p \gtrsim o$. The costs of the inputs are given by ipa and the returns on outputs by p, and so the profits by $p - ipa$. Then

$$e = \inf\{i : ipa \geq p, p \geq o, pI = 1\}$$

is the lower limit of interest factors consistent with nonpositive profits. Since p is restricted to a compact set it is attained for some p, and so is a minimum. With any prices p, the minimum interest factor is

$$e(p) = \min\{i : ipa \geq p\}$$

and then

$$e = \min\{e(p) : p \gtrsim o\}.$$

Then we have

$$epa \geq p, \ \ p \geq o, \ \ pI = 1$$

for some p, and for all p', and e',

$$e'p'a \geq p', \ \ p' \gtrsim o \ \Rightarrow \ e' \geq e.$$

Sraffa's problem now has a resolution for the case where a is irreducible; for then, moreover

$$epa = p, \ \ p \gtrsim o,$$

and such p with the normalization $pI = 1$ is unique. With this background we see Sraffa's profit rate rather as the minimum interest rate at which a positive profit is impossible at any prices.

Sraffa's profit rate is introduced in value terms without reference to a growth rate in the real terms of production. It lacks sense without such an anchor because, for all we know or have been told, there is nothing one can do with value except buy goods. In any case, we should see if Sraffa's value profit rate has, accidentally, any definite relation to the real growth rate. The growth factor is the largest multiple of inputs which can be replaced by outputs, or does not exceed them, so in Sraffa's economy it is

$$g = \max\{t : aIt \leq I\}.$$

We have $g \geq 1$ since $aI \leq I$; also $g > 1$ only if $aI \lesssim I$, otherwise $g = 1$. In any case $aIg \leq I$, and so

$$paig \leq pI = 1$$

Also, from $epa \geq p$ it follows that

$$epai \geq pI = 1,$$

and hence

$$(paI)\,g \leq (paI)\,e.$$

With $aI \gtrsim o$ and $p \gtrsim o$ we have $paI \gtrsim 0$, and it follows that $e \geq g$. The case $e \gtrsim g$ is likely in Sraffa's economy, where there is no choice of activity. Here, therefore, there might be a proof that Sraffa's economy is inflationary, were it possible to give inflation a meaning in this model.

Suppose now that some $a, b \in \Omega_m^n$ are given, and any $t \in \Omega^m$ determines m possible processes, where process j has input and output

$$x_{ij} = a_{ij}t_j, \ \ y_{ij} = b_{ij}t_j$$

of commodity i. It is required to choose processes, that is choose t, and at the same time choose prices $p \geq o$ and a profit rate r ($1 + r > 0$) at those prices, which is not permitted to exceed the rate of growth with the chosen processes, so

$$\sum_j y_{ij} \geq (1+r)\sum_j x_{ij} \quad \text{for all i,}$$

and which is to be the maximum profit rate attainable, at the prices, with any available process, so

$$\sum_i p_i b_{ij} \leq (1+r)\sum_i p_i a_{ij} \quad \text{for all } j.$$

The intention is that this profit rate be attained by the chosen processes, that is

$$\sum_i p_i y_{ij} = (1+r)\sum_i p_i x_{ij} \quad \text{for all } j.$$

But this is a consequence of the foregoing requirements. The following shows possibility of fulfilling these requirements, and at the same time identifies r with the maximum possible rate of overall real growth with the available processes.

There is the theorem that, for any a, b such that

(α) $t \geq o \Rightarrow at \geq o$, equivalently $p > o \Rightarrow pa > o$,

(β) $p \geq o \Rightarrow pb \geq o$, equivalently $t > 0 \Rightarrow bt > o$,

(γ) $a + b > o$,

there exists $t \geq o$, $p \geq o$ and unique r, $0 < 1 + r < \infty$, for which

$$pb \leq (1+r)\, pa, \; bt \geq at(1+r).$$

The value of r thus determined is the von Neumann rate associated with (a, b), and is identified with the maximum r for which there exists $t \geq o$ such that

$$bt \geq at(1+r).$$

Capability for subsistence requires $r \geq 0$.

Afriat (1974) gives a discussion and modification of von Neumann's arguments which concern this theorem, and alternative proofs.

