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# On Concept Lattices of Efficiently Solvable Voting Protocols

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**Abstract -** It is shown that concept lattices can be attached in a natural way to any voting protocol. The concept lattices of some voting protocols that are solvable with respect to some prominnent solution concepts and outcome-efficient are studied: it is proved that they typically amount to chains.

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# 1 Introduction

Voting protocols are decision mechanisms involving disclosure by participants of private information concerning their own preferences. Voting protocols are solvable with respect to some game-theoretic solution concept or rule if for any admissible preference profile there exists a non-empty set of solutions, and efficiently solvable if the resulting outcomes are Pareto efficient. Thus efficiently solvable protocols are of special interest in that they both enjoy a suitably defined strategic robustness and ensure Pareto-efficiency of the resulting strategically stable outcomes.

Concept lattices of voting protocols provide a structural representation of the a priori decision power of coalitions in terms of the events in outcome space they are able to enforce. In particular, the lenght of the concept lattice of a voting protocols does represent the layers of hierarchy among coalitions induced by the protocol, while the width of the same lattice provides some information about the extent to which coalitional decision power is distributed.

The present paper is devoted to the introduction and study of the concept lattices of some prominent classes of efficiently solvable voting protocols. That study is meant to provide some insight on the 'structural' distribution of decision power among coalitions induced by a few protocols that are 'nicely robust' with respect to some prominent game-theoretic solution concepts.

# 2 Model and results

## 2.1 Voting Protocols

Let $N = \{1, .., n\}$ denote the set of *players*, $X = \{x_1, .., x_k\}$ the set of *outcomes*, and $\mathbf{R}_X$ the set of linear (preference) orders on $X$. A strategic game form on $(N, X)$ is an array $\Gamma = (N, X, (S_i)_{i \in N}, h)$ where $S_i$ is a set, the *strategy set* of player $i, i = 1, .., n$ and $h : \prod_{i=1}^{n} S_i \rightarrow X$ is a surjective function, the *outcome function* of $\Gamma$.A *voting protocol* (in strategic form, with fixed agenda) for $(N, X, \mathbf{R}_X)$ is a strategic game form $\Gamma = (N, X, (S_i)_{i \in N}, h)$ such that $S_i \supseteq \mathbf{R}_X$ for some $i \in N$. Moreover, a voting protocol is said to be *basic* if $S_i = \mathbf{R}_X$ for any $i \in N$, i.e. the outcome function of $\Gamma$ is a social choice function.

A *solution concept* is a rule for solving games of a certain collection: e.g. if $\mathbf{R}_X$ is the set of admissible preferences for each player $i \in N$, then a solution concept for the set $\Gamma(\mathbf{R}_X^N) = \{(\Gamma, R^N) : R^N \in \mathbf{R}_X^N\}$ of games induced by game form $\Gamma$ on the domain $\mathbf{R}_X^N$ of all profiles of total preference preorders on $X$ is a correspondence $\sigma : \Gamma(\mathbf{R}_X^N) \rightarrow\rightarrow \prod_{i=1}^{n} S_i$. A few concrete examples of solution concepts including Nash equilibrium, strong equilibrium and coalitional equilibrium with threats will be introduced and discussed below.

Let $\Gamma$ be a voting protocol and $\sigma$ a solution concept for $\Gamma(\mathbf{R}_X^N)$.Then, voting protocol $\Gamma$ is said to be $\sigma$-*solvable* over preference domain $\mathbf{R}_X^N$ if $\sigma((\Gamma, R^N)) \neq \emptyset$

for any $R^N \in \mathbf{R}_X^N$.

Moreover, at any profile of preference preorders $R^N = (R_1, .., R_n) \in \mathbf{R}_X^N$ we denote $Par(R^N)$ the set of *Pareto efficient outcomes,* namely

$$Par(R^N) = \left\{ \begin{array}{c} x \in X : \text{for any } y \in X \\ \text{if } (y,x) \in \bigcap_{i=1}^{n} R_i \text{ then } (x,y) \in \bigcap_{i=1}^{n} R_i \end{array} \right\}.$$

Voting protocol $\Gamma$ will be said to be *efficiently $\sigma$-solvable* over preference domain $\mathbf{R}_X^N$ if $\emptyset \neq h[\ \sigma((\Gamma, R^N))] \subseteq Par(R^N)$ for any $R^N \in \mathbf{R}_X^N$. It should be emphasized that we do not impose any upper bound on the (finite) size of $X$.[1]

As mentioned in the Introduction, this paper will be devoted to the study of concept lattices of certain important classes of efficiently solvable voting protocols. This task will be accomplished by attaching certain coalitional game forms to voting protocols.

