UNIVERSITÀ DEGLI STUDI DI SIENA

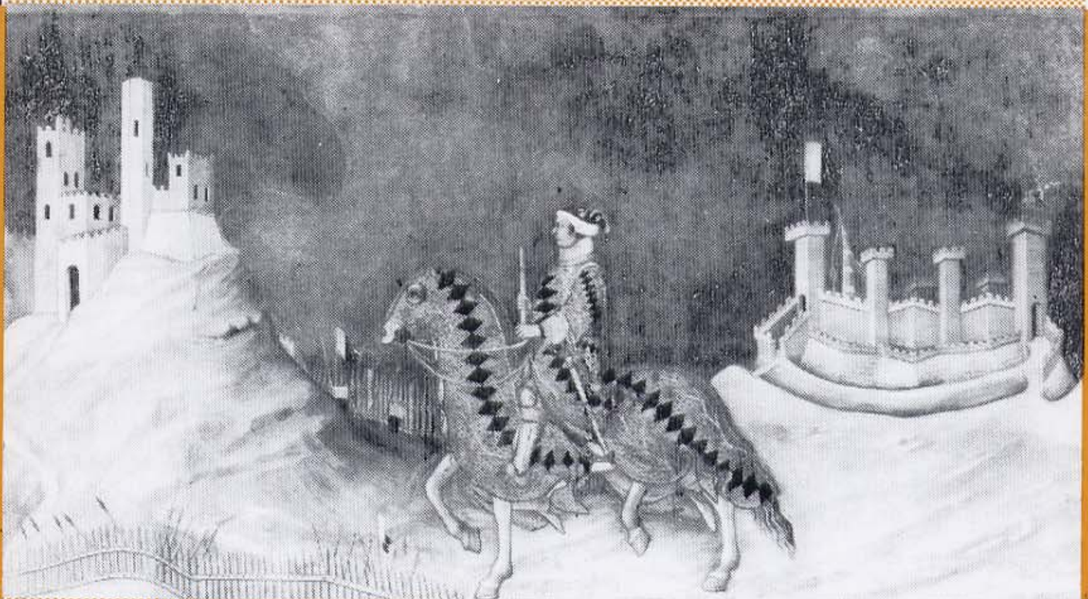
QUADERNI DEL DIPARTIMENTO DI ECONOMIA POLITICA

Marcello Basili

Stefano Vannucci

Diversity as Width

n.500 - Aprile 2007



Abstract - It is argued that if the population of options is a finite poset, diversity comparisons may be conveniently based on widths i.e. on the respective maximum numbers of pairwise incomparable options included in the relevant subposets. The width-ranking and the undominated width-ranking are introduced and characterized.

JEL Classification Numbers: D00, Q00, Z00

Marcello Basili, Dipartimento di Politica Economica, Finanza e Sviluppo, Università di Siena
Stefano Vannucci, Dipartimento di Economia Politica, Università di Siena

1 Introduction

In the last decade, the assessment and measurement of diversity has been the focus of a slowly but consistently growing body of literature. The motivations of such a remarkable growth of interest in diversity measurement come from a wide variety of concerns, including those for biodiversity and the effectiveness of conservation policies, and for several dimensions of socio-economic diversity both agreeable (e.g. diversity among feasible options in choice problems, or valuable aspects of cultural and social diversity[1]) and detrimental (e.g. severe, growth-thwarting inequalities in allocations of available opportunities). As a result, the extant literature on diversity measurement also displays a considerable variety of aims and emphases. In particular, biodiversity-oriented measures typically take into account population sizes in order to address sustainability-related issues. On the contrary, sociodiversity-oriented measures tend to disregard population sizes (see e.g. Baumgärtner (2006) for an useful survey which aptly emphasizes this point). Furthermore, as a consequence of the large variety of contexts where diversity-measurement issues arise, the formats of the relevant data structures are themselves considerably diverse. Accordingly, the current literature on the measurement of diversity deals with several and quite different formats of the characteristic/type spaces of populations, including premetric, semimetric or metric spaces (see e.g. Pattanaik and Xu (2006), Bossert, Pattanaik and Xu (2001), Pattanaik and Xu (2000) and Van Hees (2004), respectively), relational systems consisting of preference profiles as supplemented with a similarity binary relation (Peragine and Romero-Medina (2006)), and (subspaces of) suitably preordered mixture spaces (Nehring and Puppe (2002, 2003)).

The present work contributes to that literature focussing on *the measurement of 'pure' diversity when the relevant type space of population units or opportunities is just a finite partially ordered set (poset).* The foregoing 'pure' qualifier alludes to two distinct features of our analysis, namely

a) we are interested in the measurement of diversity *as such,* and work under the tentative assumption that the former can be attempted without having to commit in advance to any particular interpretation of diversity, or any value judgment concerning its desirability[2];

[1] The latter might arguably include 'undominated diversity' in the allocation of freedoms and opportunities as eloquently advocated in Van Parjis (1995).

[2] Thus, we strongly concur in that respect with Bossert, Pattanaik, and Xu (2001) (but see Baumgärtner (2006) for a quite different view).

b) in particular, we aim at disentangling diversity assessments from other aspects (including sustainability-related features such as population sizes), no matter how important.

On the other hand, focussing on (finite) posets as the relevant data structure is motivated by their somewhat unique combination of extreme informational parsimony and ubiquity: a few characteristic examples are provided in the next section to substantiate such a claim.