Application of the conclusion to the original case where each commodity is produced separately by a single process, so a is $n \times n$ and $b = 1$, gives

$$p \leq (1+r)pa, \quad t \geq at(1+r)$$

for some $p \geq o$, $t \geq o$ and r. If all goods are produced in positive amounts, so $t > o$, then

$$p = (1 + r)pa$$

and with threshold subsistence, where $r = 0$,

$$p = pa.$$

It should be noted that, while r is identified with the maximum of the growth rates for all semi-positive activity vectors t and also, because of (α), for all semi-positive quantity vectors of commodities, it is also, as follows from a theorem of McKenzie (1967) and is proved directly by Afriat (1974), identified with the upper limit of growth rates for positive quantity vectors. Thus, while only some of the commodities can grow simultaneously at the maximum rate, all can grow simultaneouslyat any rate less than that. This is an important conclusion, if the production of all commodities and not just some of them is important.

To obtain this conclusion, and also von Neumann's theorem, let

$$\bar{\rho} = \sup[\rho : bt \geq at\rho,\, t \geq o]$$

$$\dot{\rho} = \sup[\rho : bt \geq at\rho,\, t > o]$$

and

$$\bar{\sigma} = \inf[\sigma : pb \leq \sigma pa,\, p \geq o]$$

$$\dot{\sigma} = \inf[\sigma : pb \leq \sigma pa,\, p > o]$$

so immediately

$$\dot{\rho} \leq \bar{\rho}, \; \dot{\sigma} \geq \bar{\sigma},$$

and also

$$(\alpha) \Rightarrow \dot{\sigma} < \infty, \quad (\beta) \Rightarrow \dot{\rho} > 0, \quad (\gamma) \Rightarrow \overline{\rho} \le \overline{\sigma}$$

so that

$$(\alpha),(\beta),(\gamma) \;\Rightarrow\; 0 < \dot{\rho} \le \overline{\rho} \le \overline{\sigma} \le \dot{\sigma} < \infty.$$

By a theorem on systems of linear inequalities, either

$$(i) \quad bt \ge at\mu \quad \text{for some } t \ge o$$

or

$$(ii) \quad pb < \mu pa \quad \text{for some } p \ge o$$

and not both. Thus

$$\mu > \overline{\rho} \;\Rightarrow\; \sim(i) \;\Rightarrow\; (ii) \;\Rightarrow\; \mu > \dot{\sigma},$$

showing $\overline{\rho} \ge \dot{\sigma}$. Similarly $\overline{\sigma} \le \dot{\rho}$.

Accordingly,

$$(\alpha),(\beta),(\gamma) \;\Rightarrow\; 0 < \dot{\rho} = \overline{\rho} = \overline{\sigma} = \dot{\sigma} < \infty.$$

The theorem of von Neumann gives $\overline{\rho} = \overline{\sigma}$ and also that the limits $\overline{\rho}, \overline{\sigma}$ are attained.

To show $\overline{\rho}$ is attained suppose the contrary, that

$$bt \ge at\overline{\rho} \quad \text{for some } t \ge o$$

is not the case. Then

$$pb < \overline{\rho} pa \quad \text{for some } p \ge o,$$

which implies $\overline{\rho} > \dot{\sigma}$, and contradicts the conclusion $\overline{\rho} \le \dot{\sigma}$ required by (γ). Similarly $\overline{\sigma}$ is ttained.

This proves von Neumann's theorem, and also shows the enlargement of it involving $\dot{\rho}, \dot{\sigma}$. It signifies that, by permitting an arbitrary small inconsistency, through taking a smaller growth rate and a larger profit rate, both the quantities and the prices of commodities can be all positive.

4 Joint production

Instead of having *n* goods each produced separately by *n* sectors, suppose there are *n* sectors each of which jointly produces many goods from a possible *m*. Let

$$a_{ij} \ge 0, b_{ij} \ge 0$$

be the input and output of good *i* by sector *j*. In a subsistence economy the total input and output of any good *i* are equal, so with this common total taken as the unit of amount for each good we have

$$\sum\nolimits_j a_{ij} = 1, \;\; \sum\nolimits_j b_{ij} = 1 \;\; \text{for all } i,$$

that is,

$$aI = I, \;\; bI = I.$$

Also $a \geq o, \ b \geq o$ so a, b are a pair of rectangular row-distribution matrices, of order $m \times n.$ The original model of Sraffa corresponds to the case where $m = n$ and b is the unit matrix.