In order to accomplish this task a few more notions are now to be introduced.

## 2.2 Coalitional Game Forms

A *coalitional game form* is a triple $\mathbf{G} = (N, X, E)$ where $N$ and $X$ are non-empty sets denoting the sets of players and outcomes, respectively, and $E : \mathcal{P}(N) \to \mathcal{P}(\mathcal{P}(X))$ is the coalitional power function: the 'power-value' $E(S)$ of coalition $S \subseteq N$ is the collection of all events $A \subseteq X$ coalition $S \subseteq N$ is able to 'force' (under some suitable interpretation of the latter notion). We also assume $\#N \geq 2$ and $\#X \geq 2$ in order to avoid trivialities. A coalitional game form $(N, X, E)$ is a (standard) *effectivity function* (EF) if $E$ satisfies the following boundary conditions:

$EF1$) (Souvereignty) $E(N) \supseteq \mathcal{P}(X) \setminus \{\emptyset\}$ ;

$EF2$) (Null Set Normalization) $E(\emptyset) = \emptyset$;

$EF3$) (Exhaustiveness) $X \in E(S)$ for any $S$, $\emptyset \neq S \subseteq N$.

$EF4$) (Null Event Unenforceability) $\emptyset \notin E(S)$ for any $S$ , $\emptyset \subset S \subseteq N$ .

A CGF is *monotonic* if for any $S, T \subseteq N$ and any $A, B \subseteq X$

$[A \in E(S)$ and $S \subseteq T$ entail $A \in E(T)]$ and

$[A \in E(S)$ and $A \subseteq B$ entail $B \in E(S)]$ .

In what follows we shall confine ourselves to *monotonic* CGFs.

Moreover, it is *regular* if for any $S, T \subseteq N$ and $A \subseteq X$, $A \in E(S)$ entails $X \setminus A \notin E(N \setminus S)$, *maximal* if for any $S, T \subseteq N$ and $A \subseteq X$, $A \notin E(S)$ entails $X \setminus A \in E(N \setminus S)$, *superadditive* if for any $S, T \subseteq N$ and $A, B \subseteq X$, $A \in E(S)$, $B \in E(T)$ and $S \cap T = \emptyset$ entail $A \cap B \in E(S \cup T)$, *convex* if for any $S, T \subseteq N$ and $A, B \subseteq X$, $A \cap B \in E(S \cup T)$ or $A \cup B \in E(S \cap T)$ whenever $A \in E(S)$ and $B \in E(T)$, and *additive* if there exist positive probability measures $p, q$ on $\mathcal{P}(N), \mathcal{P}(X)$ respectively s.t. $A \in E(S)$ iff $p(S) + q(A) > 1$ (an additive EF is also convex).

A *(monotonic) simple game* on $N$ is an order filter of $(\mathcal{P}(N), \supseteq)$ i.e. a set $W$, $\mathcal{P}(N) \supseteq W \neq \emptyset$, such that $S \in W$ and $T \supseteq S$ entail $T \in W$. The coalitions

[1] That kind of generality rules out simple majority voting protocols, which are strategically robust only with (not more than) two outcomes.

belonging to $W$ are meant to represent the *winning* or all-powerful ones. Finally, a CGF $(N, X, E)$ is *simple* if there exists an order filter $W$ of $(\mathcal{P}(N), \supseteq)$ such that for any $S \subseteq N$, $A \subseteq X$, $A \in E(S)$ if and only if either $A = X$ and $S \neq \emptyset$ or $A \neq \emptyset$ and $S \in W$ (notice that a simple EF is –by definition– both and monotonic). Indeed, simple EFs amount to simple games as endowed with a fixed outcome set.

We are mainly interested in those CGFs -and EFs- that can represent the decision power of coalitions under a certain (deterministic) decision mechanism, or *strategic game form* as defined above.

