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The Proportional Lottery Protocol is Strongly Participatory and VNM-Strategy-Proof

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Abstract - A voting protocol is said to be *strongly participatory* if for any player i and any strategy profile either the outcome is i's preferred one or i has a strategy which would ensure her a better outcome, and *VNM-strategy proof* if at any preference profile the set of sincere strategies of each player is a VNM-stable set. It is shown that the proportional lottery (PL) modular voting protocol is both strongly participatory and VNM-strategy proof.

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1 Introduction

It is widely agreed that, ideally, a nice voting protocol should provide prospective voters with incentives to participate and to cast votes consistently with their 'true' preferences. But then, does there exist a voting protocol invariably providing each voter with a positive incentive to participate *and* cast a forthright vote, independently of the size of the relevant outcome set and of the population of voters?

In the extant literature the foregoing 'forthrightness' and 'participation' issues have been typically addressed separately, focussing in turn on 'participatory' and 'strategy-proof' protocols. Thus, a voting protocol is usually said to be *participatory* if casting a sincere vote can never induce a worse outcome than abstention, and *strategy-proof* if a sincere vote can never induce a worse outcome than an insincere, manipulative one. Notice that, if voters are allowed to express indifference between distinct outcomes, then under that formulation the 'participation' requirement may also be regarded as a consequence of 'strategy-proofness' by taking abstention to be equivalent to a special case of manipulative voting behaviour. Hence, the existence problem stated above might arguably reduce eventually to addressing the 'strategy-proofness' issue. Now, it is well-known that, by the Gibbard-Satterthwaite theorem, *dictatorial* mechanisms are the only available nonmanipulable or 'strategy-proof' protocols when every conceivable profile of preference rankings is allowed and the range of possible voting outcomes includes three items at least. Observe that dictatorial protocols are clearly *participatory* according to the definition introduced above, even under ballot formats that preclude the expression of nontrivial indifference. It follows that, *if the existence issue concerning forthright and participatory protocols is thus formulated via the 'participation' and 'strategy-proofness' requirements* then the answer is after all already known, thanks to the Gibbard-Satterthwaite theorem. Yes, forthright participatory voting protocols do exist, but they are unfortunately neither nice nor interesting.

However, I would like to suggest that the foregoing formulation, far from being compelling, is indeed rather inadequate. Concerning incentives to vote, the 'participation' requirement as defined above is clearly exceedingly weak: it only requires that voting is never worse than abstaining, allowing for the possibility that those two options be invariably indifferent. That is why dictatorships turn out to be 'participatory' *in that weak sense.* But, as a matter of fact, dictatorial mechanisms are hardly endowed with positive incentive to participate for all voters since the dictator's ballot is by definition the only one to count. Clearly, a stronger property is needed if the notion of positive incentives to participate is to be properly expressed in our model. Here, I opt for the *strongest possible version* of such a notion one may possibly conceive of, namely the requirement that there be for *each* player and under *any* circumstance the possibility to improve *any* improvable outcome by participating. This strong requirement I call *strong participation,* and it is immediately checked that dictatorships are as expected *not* strongly participatory. Hence, any possible positive answer to our search for voting protocols enjoying both (strong) participation and forthrightness clearly also requires some relaxation of the standard 'strategy-proofness' requirement. In that vein, I introduce a mild weakening of the usual strategy-proofness requirement based upon the notion of Von Neumann-Morgenstern stability and therefore denoted as *VNM-strategy-proofness.* VNM-strategy-proofness refers to voting protocols that require submission of information on preferences plus possibly some extra-input: as a result at each preference profile an entire set of 'sincere' strategies may be identified for each voter, and that set is typically not a singleton. VNM-strategy-proofness simply requires the set of sincere strategies of each player to be a *stable standard of behaviour*, namely a *VNM-stable set* with respect to the dominance relation on the player's strategy set. It should be emphasized that strategy-proofness itself, asking the (unique) sincere strategy of any player to be a dominant strategy i.e. the essentially unique maximum of the dominance relation on the player's strategy set, may be regarded as a specialization of VNM-strategy-proofness.[1]

Thus, the present paper combines a significant strenghtening of the participatory requirements with a relatively mild weakening of 'strategy-proofness'. Its main result consists in showing that the proportional lottery protocol (a modular-arithmetical deterministic version of random dictatorship) is indeed both strongly participatory and 'VNM-strategy-proof'. Thus, under the new suggested formulation (strong) participation, forthrightness and a reasonably uniform allocation of decision power are after all consistent.