Any prices p are required to be non-negative and with sum 1,

$$p \geq o, \ pI = 1,$$

and the value equation between input and output in every sector requires

$$pa = pb.$$

In the case of Sraffa's subsistence economy such prices would be ordinary Sraffa prices, and their existence is assured. But in the more general case with joint production there is no such assurance. With

$$\Pi = \{ p : p \geq o, \ pI = 1 \}$$

as the price simplex, consider the polytopes

$$A = \{ pa : p \in \Pi \}, \ B = \{ pb : p \in \Pi \}$$

which are the convex closures of the rows of a, b lying in the distribution simplex

$$\Delta = \{ d : d \geq o, \ Jd = 1 \}$$

where J is the row vector with n elements all equal to 1. The existence of consistent prices immediately implies that A and B intersect, and there is no general reason why they should. One could divide the simplex A into two parts linearly, so that both parts are convex, and take the rows of a in one part and of b in the other. Their convex closures would then be disjoint. In the special case of Sraffa we have $\Delta = B$ so that $A \subset B$, and so of course this cannot be done. Sraffa's economy has generalizations in which consistent prices still must exist. One is where $m > n$ and some n of the goods are produced entirely by some n different sectors; in other words, each sector has a monopoly in the production of at least one good.

There is now no general necessity for the existence of consistent prices, but a specia1 condition which assures their existence is that $A \subset B$. In the first considered case, where $m = n$, so $A \subset \Delta,$ and b is the unit matrix, so $B = \Delta,$ this condition is automatically satisfied. The condition requires the input distribution of any commodity to be a mixture of the output distributions of all the commodities

$$a_i = c_i b = \sum_k c_{ik} b_k$$

where $c_i \in \Omega_n, \ c_i I = 1,$ that is

$$a = cb$$

where $c \in \Omega_m^m, \ cI = I.$ Then $pc = p$, for some $p \in \Delta$, so

$$pa = pcb = pb,$$

as required for consistent prices p. Again, an interdependence condition assures $p > o$.

An alternative reading of $a = cb$ is

$$a_j = cb_j = \sum_k c_k b_{kj}$$

where j suffixes denote columns, corresponding to processes. Thus for any process j the n input quantities a_{ij} are obtained by taking n averages of the output quantities b_{kj}, the averaging coefficients being independent of the process, and given by the rows of c. Alternatively, the columns of c describe n composite commodities having a correspondence to the n simple ones. The input of a process is derived from the output by exchanging the bundle of simple commodities in it for the corresponding bundle of composite commodities and then aggregating this into a bundle of simple commodities. So to speak, inputs are recoverable from outputs by substituting the quantities of simple commodities in it by the same quantities of the corresponding composites, and then collecting the quantities of simple commodities in the result. It is as if there were a shadow system of production where each simple commodity is produced separately by a single process, using up a composite commodity, c being the distribution matrix for this system, and consistent prices for it then give consistent prices for the original.

5 Variable activity

A principle about value is invalidated as a general principle if it requires very special circumstances for its applicability, and we saw that Sraffa's cannot generally be applied to a subsistence economy with joint production. Also, a rate of profit for production with a surplus is introduced purely in value or money terms, without any explicit relation to the real terms of production. In his model the profit rate can exceed the physical growth rate. From experience, this might signify an inflationary situation, but here it cannot, since there is no sure way to interpret inflation in this model, where prices have significance only through their ratios. We are not told what happens to the profit and surplus, and without other guidance they seem useless. Sraffa in his preface emphasizes that the production plan is fixed, in order to guard against any presumption that he is dependent on constant returns. Then the surplus cannot be used to expand production, and we do not know what happens to it. Sraffa might be forced to allow variable activity to give a destination to the surplus and profit, and we are also. Another observation is that his positive profit rate has an alternative meaning: it is also the minimum interest rate which makes positive profit impossible at any prices. Zero profit is associated in theory with perfect competition—an uncongenial model in this setting—but if one adopts the latter meaning a way is open for resolving these difficulties.