Now, the notion of decision power admits at least two distinct interpretations, namely "guaranteeing power" and "counteracting power" that in turn correspond to the ability to force maximin and minimax outcomes, respectively. Indeed, let $\Gamma = (N, X, (S_i)_{i \in N}, h)$ be a voting protocol (or, for that matter, any strategic game form). Then, the allocation of "guaranteeing power" under voting protocol $\Gamma$ is represented by the $\alpha - EF$ of $\Gamma$- denoted by $E_\alpha(\Gamma)$- as defined by the following rule: for any non-empty $S \subseteq N$,

$$(E_\alpha(\Gamma))(S) = \left\{ \begin{array}{c} A \subseteq X\text{: a } t^S \in \prod_{i \in S} S_i \text{ exists such that} \\ h(t^S, s^{N \setminus S}) \in A \\ \text{for any } s^{N \setminus S} \in \prod_{i \in N \setminus S} S_i, \end{array} \right\}.$$

Conversely, the allocation of "counteracting power" under voting protocol $\Gamma$ is represented by the $\beta - EF$ of $\Gamma$, denoted by $E_\beta(\Gamma)$ and defined as follows :

for any non-empty $S \subseteq N$

$$(E_\beta(\Gamma))(S) = \left\{ \begin{array}{c} A \subseteq X : \text{ for any } s^{N \setminus S} \in \prod_{i \in N \setminus S} S_i \text{ some } t^S \in \prod_{i \in S} S_i \\ \text{and } h(t^S, s^{N \setminus S}) \subseteq A \end{array} \right\}.$$

It is easily checked that $(N, X, E_\alpha(G))$ is regular, $(N, X, E_\beta(G))$ is maximal, and both of them are *monotonic* satisfy *null-event-unenforceability.* Conversely, it is well-known that superadditivity and monotonicity of an EF $\mathbf{G} = (N, X, E)$ imply that it is *$\alpha$-playable* i.e. a strategic game form $\Gamma'$ exists such that $E = E_\alpha(\Gamma')$ : see Moulin(1983), and Otten,Borm,Storcken,Tijs(1995)).

## 2.3 Concept lattices of coalitional game forms and voting protocols

We are now ready to introduce concept lattices of voting protocols.

The *concept lattice* of a CGF $\mathbf{G}$ can be defined through the following steps.

First, define the functions $\curlywedge_E : \mathcal{P}(\mathcal{P}(N)) \to \mathcal{P}(\mathcal{P}(X))$ and $\curlyvee_E : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(\mathcal{P}(N))$ for any $\mathbf{S} \subseteq \mathcal{P}(N)$ , $\mathbf{A} \subseteq \mathcal{P}(X)$ define

$\curlywedge_E(\mathbf{S}) = \{A \subseteq X : A \in E(S) \text{ for all } S \in \mathbf{S}\}$ and

$\curlyvee_E(\mathbf{A}) = \{S \subseteq N : A \in E(S) \text{ for all } A \in \mathbf{A}\}$.

It is easily seen that $(\curlywedge_E, \curlyvee_E)$ is a *Galois connection* between $(\mathcal{P}(\mathcal{P}(N)), \subseteq)$ and $(\mathcal{P}(\mathcal{P}(X)), \subseteq)$ i.e. for any $\mathbf{S}, \mathbf{T} \subseteq \mathcal{P}(N)$, and $\mathbf{A}, \mathbf{B} \subseteq \mathcal{P}(X)$,

i) if $\mathbf{S} \subseteq \mathbf{T}$ then $\curlywedge_E(\mathbf{S}) \subseteq \curlywedge_E(\mathbf{T})$, and if $\mathbf{A} \subseteq \mathbf{B}$ then $\curlyvee_E(\mathbf{B}) \subseteq \curlyvee_E(\mathbf{A})$, and

ii) $(\curlyvee_E \circ \curlywedge_E)(\mathbf{S}) \supseteq \mathbf{S}$,$(\curlywedge_E \circ \curlyvee_E)(\mathbf{A}) \supseteq \mathbf{A}$.