The paper is organized as follows. Section 2 introduces the basic formal framework, and defines strong participation and VNM-strategy-proofness. Section 3 is devoted to the main result of the paper on strong participation and VNM-strategy-proofness of the proportional lottery protocol. Section 4 includes some comments on the sparse related literature with a few concluding remarks.

2 Strong participation and VNM-strategy-proofness

Let $\mathbb{N}$ denote the countable population of all possible voters, $\mathcal{P}_f(\mathbb{N})$ the set of all *finite* subsets of $\mathbb{N}$, X the *finite* set of all possible outcomes and L_X the set of all linear orders - i.e. total transitive and antisymmetric binary relations - on X. For each voter $i \in \mathbb{N}$, $R_i = L_X$ denotes the set of her possible preferences on X: as usual, the actual preference of a voter is regarded as a nonverifiable private characteristic of the latter.

A *signalling function* of *type* Σ for L_X is an injective function $\sigma : \Sigma \rightarrow \mathcal{P}(L_X)$, while a set U is said to be σ-direct if there exists another set V such that $U = V \times \Sigma$, and *superdirect* on L_X if it is σ-direct with respect to some signalling function σ for L_X. For any *signalling function* $\sigma : \Sigma \rightarrow \mathcal{P}(L_X)$ and preference $\succcurlyeq_i \in L_X$, a σ-signal $x \in \Sigma$ is said to be *sincere* at $\succcurlyeq_i$ if $\succcurlyeq_i \in \sigma(x)$.

[1] One should consider the possible existence of equivalent strategies. Therefore, strictly speaking, strategy-proofness is a special case of VNM-strategy-proofness as defined in this paper on the class of protocols consisting of *reduced* game forms (where equivalent strategies are identified, and the sincere strategy is identified with its equivalence class). Alternatively, one might slightly modify the definition of VNM-strategy proofness, requiring that the set consisting of sincere strategies *and their equivalent strategies* be a VNM-stable set.

A (uniform, strategic) *voting protocol* for $\mathcal{P}_f(\mathbb{N})$ on X is a family $\mathbf{G} = \left\{G^N = (N, X, (S_i)_{i\in N}, h^N)\right\}_{N\in\mathcal{P}_f(\mathbb{N})}$ of strategic game forms (where N, X denote the player and outcome sets, respectively, $(S_i)_{i\in N}$ is the profile of strategy sets, and $h^N \in X^{\Pi_{i\in N} S_i}$ is the (surjective) outcome function) such that for all $N \in \mathcal{P}_f(\mathbb{N})$ and $i \in N$ there exists $s_i^* \in S_i$ satisfying $h^N((s_j)_{j\in N\setminus\{i\}}, s_i^*) = h^{N\setminus\{i\}}((s_j)_{j\in N\setminus\{i\}})$ for all $(s_j)_{j\in N\setminus\{i\}} \in \prod_{j\in N\setminus\{i\}} S_j$.[2]