As usual, irreducibility, the nonexistence of an independent subeconomy, will play a part, and this is suitable if an economy is a proper unit arising from an interdependence between the parts. Sraffa distinguishes *basic goods* essential to the production of all others and *luxury goods*, which are not essential to any. In an irreducible economy all goods are basic if not from direct dependence of other goods on them then indirectly from chains of dependence. He remarks that there are no luxury goods in a subsistence economy because every output immediately becomes an input, although they can arise when there is a surplus. There cannot be any luxury goods if the economy is irreducible. But this would imply that the smoke issuing from factory chimnies is a luxury! There can be some grievance about the smoke, but not that sort. The obvious way of introducing variable activity is to turn the Sraffa economy into a Leontief economy. This is especially easy, since the Sraffa matrix is already, with the quantity units that have been adopted, also a Leontief matrix.

6 Sraffa and Leontief

Since the output quantities have been made the units, the vector I with all elements 1 is the output vector, and aI is the input vector. The subsistence case is where $aI=I$. When a is regarded as a Leontief matrix the output can be any $x \geq o$, and ax is the input required for it. The Sraffa model then corresponds to the case where only $x = I$ is allowed.

In reality, production takes time, and inputs come before outputs. If the outputs supply inputs it must be for the next round of production. The output x_t in period t is the resource for the input ax_{t+1} in the next period, and cannot be exceeded by it. The output in one period puts a condition on the possible output in the next, and for a series of outputs to be feasible it is required that

$$x_t \geq ax_{t+1} \quad t = 0,1,2,\ldots$$

The condition for successive outputs x, y to be feasible is that $x \geq ay$, and for growth by a factor θ it is required that $y \geq x\theta,$ so we have

$$x \geq ay \geq ax\theta.$$

Thus $x \geq ax\theta$ is a necessary condition for growth θ of an output x. Also it is sufficient since, given this condition, we can take $y = x\theta,$ and then $x \geq ay$, $y \geq x\theta$ as required.

Any output $x \gtrsim o$ is associated with a growth factor

$$\begin{aligned} g(x) &= \max\{\theta : x \geq ax\theta\} \\ &= \min\{x_i / a_{(i}x : a_{(i}x \gtrsim o\} \end{aligned}$$

so

$$x \geq ax\theta \;\Leftrightarrow\; \theta \leq g(x).$$

The function $g(x)$ depends only on the ratios of the elements of x. Also, $x \gtrsim o$ is equivalent to $x \geq o$ and $Jx \gtrsim 0,$ J being a row vector with elements all 1. The range of $g(x)$ therefore is unaltered by restriction to the set

$$X = \{x : x \geq o, Jx = 1\}.$$

The set X is compact and $g(x)$ is continuous in it, and so attains a maximum. The system therefore has a maximum growth factor, for all possible outputs $x \gtrsim o,$ given by

$$\begin{aligned} g &= \max\{g(x) : x \gtrsim o\} \\ &= \max\{\theta : x \geq ax\theta, x \geq o, Jx = 1\}. \end{aligned}$$

With a Leontief matrix a, we consider conditions

$$ax \leq x \text{ for some } x \gtrsim o$$
$$ax \gtrsim x \text{ for some } x \geq o$$
$$ax < x \text{ for some } x \geq o$$

These are required for the economy to be capable of, respectively, at least subsistence, to be semi-productive, maintaining levels of all goods with a surplus of some, and to be productive, with a surplus of all, respectively. If the economy is irreducible, the last two conditions are equivalent. A main input-output theorem is that the last condition is necessary and sufficient for the inverse of $1-a$ to exist and be non-negative. Now that growth has been brought in, the conditions are just telling us about g, the first that $g \geq 1$ and the last that $g \gtrsim 1$. Also, the way the theorem is formulated is artificial. It makes sense when production has no reference to time; if it does, then outputs are required to supply the inputs that produced them as if they were available for that purpose in advance of their own production. Another form for the theorem is that

$$|1-ga| = 0,$$

and a necessary and sufficient condition for the inverse of $1-\theta a$ to exist and be non-negative is that $\theta < g$.