Now, consider

$$\mathbb{C}(\mathbf{G}) = \{(\mathbf{S}, \mathbf{A}) \in \mathcal{P}(\mathcal{P}(N)) \times \mathcal{P}(\mathcal{P}(X)) : \mathbf{S} = \curlyvee_E(\mathbf{A}), \text{ and } \mathbf{A} = \curlywedge_E(\mathbf{S})\}.$$

In the language of formal concept analysis (see e.g. Ganter and Wille(1999)) an element $(\mathbf{S},\mathbf{A})$ of $\mathbb{C}(\mathbf{G})$ is said to be a *concept* of the context $\mathbf{G}$, with *extent* $\mathbf{S}$ and *intent* $\mathbf{A}$ (the latter notions are amenable to straightforward dualizations).

Thus, the (dual)[2] *concept lattice* of $\mathbf{G}$ (sometimes also referred to as its *Galois lattice*) is $\mathbf{L}(\mathbf{G}) = (\mathbb{C}(\mathbf{G}), \geqslant)$ where for any $(\mathbf{S}_1,\mathbf{A}_1),(\mathbf{S}_2,\mathbf{A}_2) \in \mathbb{C}(\mathbf{G})$

$(\mathbf{S}_1,\mathbf{A}_1) \geqslant (\mathbf{S}_2,\mathbf{A}_2)$ iff $\mathbf{A}_1 \supseteq \mathbf{A}_2$ (which is provably equivalent to $\mathbf{S}_2 \supseteq \mathbf{S}_1$), and

$$(\mathbf{S}_1,\mathbf{A}_1) \wedge (\mathbf{S}_2,\mathbf{A}_2) = (\leftthreetimes_E(\rightthreetimes_E(\mathbf{S}_1 \cup \mathbf{S}_2)), \mathbf{A}_1 \cap \mathbf{A}_2)$$
$$(\mathbf{S}_1,\mathbf{A}_1) \vee (\mathbf{S}_2,\mathbf{A}_2) = (\mathbf{S}_1 \cap \mathbf{S}_2, \rightthreetimes_E(\leftthreetimes_E(\mathbf{A}_1 \cup \mathbf{A}_2)).$$

It is also well-known and easily shown that both $(\leftthreetimes_E \circ \rightthreetimes_E) : \mathcal{P}(\mathcal{P}(N)) \to \mathcal{P}(\mathcal{P}(N))$ and $(\rightthreetimes_E \circ \leftthreetimes_E) : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(\mathcal{P}(X))$ are closure operators with respect to set-inclusion (recall that a *closure operator* $K$ on a preordered set $(Y,\geqslant)$ is a function $K : Y \to Y$ such that for any $y, x \in Y : K(y) \geqslant y, K(y) \geqslant K(x)$ whenever $y \geqslant x$, and $K(y) \geqslant K(K(y))$ ), and *extents* and *intents* of concepts are precisely the *closed* elements -or fixed points- of $(\leftthreetimes_E \circ \rightthreetimes_E)$and $(\rightthreetimes_E \circ \leftthreetimes_E)$ respectively (i.e. $(\mathbf{S},\mathbf{A}) \in \mathbb{C}(\mathbf{G})$ iff $\mathbf{S} = \leftthreetimes_E(\rightthreetimes_E(\mathbf{S}))$ and $\mathbf{A} = \rightthreetimes_E(\leftthreetimes_E(\mathbf{A})))$.We shall also denote $(\leftthreetimes_E \circ \rightthreetimes_E)$ and $(\rightthreetimes_E \circ \leftthreetimes_E)$ by $K_{\mathbf{G}}$ and $K^*_{\mathbf{G}}$, respectively. The sets of all fixed points of $K_{\mathbf{G}}$ and $K^*_{\mathbf{G}}$, are also called the *(Galois) closure systems* of CGF $\mathbf{G}$, and denoted by $\mathcal{C}$ and $\mathcal{C}^*$, respectively.

Clearly enough, the concept lattice $\mathbf{L}(\mathbf{G})$ -that is also sometimes called the *Galois lattice* of $\mathbf{G}$ (see e.g. Barbut and Monjardet (1970), chpt. V, vol.2) - is lattice-isomorphic to the lattices of inclusion-ordered closure systems $(\mathcal{C}, \subseteq)$ and $(\mathcal{C}^*, \supseteq)$, respectively (see Davey and Priestley (1990) , chpt. 11, p.227). Hence, $\mathbf{L}(\mathbf{G})$ is complete, has a unique atom if $\mathbf{G}$ is null-set-normalized and a unique co-atom if $\mathbf{G}$ satisfies null-event-unenforceability. Moreover, if $\mathbf{G}$ is linear then $\mathbf{L}(\mathbf{G})$ is also linearly ordered.