A voting protocol $\mathbf{G} = \left\{G^N = (N, X, (S_i)_{i\in N}, h^N)\right\}_{N\in\mathcal{P}_f(\mathbb{N})}$ is said to be *order-enriched* if both $\mathbb{N}$ and X are endowed with linear orders $\leqslant_{\mathbb{N}}$and $\leqslant_X$, respectively, *superdirect* on $\mathcal{L} := \bigcup_{N\in\mathcal{P}_f(\mathbb{N})} \prod_{i\in N} R_i$ if for any $N \in \mathcal{P}_f(\mathbb{N})$, and any $i \in N$ the strategy set S_i is superdirect on R_i, *symmetric* if $S_i = S_j$ for any $N \in \mathcal{P}_f(\mathbb{N})$, and any $i, j \in N$. Moreover, a voting protocol $\mathbf{G}$ which is superdirect on $\mathcal{L}$ (w.r.t. signalling function σ) is *Pareto efficient* (w.r.t. σ) if for any $y \in X, N \in \mathcal{P}_f(\mathbb{N})$, preference profile $(\succcurlyeq_i)_{i\in N} \in L_X^N$, profile $(x_i)_{i\in N}$ of sincere σ-signals at $(\succcurlyeq_i)_{i\in N}$, and preference profile $(\succcurlyeq'_i)_{i\in N}$ such that $\succcurlyeq'_i\in \sigma_i(x)$ for any $i \in N$:

if there exists $z \in X\setminus\{y\}$ such that $z \succcurlyeq'_i y$ for each $i \in N$, then $y \notin h^N[\Pi_{i\in N}(V \times \{x_i\})]$.

We shall be henceforth interested in order-enriched, superdirect, and symmetric voting protocols.

Let us now turn to the definitions of the two basic requirements which constitute the focus of the present paper.

A voting protocol $\mathbf{G}$ for for $\mathcal{P}_f(\mathbb{N})$ is *(strongly) participatory* w.r.t. $\mathcal{L}$ if for all $N \in \mathcal{P}_f(\mathbb{N})$ with $|N| \geqslant 2$, for all $i \in N$, $(\succcurlyeq_i)_{i\in N} \in \prod_{i\in N} R_i$ and $s' \in \prod_{j\in N\setminus\{i\}} S_j$ there exists $s_i \in S_i$ such that $h^N((s_i, s')) \succcurlyeq_i h^{N\setminus\{i\}}(s')$ (respectively, such that either $h^N((s_i, s')) \succ_i h^{N\setminus\{i\}}(s')$ or $\left\{h^N((s_i, s'))\right\} \cup \left\{h^{N\setminus\{i\}}(s')\right\} \subseteq$ *top*$(\succcurlyeq_i)$ where $\succ_i$denotes the asymmetric component of $\succcurlyeq_i$and *top*$(\succcurlyeq_i)$ the set of all $\succcurlyeq_i$-maximal outcomes).

Let $\mathbf{G} = \left\{G^N = (N, X, (S_i)_{i\in N}, h^N)\right\}_{N\in\mathcal{P}_f(\mathbb{N})}$ be a voting protocol for $\mathcal{P}_f(\mathbb{N})$ that is superdirect w.r.t. a signalling function $\sigma : \Sigma \to \mathcal{P}(L_X)$: a strategy $s_i = (t, x) \in S_i = T \times \Sigma$ of player $i \in N \in \mathcal{P}_f(\mathbb{N})$ is *sincere* at $\succcurlyeq_i\in R_i$ if x is sincere at $\succcurlyeq_i$i.e. if $\succcurlyeq_i\in \sigma_i(x)$: we denote $S_i^*(\succcurlyeq_i) \subseteq S_i$ denote the set of all sincere strategies at $\succcurlyeq_i$of player i in G^N. For each $i \in N \in \mathcal{P}_f(\mathbb{N})$, and $\succcurlyeq_i\in R_i$ an individual *dominance relation* $\triangle_{\succcurlyeq_i}(G^N) \subseteq S_i \times S_i$ is defined as follows: for any $s_i, t_i \in S_i$, $s_i\triangle_{\succcurlyeq_i}(G^N)t_i$ iff $h^N(s_i, s_{-i}) \succcurlyeq_i h^N(t_i, s_{-i})$ for each $s_{-i} \in \prod_{j\in N\setminus\{i\}} S_j$ and there exists $s_{-i} \in \prod_{j\in N\setminus\{i\}} S_j$ such that $h^N(s_i, s_{-i}) \succ_i h^N(t_i, s_{-i})$.Then,

[2] Thus, it is required that for any set of participants all players have the option of an entirely 'neutral' i.e. inconsequential abstention.