Outputs replace inputs, and for the fixed production economy the replacement is stated to be physical; then the prices which are introduced have no function. An alternative view is that the cost $pa_{j)}x_j$ of input of any industry j is borrowed at the beginning of the production period, and paid back with interest from the return $p_j x_j$ on output x_j at the end. If the interest factor is θ the profit is

$$\left(p_j - \theta pa_{j)}\right)x_j .$$

Perfect competition denies positive profit in equilibrium, so θ is an admissible interest factor if

$$\theta pa_{j)} \geq p_j \quad \text{for all } j,$$

that is, if $\theta pa \geq p$. Solvency of industry j requires a non-negative profit, and so if the rate of profit on output is negative it will not produce; that is,

$$\theta pa_{j)} \geq p_j \;\; \Rightarrow x_j = 0.$$

With $\theta pa \geq p$ this condition is equivalent to $\theta pax = px$.

The considerations given to the maximum growth factor apply similarly to the minimum interest factor for the system, given by

$$h = \min\left\{\theta : \theta pa \geq p,\, p \geq o,\, pI = 1\right\}.$$

We now have a maximum growth factor g and a minimum interest factor h and some quantities x and prices p with which they are achieved. For these we have

$$axg \leq x, \quad hpa \geq p.$$

Industry solvency requires, moreover, that $hpax = px$.

Having introduced prices and hypothetical criteria for an equilibrium such as nonpositive profit and solvency, we can proceed similarly with the growth factor. Equilibrium is not significant in the absence of a mechanism with forces that produce and maintain it, and ideas associated with perfect competition are relevant here. Such a picture would amount to a computational algorithm for the equilibrium, in the way that the Walrasian *tâtonnement* is an algorithm for prices that clear markets, though here we deal with a different model.

Output goods in one period are demanded only as inputs in the next, and if the growth of any good exceeds the maximum overall rate there would be an unusable surplus of it. Excess supply in equilibrium makes a free good, so we have

$$a_{(i}xg \leq x_i \Rightarrow p_i = 0 \quad \text{for all } i.$$

With $axg \leq x$, this condition is equivalent to $paxg = px$.

We now have

$$axg \leq x, \quad paxg = px,$$
$$hpa \geq p, \quad hpax = px,$$

and consequently also

$$paxg = px = hpax.$$

If $pax = 0$, it follows that also $px = 0$. But with a irreducible, this combination is impossible. Therefore $pax \underset{\sim}{>} o$, and it follows that $g = h$. With θ as the common value of g and h, our conditions imply

$$(M) \quad ax\theta \leq x, \quad x \underset{\sim}{>} o,$$
$$(W) \quad \theta pa \geq p, \quad p \underset{\sim}{>} o,$$

and these imply

$$px \geq pax\theta \geq px,$$

so all the conditions follow from these.

The issue now is whether all the conditions entertained can be satisfied simultaneously, or, what now is the same, whether there exist p, x and θ which satisfy M and W.

It is noted that M and W are the Kuhn-Tucker conditions for the function

$$r(p,x) = px / pax \quad (p \geq o, x \geq o)$$

to have a saddle point, with saddle value θ. This function is well defined provided

$$pax = 0, px = 0 \quad (p \underset{\sim}{>} o, x \underset{\sim}{>} o)$$

is impossible as it is since a is irreducible. With the function well defined, the question is whether it has a saddle point.

The conditions make sense even when a is a rectangular matrix, and in fact the existence question is unaffected. It is a special case of the similar question for the von Neumann model dealt with in the next section. That model incorporates joint production, and the numbers of goods and of industries, or activities, that produce them are not restricted to equality. But now we have an irreducible square matrix, and an appeal can be made to the Perron-Frobenius theorem. The conclusion is that the conditions can be satisfied, moreover, with the equalities

$$ax\theta = x, \quad \theta pa = p,$$

and θ is identified not only with the maximum growth factor and the minimum interest factor, but also with Sraffa's profit factor. Also, p provides Sraffa's prices. The profit rate is at least the rate of surplus, and, with the restriction to the single activity $x = I$ of the fixed production economy, it cannot be granted that it is not greater.