Those basic facts concerning $\mathbf{L}(\mathbf{G})$ - which rely on the classic Birkhoff's theorem on concrete -i.e. polarity-induced- Galois connections (see Birkhoff(1967))- can be summarized by the following[3]:

**Proposition 1** *Let $\mathbf{G} = (N, X, E)$ be a CGF. Then, a complete lattice $\mathbf{L}(\mathbf{G})$ - the concept lattice of $\mathbf{G}$, uniquely defined up to isomorphisms- can be canonically attached to $\mathbf{G}$. Moreover, i) if $\mathbf{G}$ is null-set-normalized, then $\mathbf{L}(\mathbf{G})$ is dense i.e. has a minimum that is meet-irreducible ; ii) if $\mathbf{G}$ satisfies null-set-unenforceability, then $\mathbf{L}(\mathbf{G})$ is co-dense i.e. has a maximum that is join-irreducible; iii) $\mathbf{L}(\mathbf{G})$ is finite whenever either $N$ or $X$ is finite; iv) if $\mathbf{G}$ is linear then $\mathbf{L}(\mathbf{G})$ is a chain.*

[2] To be sure, the concept lattice as defined below is indeed endowed with the reverse ordering of the concept lattice as usually defined in the literature. Therefore, as mentioned above, what is referred to as a 'concept lattice' in the text is in fact the *dual* of a concept lattice as usually defined. The reason I insist on dual concept lattices is my intention to focus on rankings of coalitions in terms of decision power, relying on the ability to 'force' events as the relevant criteria/attributes. By contrast, the concept lattice of a CGF in the standard sense is best regarded as a classification of the 'resilience' of (families of) events with respect to coalitional capabilities to act.

[3] A similar result for the slightly more specialized case of EFs is presented and discussed in Vannucci(1999).

**Remark 2** *In particular, Proposition 1 ii) says that* $\mathbf{L}(\mathbf{G}) = \mathbf{1} \oplus B(\mathbf{L}(\mathbf{G})) \oplus \mathbf{1}$ *for some lattice* $B(\mathbf{L}(\mathbf{G}))$ *if* $\mathbf{G}$ *satisfies null-set-unenforceability and null-set-normalization and* $\mathbf{L}(\mathbf{G}) = \mathbf{1} \oplus B(\mathbf{L}(\mathbf{G}))$ *if it is just null-set-normalized (where* $\mathbf{1}$ *denotes the degenerate* 1*-element lattice, and* $\oplus$ *denotes the linear or ordinal sum operation: see e.g. Birkhoff(1967), or Davey,Priestley(1990)). In any case, we shall refer to the lattice* $B(\mathbf{L}(\mathbf{G}))$ *as the* bulk *of* $\mathbf{L}(\mathbf{G})$ .

#### 2.3.1 Efficiently dominant-strategy solvable voting protocols

To begin with, we consider an easy case namely the class of dominant-strategy solvable voting protocols.

Let $g = (N, X, (S_i)_{i \in N}, h, (R_i)_{i \in N})$ -with $R_i \in \mathbf{R}_X$ for any $i \in N$- a game in strategic form, and $i \in N$. A dominant strategy for $i$ is a strategy $s_i \in S_i$ such that for any $s'_i \in S_i$

$h(s_i, s_{N \setminus \{i\}}) R_i h(s'_i, s_{N \setminus \{i\}})$ for all $s_{N \setminus \{i\}} \in \prod_{i \in N \setminus \{i\}} S_i$

and there exists $s'_{N \setminus \{i\}} \in \prod_{i \in N \setminus \{i\}} S_i$ such that

*not* $h(s_i, s'_{N \setminus \{i\}}) R_i h(s'_i, s'_{N \setminus \{i\}})$.

Let us denote $DS_i(g)$ the set of dominant strategies for $i$, and $DS(g) = \prod_{i \in N} DS_i(g)$ the set of all profiles of dominant strategies of game $g$. Then, a voting protocol $\Gamma = (N, X, (S_i)_{i \in N}, h)$ is dominant-strategy solvable (or DS-solvable) over preference domain $\mathbf{R}_X^N$ if $DS(g) \neq \emptyset$ for any game $g = (\Gamma, R^N)$ with $R^N \in \mathbf{R}_X^N$.