$\mathbf{G}$ is said to be *VNM-strategy-proof* on $\prod_{i\in N} R_i$ if for any $N \in \mathcal{P}_f(\mathbb{N})$, $i \in N$, $\succcurlyeq_i \in R_i$, $S_i^*(\succcurlyeq_i)$ is a VNM-stable set of $(S_i, \triangle_{\succcurlyeq_i}(G^N))$, i.e. $S_i^*(\succcurlyeq_i)$ satisfies the following properties:

(i) Internal stability : for any $s_i, t_i \in S_i^*(\succcurlyeq_i)$, *not* $s_i \triangle_{\succcurlyeq_i}(G^N) t_i$;

(ii) External stability: for any $t_i \in S_i \backslash S_i^*(\succcurlyeq_i)$ there exists $s_i \in S_i^*(\succcurlyeq_i)$ such that $s_i \triangle_{\succcurlyeq_i}(G^N) t_i$.

3 The proportional lottery protocol: main result

We are interested in finding out whether strong participation and VNM-strategy-proofness are consistent properties of a voting protocol. In view of some well-known results on strategy-proof protocols on restricted domains, the most natural candidates for a positive solution are those protocols which arise from suitable 'combinations' of dictatorial mechanisms (see Danilov and Sotskov (2002)). In that vein, we focus on the proportional lottery protocol, a special deterministic version of 'random dictatorship' which relies on modular arithmetic.

The *proportional lottery (PL)* protocol

$\mathbf{G}_{PL} = \left\{G_{PL}^N = (N, X, (S_i)_{i\in N}, h_{PL}^N)\right\}_{N\in\mathcal{P}_f(\mathbb{N})}$ for $\mathcal{P}_f(\mathbb{N})$ on the domain $\mathcal{L} = \bigcup_{N\in\mathcal{P}_f(\mathbb{N})} \prod_{i\in N} L_i^X$ of all profiles of linear orders on X is defined as follows: for any $N = \{1, .., n\} \in \mathcal{P}_f(\mathbb{N})$, $i \in N$, and $(\succcurlyeq_i)_{i\in N} \in \prod_{i\in N} L_i^X$, $S_i = X \times \mathbb{Z}_+$, and for any $s = ((x_1, z_1), .., (x_n, z_n)) \in \prod_{i\in N} S_i$, $h_{PL}^N(s) = x_{i^*}$ where $i^* = \sum_{i\in N} z_i (\operatorname{mod} n)$.

Remark 1 *Notice that PL is a Pareto-efficient voting protocol. To see this, observe that PL as defined above is by definition superdirect on $\mathcal{L}$ w.r.t. the signalling function σ mapping each outcome $x \in X$ into the set of linear orders $\succcurlyeq \in L_X$ such that top$(\succcurlyeq) = \{x\}$. Therefore, for any $y \in X$, $N \in \mathcal{P}_f(\mathbb{N})$, preference profile $(\succcurlyeq_i)_{i\in N} \in \prod_{i\in N} R_i$, profile $(x_i)_{i\in N}$ of sincere σ-signals at $(\succcurlyeq_i)_{i\in N}$, if preference profile $(\succcurlyeq_i')_{i\in N}$ is such that $\succcurlyeq_i' \in \sigma(x_i)$ for any $i \in N$ then by construction top$(\succcurlyeq_i')$ = top$(\succcurlyeq_i)$ for all $i \in N$. Therefore, if there exists $z \in X \backslash \{y\}$ such that $z \succcurlyeq_i' y$ for each $i \in N$, then $y \notin$ top$(\succcurlyeq_i') = \{x_i\}$ i.e. $y \neq x_i$ for any $i \in N$ whence by definition $y \notin h_{PL}^N[\Pi_{i\in N}(V \times \{x_i\})]$ as required.*