7 Sraffa and von Neumann

With the input and output matrices a and b of section 4, suppose now they are rectangular of order $m \times n$, so m goods are produced by n sectors without the restriction $m = n$. The Sraffa economy with separate production is now the case where $m = n$ and $b = 1$. The subsistence case is where $aI = bI$ and there is a surplus of output over input if $aI \lesssim bI$. With variable activity these conditions become less important, and only signify the existence of some quantities with a growth factor of *at* least 1.

Sraffa's prices p and profit π are subject to conditions $\pi pa = pb,$ but these are not generally consistent, and no such prices and profit need exist. An alternative is to think in terms of an interest factor as in the last section. With zero profit, the criterion for solvency, as the maximum profit there are conditions $\pi pa \geq pb$ instead. These are easier to solve, and even too easy. For any prices there exists an interest factor which makes them satisfied. There is no prospect of using these conditions to determine prices, since any prices will do. A limitation on the interest rate is needed. One that is suitable—perhaps even for a capitalist economy—is that it should not exceed the real rate of growth. Without an objective for growth, there is an ambiguity about the rate of growth, by any measure, and the proportional sense used here has a limited significance. But at least, if positive quantities of all goods are growing at a positive rate, then every quantity for every good will be exceeded eventually—or the opposite if the rate is negative, so that there is contraction instead of expansion.

Now we shall describe the linear activity model with joint production due to von Neumann, which extends the Sraffa and Leontief models. Growth is defined with it as reference. The growth factor, and quantity side of the model, fit symmetrically, as a dual, to the interest factor and price side, which have already been touched. A central point is the feasibility of making an interest factor, with some prices, not exceed a growth factor with some quantities, and the uniqueness which results. The theorem of von Neumann offers this with some provisors. At first, with activities fixed, each sector j has an input vector $a_{j)}$ and output vector $b_{j)}$. One way to make this variable is by introducing an activity parameter t_j, making an input $a_{j)}t_j$ and output $b_{j)}t_j$. The parameter, or activity intensity, is not now restricted to the value 1 but can take any value $t_j \geq 0.$ With t as the activity vector, the total input x and output y of the economy are given by $x = at,\ bt = y$. The system so described is called a *linear activity system*, and is an innovation of von Neumann. By taking b with a single row we have many inputs and one output, and so a production function for one good. But the system gives service especially as a model for joint production. The output goods need not be the same as the input goods, though here the m goods listed can include all goods.

Another way to read this system is that at is the vector of minimum inputs required to perform the activity t, and bt the vector of maximum outputs from the activity. Then, with x as the vector of quantities available to serve as inputs, they must be at least enough to support the activity t, so that we have the constraint $x \geq at.$ Another understanding of this constraint is that it expresses free disposal on the input side, or that the excess of availability over requirement can be eliminated without constraint or cost. One could have this and keep the formulation with equations instead of inequalities by introducing disposal activities with input and no output, but this way is more suitable. Similarly, we have the constraint $bt \geq y$ on the output side, showing that outputs can be bt or anything less, and so incorporating free disposal on that side. Thus, in order that any input and ouput x and y with an activity t to be feasible, it is required that

$$x \geq at, \quad bt \geq y.$$

Therefore, for any given x, y, the output y with input x is feasible provided there exists an activity t which satisfies these simultaneous constraints. Thus the input-output relation R for the economy across a single production period is defined by

$$xRy \equiv x \geq at, bt \geq y \text{ for some } t.$$

In particular, $a_{j)}Rb_{j)}$, this corresponding to the case where only sector j is active; and $(at)R(bt)$ is a further case. Free disposal on both sides is expressed by

$$x' \geq xRy \geq y' \Rightarrow x'Ry'$$

or, using the relation product, by

$$\geq R \geq \subset R.$$

Growth can be formulated as for input or for output, or for activity. It makes no difference to growth factors, and activity suits best. Activities t admit a growth factor θ if $at\theta \leq bt$. We already have that prices p admit an interest factor π if $\pi pa \geq pb$. With the constraint $\pi \leq \theta$ making the interest rate at most the growth rate, a question of consistency arises, whether such p, t, π and θ exist.