It is easily checked that, as a Corollary to the well-known Gibbard-Satterthwaite theorem (see e.g. Danilov, Sotskov (2002)) as combined with the so-called 'revelation principle' that whenever $\#X \geqslant 3$, *dictatorial* voting protocols are the only efficiently DS-solvable ones ( a dictatorial voting protocol invariably selects the most preferred outcome of a fixed player, the dictator). As a consequence of that fact, we have

**Proposition 3** *Let* $\Gamma = (N, X, (S_i)_{i \in N}, h)$ *be an efficiently DS-solvable voting protocol,* $\mathbf{G}_\alpha(\Gamma) = (N, X, E_\alpha(\Gamma)), \mathbf{G}_\beta(\Gamma) = (N, X, E_\beta(\Gamma))$. *Then* $\mathbb{C}(\mathbf{G}_\alpha(\Gamma)) = \mathbb{C}(\mathbf{G}_\beta(\Gamma)) \simeq \mathbf{4}$.

***Remark 4*** *Notice that the concepts attached to a DS-solvable voting protocol may be described by the following intents: 'omnipotent' coalitions (i.e. coalitions able to enforce any event including the null event), 'all-powerful' coalitions (i.e. coalitions able to enforce any event* except for *the null event), 'essentially powerless' coalitions, and 'absolutely powerless' coalitions. The corresponding extents are, respectively, the empty set, the ultrafilter of all coalitions including the* dictator *among their members, the set of all nonempty coalitions, and the set of all coalitions.*

### 2.3.2 Acceptable voting protocols

Next, we consider efficiently Nash equilibrium solvable voting protocols (such protocols are usually referred to as *acceptable*). Let $g = (N, X, (S_i)_{i\in N}, h, (R_i)_{i\in N})$ be a game in strategic form. A *Nash equilibrium* of $g$ is a strategy profile $s = (s_i)_{i\in N}$ such that $h(s_i, s_{N\setminus\{i\}})R_i h(s'_i, s_{N\setminus\{i\}})$ for any $i \in N$ and any $s'_i \in S_i$. The set of Nash equilibria of game $g$ is denoted by $NE(g)$.

No general characterization of acceptable voting protocols is available. The simplest example of an acceptable voting protocol is perhaps the *kingmaker protocol* $\Gamma^K = (N, X, (S_i^K)_{i\in N}, h^K)$ defined as follows:

$S_1 = N\setminus\{1\}$ , $S_i = \mathbf{R}_X$ for any $i \in N\setminus\{1\}$, and

$$h^K((s_i)_{i\in N}) = \max R_j \text{ for any } (s_i)_{i\in N} \equiv (j, R_2, .., R_n) \in \prod_{i=1}^{n} S_i.$$

Hence, under the kingmaker protocol as presented above player 1 selects a dictator or 'king' who in turn proceeds to select the final outcome.

Moreover, a kind of 'universal' family of efficiently Nash-equilibrium-solvable voting protocols $\Gamma^{M(.)} = (N, X, (S_i^{M(.)})_{i\in N}, h^{M(.)})$ -the family of so-called *Maskin protocols* (see again Danilov, Sotskov(2002)) - can be defined as follows. A correspondence $f : \mathbf{R}_X^N \to\to X$ is *Maskin-monotonic* if for any $R^N, Q^N \in \mathbf{R}_X^N$ and any $y \in X$,

if $y \in f(R^N)$ and $\{x \in X : yR_i x\} \subseteq \{x \in X : yQ_i x\}$for any $i \in N$, then $x \in f(Q^N)$.