Remark 2 *Notice that PL is clearly 'coalitionally anonymous' namely under PL coalitions of the same size can enforce the same outcome-subsets. To check this, notice that each player can subvert any outcome through a judicious choice of her strategy: therefore, while the grand coalition N can obviously enforce any single outcome, any other coalition is unable to enforce any proper subsets of* *outcome. That circumstance is by no means coincidental. It can be shown that* any *strongly participatory protocol must induce such an allocation of decision power among coalitions. The details however will not be spelled here.*

Proposition 3 *The proportional lottery protocol* $\mathbf{G}_{PL} = \left\{G^N_{PL} = (N, X, (S_i)_{i\in N}, h^N_{PL})\right\}_{N\in\mathcal{P}_f(\mathbb{N})}$ *for* $\mathcal{P}_f(\mathbb{N})$ *is strongly participatory and VNM-strategy-proof on* $\mathcal{L}$.

Proof. *Let* $N \in \mathcal{P}_f(\mathbb{N})$, $(\succcurlyeq_i)_{i\in N} \in \prod_{i\in N} R_i$,

$s' = ((x_1, z_1), .., (x_{j-1}, z_{j-1}), (x_{j+1}, z_{j+1}), .., (x_n, z_n)) \in \prod_{i\in N\setminus\{j\}} S_i$, $j \in N$, *and* $x = top(\succcurlyeq_j) \notin h^{N\setminus\{j\}}_{PL}(\sigma)$*. Then, consider* $\sum_{i\in N\setminus\{j\}} z_i (\bmod n)$*: of course, there exists* $z \in \mathbb{Z}_+$ *such that* $(z + \sum_{i\in N\setminus\{j\}} z_i)(\bmod n) = j$*. Then, posit* $s_j = (x, z)$*. Clearly,* $h^N_{PL}((s_j, s')) = x$*. Thus,* $\mathbf{G}_{PL}$ *is indeed strongly participatory.*

Next, take a preference profile $(\succcurlyeq_i)_{i\in N} \in \prod_{i\in N} R_i$ *with* $N \in \mathcal{P}_f(\mathbb{N})$*, and consider the set* $\prod_{i\in N} S^*_i(\succcurlyeq_i)$ *of* sincere *strategy profiles of* G^N_{PL} *at* $(\succcurlyeq_i)_{i\in N} \in \mathcal{D}$*. Let* $s = ((x_1, z_1), .., (x_n, z_n)), t = ((x_1, z'_1), .., (x_n, z'_n)) \in \prod_{i\in S} S^*_i(\succcurlyeq_i)$*, and* $i \in N$*. If* $z_i(\bmod n) = z'_i(\bmod n)$ *then* $h^N(s) = h^N(t) = x \in \prod_{i\in N} S^*_i(\succcurlyeq_i)$ *for any* $s_{N\setminus\{i\}} \in \prod_{j\in N\setminus\{i\}} S_j$ *, therefore -by definition-* not $s_i \triangle_{\succcurlyeq_i}(G^N_{PL})t_i$*. If* $z_i(\bmod n) \neq z'_i(\bmod n)$ *then -by definition of the PL protocol- there exist* $s'_{N\setminus\{i\}}, s''_{N\setminus\{i\}} \in \prod_{j\in N\setminus\{i\}} S_j$ *such that* $h^N(s_i, s'_{N\setminus\{i\}}) = x_i \neq h^N(t_i, s'_{N\setminus\{i\}})$ *and* $h^N(t_i, s''_{N\setminus\{i\}}) = x_i \neq h^N(s_i, s''_{N\setminus\{i\}})$*, whence* not $s_i \triangle_{\succcurlyeq_i}(G^N_{PL})t_i$ *again. Moreover, let* $t \in \prod_{i\in N} S_i$ *a strategy profile that is* not *sincere at* $(\succcurlyeq_i)_{i\in N}$*. Then by definition there exists* $i \in N$ *such that* $t_i = (y_i, z_i) \notin S^*_i(\succcurlyeq_i)$*, namely* $y_i \neq x_i = top(\succcurlyeq_i)$*. Thus, consider strategy* $s_i = (x_i, z_i)$*: clearly, for any* $s_{N\setminus\{i\}} = ((x_1, z_1), .., (x_{i-1}, z_{i-1}), (x_{i+1}, z_{i+1}), .., (x_n, z_n)) \in \prod_{j\in N\setminus\{i\}} S_j$ *either* $h^N(s_i, s_{N\setminus\{i\}}) = x_i$ *and* $h^N(t_i, s_{N\setminus\{i\}}) = y_i$ *hence* $h^N(s_i, s_{N\setminus\{i\}}) \succ_i h^N(t_i, s_{N\setminus\{i\}})$ *or* $h^N(s_i, s_{N\setminus\{i\}}) = h^N(t_i, s_{N\setminus\{i\}}) \neq x_i$*. It follows that* $s_i \triangle_{\succcurlyeq_i}(G^N_{PL})t_i$ *as required.* ■