The conditions

$$\pi pa \geq pb, \ p \gtrsim o,$$

$$at\theta \leq bt, \ t \gtrsim o,$$

imply

$$\pi pat \geq pbt \geq pat\theta.$$

Therefore if it can be granted that $pat > 0$, it would follow that $\pi \geq \theta$ and hence $\pi = \theta$ and hence also

$$\pi pat = pbt = pat\theta.$$

It would be enough to know that

$$pat = 0, \ pbt = 0 \quad (p \gtrsim o, t \gtrsim o)$$

is impossible; for from the last relation $pbt = 0$ if $pat = 0$, and so $pat = 0$ would be denied. This wanted impossibility amounts to a generalization applicable to a rectangular matrix pair a, b of the condition for a single square matrix a to be irreducible. It reduces to that condition when these are square matrices and $b = 1$. It is a generalization arrived at by pursuing the economic sense of irreducibility, the non-existence of an independent subeconomy, with this more general model. For his existence theorem von Neumann required the stronger condition $a + b > o$, which amounts to saying that every good is either an input or an output in every activity, but this condition can replace it. Conditions that are not mentioned, but are also needed, are

$$p > o \Rightarrow pa > o, \quad t > o \Rightarrow bt > o.$$

For another view, the conditions being considered are equivalent to

$$(W)\quad \pi pa \geq pb, \quad \pi pat = pbt,$$
$$(M)\quad at\theta \leq bt, \quad pat\theta = pbt,$$

together with $\pi \leq \theta$, which with irreducibility implies that $\pi = \theta$. Part of (*W*) is the condition

$$\pi pa_{j)} \geq pb_{j)} \text{ for all } j,$$

for π to be a permissible interest factor with the prices *p*, making zero profit the maximum attainable by any sector *j*. With that, the second part is equivalent to

$$\pi pa_{j)} > pb_{j)} \Rightarrow t_j = 0,$$

that is, sectors which do not achieve solvency cease activity. Here are equilibrium conditions of perfect competition. The total profit in all sectors of the economy when the activities are *t* is

$$\left(pb - \pi pa\right)t = 0,$$

so if not all profits are zero then some will be positive and some negative. With free movement of resources from the insolvent sectors to profitable ones the economy will come to rest only when these conditions are satisfied. The survivors will all be solvent, each with zero profit since the total is zero.

Another competitive mechanism is on the side of the goods. They are supplied by output and demanded by input, which are related by the growth factor θ. The total value for all goods of the difference between supply and demand at the prices *p* is

$$p\left(bt - at\theta\right) = 0.$$

Total excess supply value being zero, if it is not zero for all then for some it will be positive and others negative. If the prices are free to rise and fall according to the law of supply and demand, when the economy is at rest any good *i* still in excess supply must be a free good; that is, $p_i = 0$, so there is the condition

$$b_{(i}t > a_{(i}\theta \;\Rightarrow\; p_i = 0.$$

With the condition

$$b_{(i}t \geq a_{(i}\theta \text{ for all } i$$

for θ to be a possible growth factor, with the activity *t*, this is equivalent to $pbt = pat\theta$. The considered conditions are interpreted in this way as equilibrium conditions for perfect competition. When the growth rate determined by the conditions is identified with the maximum rate, some sort of realization for the offerings of the Maximum Doctrine of Perfect Competition is obtained.

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Notation

o, O	vector or matrix with elements all 0
1	unit matrix: 1s on diagonal 0s off diagonal
I, J	column, row vector with elements all 1
Ω	the non-negative numbers
Ω_n	the non-negative row *n*-vectors
Ω^n	the non-negative column *n*-vectors
Ω_m^n	the non-negative $m \times n$-matrices
$a_{(i}, a_{j)}$	row-*i*, column-*j* of matrix *a*
$p \geq o$	non-negative vector
$p \gtrsim o$	semi-positive vector $p \geq o,\ p \neq o$
$p > o$	positive vector
xABy	*xAzBy* for some *z*