Let $f : \mathbf{R}_X^N \to\to X$ be a *Maskin-monotonic* non-empty-valued correspondence, then $\Gamma^{M(f)} = (N, X, (S_i^{M(f)})_{i\in N}, h^{M(f)})$ is specified by the following rules:

$S_i^{M(f)} = gr(f) \times \mathbb{Z}_+$for any $i \in N$

(where $gr(f) \equiv \{(R^N, x) : R^N \in \mathbf{R}_X^N, x \in f(R^N)\}$ and $\mathbb{Z}_+$is the set of non-negative integers) and,

for any $s = (s_i)_{i\in N} \equiv ((R_i^N, x_i), z_i) \in \prod_{i=1}^{n} S_i^{M(f)}$

$$h^{M(f)}(s) = \left\{ \begin{array}{c} x : \text{if } (R_i^N, x_i) = (R^N, x) \text{ for all } i \in N \\ y : \text{ if there exists } k \in N \text{ such that} \\ (R_i^N, x_i) = (R_j^N, x_j) = (R^N, x) \neq (Q_k^N, x_k) \text{ for all } i, j \in N\setminus\{k\} \\ \text{and } xR_k x_k \\ x_{i^*} : \text{ where } i^* = \sum_{i\in N} z_i (\text{mod}\,\#N), \text{ otherwise} \end{array} \right\}.$$

Thus, under the Maskin protocol for choice rule $f$ every player chooses a point of the graph of $f$ and a nonnegative integer. If all players happen to choose the same point $(R^N, x)$ the final outcome is $x$. If all players choose the same point $(R^N, x)$ except for one 'dissident' $j$ who chooses $(Q_k^N, x_k)$ then the outcome is $x_k$ if $x_k$ is no better than $x$ *according to preference order* $R_k$, and $x$ otherwise. Under any other circumstance the modular sum $i^* = \sum_{i\in N} z_i(\text{mod}\,\#N)$ is computed to identify the player $i^*$having that number as her identity number: the final outcome is the outcome corresponding to the point of $f$ chosen by player

$i^*$.

It can be shown that such a family of Maskin protocols is 'universal' for acceptable voting protocols in that for each of the latter there is a Maskin protocol which is behaviorally equivalent to the latter (to the extent that Nash equilibrium predicts correctly the participants' behaviour). It is readily checked that the following proposition holds true.

**Proposition 5** *Let $f : \mathbf{R}_X^N \to\to X$ be a Maskin-monotonic and Pareto efficient non-empty valued correspondence*[4] *and $\Gamma^{M(f)} = (N, X, (S_i^{M(f)})_{i\in N}, h^{M(f)})$ the corresponding Maskin voting protocol. Then,*
$\mathbb{C}(\mathbf{G}_\alpha(\Gamma^{M(f)})) = \mathbb{C}(\mathbf{G}_\beta(\Gamma^{M(f)})) \simeq \mathbf{4}$.[5]

***Remark 6*** *The concepts attached to the Maskin voting protocol may also be described by the following intents: 'omnipotent' coalitions (i.e. coalitions able to enforce any event including the null event), 'all-powerful' coalitions (i.e. coalitions able to enforce any event* except for *the null event), 'essentially powerless' coalitions, and 'absolutely powerless' coalitions. The corresponding extents, are respectively the empty set,* the set of all coalitions including at least $n-1$ *agents, the set of all nonempty coalitions, and the set of all coalitions.*

Indeed, both efficiently Nash equilibrium solvable and DS-solvable voting protocols as considered above induce a sharp distribution of decision power among coalitions, and the corresponding effectivity functions are *simple.* This is suitabbly reflected in the structure and 'short' length of their concept lattices.

### 2.3.3 Strong-equilibrium solvable and core solvable voting protocols

Finally, we turn to some solution concepts implying coalition formation and coordinated coalitional behaviour. Again, let $g = (N, X, (S_i)_{i\in N}, h, (R_i)_{i\in N})$ be a game in strategic form. A *strong equilibrium* of $g$ is a strategy profile $s = (s_i)_{i\in N}$ such that for any coalition $T \subseteq N$ and any $s'_T \in \prod_{i\in T} S_i$ there exists $i \in T$ such that $h(s_T, s_{N\setminus T}) R_i h(s'_T, s_{N\setminus T})$. A *coalitional equilibrium with threats of* $g$ is a strategy profile $s = (s_i)_{i\in N}$ such that for any coalition $T \subseteq N$ and any $s'_T \in \prod_{i\in T} S_i$ there exists $s'_{N\setminus T} \in \prod_{i\in N\setminus T} S_i$ and $i \in T$ such that $h(s_T, s_{N\setminus T}) R_i h(s'_T, s'_{N\setminus T})$ (notice that coalitional equilibrium *outcomes* with threats of $g$ do exactly coincide with the *core*[6] outcomes of $g$). Clearly, since

[4] Notice that Maskin-monotonic, Pareto efficient and nonempty valued correspondences defined over $\mathbf{R}_X^N$ do clearly exist, the Pareto correspondence being an obvious example (under mild restrictions on the outcome set $X$).