Remark 4 *It should also be emphasized here that 'Strong Participation' and 'VNM-Strategy-Proofness' are indeed two mutually independent properties for voting protocols. To check this, consider first any standard direct dictatorial mechanism* $\mathbf{G}_{DIC(j)}$*with* $S_i = X$ *as strategy set for each player* i*. Clearly,*

$\mathbf{G}_{DIC(j)}$ *is reduced and strategy-proof hence in particular VNM-strategy-proof, but it is obviously not strongly participatory. Next, take the 'permuted' variant of PL induced by the outcome functions* $h^N_{PPL}(.)$ *as defined by the following rule: for any strategy profile* s, $h^N_{PPL}(s) = x_{\pi(x_{i^*})+1(\bmod n)}$ *where* $i^* = \sum_{i \in N} z_i (\bmod n)$ *and* $\pi : X \rightarrow \{1, .., |X|\}$ *is a function that respects* $\leqslant_X$ *i.e.* $\pi(x) \leqslant \pi(y)$ *if and only if* $x \leqslant_X y$. *It can be immediately checked that such a 'permuted' PL is strongly participatory but not VNM-strategy-proof.*

4 Related literature and concluding remarks

Under the heading of 'strategy-proofness', problems pertaining to incentives for forthrightness of voters' behaviour have spawned a huge literature. A discussion of some selected topics most germane to our present concerns as well as a presentation of the proportional lottery mechanism (under the simpler denotation 'lottery mechanism') is provided e.g. in Danilov and Sotskov (2002).

The paradoxical fact that participation may be occasionally self-damaging - the 'No Show Paradox'- under some standard voting protocols (e.g. plurality voting with runoff) was first noticed by Brams and Fishburn (1983), Fishburn and Brams (1984)). Subsequently Moulin (1988) addressed the issue in a fairly general setting: he introduced a Participation axiom for voting mechanisms requiring that by participating and submitting a sincere preference ranking a voter can never induce a worse outcome than by abstaining. Then, he proceeded to show that when there are at least four outcomes no voting mechanism selecting Condorcet winners when the latter exist do satisfy Participation (a 'No Show Paradox' again), while (essentially) simple scoring mechanisms (with no runoff) always do. Pérez (2001) extends those results to Condorcet voting correspondences, while Saari (1994) relates No Show Paradox-type results to manipulability and (non)monotonicity problems. Holzman(1988/1989) studies Participation under mechanisms with special majority quotas and proves that if the cardinality of the outcome set is four or larger, then the only winning quota consistent with Participation is quasi-unanimity i.e. $(n-1)/n$.

As mentioned in the Introduction above, however, Participation amounts to a rather weak participation incentive. Indeed, dictatorial mechanisms are perfectly consistent with the Participation requirement. That is why this paper focusses on the more demanding Strong Participation property, requiring that under any circumstance each player be able to *positively improve* the outcome by participating and choosing some 'forthright' voting strategy. It would certainly be of some interest a characterization of *all* voting protocols enjoying both VNM-Strategy-Proofness and Strong Participation (or some other weaker property of 'intermediate' strenght i.e. stronger than Participation) over the universal domain as well as over other interesting large domains. I leave it as a topic for further research.

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