[5] Similarly, $\mathbb{C}(\mathbf{G}_\alpha(\Gamma^K)) = \mathbb{C}(\mathbf{G}_\beta(\Gamma^K)) \simeq \mathbf{4}$

[6] An outcome $x \in X$ is dominated within a game in coalitional form $G = (N, X, E, (\succcurlyeq_i)_{i\in N})$, where $(N, X, E)$ is a coalitional game form and $(\succcurlyeq_i)_{i\in N}$ is the profile of total preference preorders on $X$, if there exist $A \subseteq X$ and $S \subseteq N$ such that $A \in E(S)$ and for any $y \in A$ and $i \in S$ both $y \succcurlyeq_i x$ and *not* $x \succcurlyeq_i y$. The *core* of $G$ is the set of outcomes of $X$ which are *not* dominated in $G$.

the grand coalition $N$ is one of the feasible coalitions, both strong equilibrium outcomes and coalitional equilibrium outcomes with threats are by definition Pareto efficient.

It turns out that certain *voting-by-limited-veto* protocols ( as briefly introduced in section 2 above, and thoroughly analyzed elsewhere, e.g. Danilov,Sotskov(2002)) enjoy both strong equilibrium solvability and coalitional equilibrium with threats solvability over preference domain $\mathbf{R}_X^N$. Of course, such voting-by-veto protocols rely on a considerably more complex allocation of decision power than the voting protocols considered previously. This is neatly reflected by the properties of their concept lattices. As a prominent example of a voting-by-limited-veto procedure that shares anonymity and neutrality properties with majoritarian-like schemes we shall focus on a version of the *proportional veto protocol*, first introduced by Moulin (see e.g. Moulin,Peleg(1982), Moulin(1983), Peleg(1984), Abdou,Keiding(1991), Danilov,Sotskov(1993,2002)). Namely, we consider a *proportional veto protocol with endogenous agenda formation* that can be informally described as follows. A distinguished outcome $x^*$- the *"status quo"*- is identified. Then, each player *makes k proposals,* is informed on the resulting set of outcomes, and *issues k vetos* - according *to* a *prefixed order* - on non-vetoed alternatives. The unique non-vetoed outcome is selected. The corresponding EF $E^{PV}$ (which is regular and maximal, hence unambiguously determined) is defined by the following rule:

for any $S \subseteq N$, $A \subseteq X$ , $A \in E^{PV}(S)$ if and only if

$$\lceil (kn+1)\tfrac{s}{n} \rceil > kn + 1 - a$$

where $s = \#S$, $n = \#N$, and $a = \#A$ .

Since each coalition-size corresponds to a distinctive "degree" of decision power, the concept lattice $\mathbf{L}(\mathbf{G}^{PV})$ is easily computed. Thus, it is straightforward to establish validity of the following

**Proposition 7** *Let $\mathbf{G}^{PV} = (N, X, E^{PV})$ be the proportional veto EF as defined above. Then, $\mathbf{L}(\mathbf{G}^{PV}) = \mathbf{1} \oplus \mathbf{n} \oplus \mathbf{1}$ (where $\mathbf{n}$ denotes the chain of size $n$ ).*

Clearly, the intents of the concepts attached to the proportional veto protocol may be described as 'the coalitions that are able to veto at least $k \cdot l$ outcomes', with $k \cdot l \leqslant k \cdot n$. The corresponding extents amount to player (sub)sets of cardinality $l$, $0 \leqslant l \leqslant n$.

It should also be remarked that the concept lattices of non-anonymous versions of that voting by veto protocol are also chains.

# 3 Concluding remarks

The concept lattices of some prominent efficiently solvable voting protocols have been introduced and described. It turns out that all of them are chains. Moreover, in nontrivial cases their sizes do not depend on the cardinality of either the player set or the outcome set, except for the voting by veto protocol. Therefore, even allowing for quite different choices of the solution concept, the distribution of coalitional decision power induced by known efficiently solvable voting protocols apparently share a rather simple (concept-latticial) structure. It remains to be seen whether efficiently solvable voting protocols endowed with more general concept lattices do exist.

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