Once the general relevance of the foregoing posetic framework is established, one is left wondering what is the proper way to introduce diversity assessments within such a parsimonious data format. Following the standard approach in the literature on diversity measurement, we subscribe to the notion that the diversity of an opportunity set should somehow depend on the assessment of dissimilarities between pairs of its members. However, we maintain that starting from a poset we do not really have to introduce extra information such as a (pre)metric on the option space: we propose to regard *incomparability* between two options as strong evidence of *dissimilarity* between them. From this suggestion, it follows that the comparative *diversity* of subposets should be assessed by aggregating somehow such information about their respective incomparabilities. Of course, there are many possible ways to do that. The present work is devoted to exploring one particularly simple way to evaluate diversity via incomparability as summarized by the following rule: *just rely on the width of subposets to assess their diversity.* Indeed, the width of a subposet counts the maximum number of pairwise incomparable points it contains. Thus, such a basic parameter of any poset arguably conveys in a very effective way some essential information about its intrinsic diversity.

In particular, we define two distinct width-based rankings for subsets of the population space. The first one is the plain width-ranking of subposets. The second one is the undominated width-ranking of subposets or equivalently the ranking induced by the size of undominated subposets. We provide a simple axiomatic characterization for each one of them. Moreover, in order to illustrate the possible uses of width-based rankings we consider their application to a knapsack-problem variant of the so-called 'Noah's ark problem' as introduced by Weitzman (1998) to motivate his own approach to constrained optimality of biodiversity conservation policies.

The paper is organized as follows: Section 2 discusses a few prominent examples of characteristic spaces with a poset structure; Section 3 introduces the basic notations and definitions, and presents the basic model; Section 4 includes the main results; Section 5 presents a knapsack version of the 'Noah's ark problem'; Section 6 provides a discussion of some related literature, and a few concluding remarks.

2 Populations of options as posets: motivation and examples

We shall be concerned with a population of options or opportunities whose type space is a partially ordered set or *poset*[3]. As mentioned in the Introduction, that approach amounts to a significant departure from most of the relevant literature on the measurement of diversity, which is mainly focused on richer environments including metric spaces and similar structures. The present section is meant to motivate such a modelling choice and advocate its relevance by means of a few prominent examples.

2.1 The general case: the dominance order in ordinal multi-attribute spaces

Let us consider a finite population X of opportunities that are described by means of a finite family $\{c_1, ..c_k\}$ of ordinal attributes or criteria[4]. The latter may be aptly represented by a finite family $K = \{f_1, .., f_k\}$ of real-valued ordinal scales $f_i : X \to \mathbb{R}$, $i = 1, .., k$. Then, one may define the *dominance order* $\leqslant_K$induced by K and correspondigly the dominance poset $(X, \leqslant_K)$ by the following rule:

for any $x, y \in X$, $x \leqslant_K y$ iff $f_i(x) \leq f_i(y)$ for any $f_i \in K$.

It should be remarked that the foregoing interpretation of posets is indeed quite general since (as it is well-known by the so called Dushnik-Miller theorem) for any finite poset $(X, \leqslant)$ there exists a finite family $\{\leqslant_1, .., \leqslant_h\}$ of linear orders i.e. total, transitive and antisymmetric binary relations on

[3] It should be remarked that the present approach to diversity as width might be easily extended to *preordered sets*, namely to sets endowed with a preorder i.e. a reflexive and transitive binary relation. That is so, because such a preorder induces in an obvious way a poset on the set of its indifference/equivalence classes. Nevertheless, in the current paper we stick to posets just for the sake of simplicity of presentation.

[4] We identify those opportunities which cannot be distinguished by criteria of the given family. Hence, strictly speaking the elements of X are equivalence classes of opportunities.

X such that $\leqslant = \cap_{i=1}^{h} \leqslant_i$(see e.g. Trotter (1992)). Also, it includes as a particular case the starting point of the Nehring-Puppe approach to diversity analysis, that amounts to a binary multi-attribute description of the population of opportunities (see Nehring and Puppe (2002) and the subsection on relational databases below).

Of course, multi-criteria analysis and its applications in operations research suggest a wealth of examples where ordinal multi-attribute spaces are the basic data format.

But in fact the foregoing ordinal multi-attribute description of options is a remarkably general and flexible format which also accommodates an impressively wide range of interesting issues arising in resource management and environmental economics (we refer the reader to Brüggemann and Carlsen (2006) for an authoritative up-to-date discussion of some relevant examples, including analysis of habitat diversity across landscapes, analysis of water sediment data, risk assessment of chemicals, pollution monitoring data).

Notice that, in view of the Dushnik-Miller theorem mentioned above, the poset $(X, \leqslant)$ can always be regarded as a handy, succinct representation of some instance(s) of the foregoing multi-attribute model.

2.2 Trees

A *tree* $\mathcal{T} = (X, \leqslant)$ is a poset such that for any $x \in X$, the set $x \downarrow = \{y \in X : y \leqslant x\}$ of ancestors of x is a chain i.e. is linearly ordered by $\leqslant$. Populations endowed with a tree structure have been widely considered in the extant biodiversity-oriented literature on diversity measurement, due to the prominent role of evolutionary or philogenetic trees in evolutionary biology (see Weitzman (1992, 1998), Nehring and Puppe (2002, 2003)).

2.3 Concept lattices of relational databases

Let us consider a finite population Y of options/objects that are described by a finite set C of binary attributes (again we may choose to identify those objects which cannot be distinguished by means of those attributes). Then, one may define a binary relation $\rho \subseteq Y \times C$ by the following rule: for any $y \in Y$ and $c \in C$, $y\rho c$ iff $c(y) = 1$ i.e. 'x has attribute c'. The database $((Y, C), \rho)$ is also referred to as a *context*. The information content of $((Y, C), \rho)$ is aptly summarized by its *concept lattice* $\mathbb{C}(Y, C) = (\mathbf{X}, \leqslant)$ as defined by the following construction.

First, define the functions $\leftthreetimes_\rho : \mathcal{P}(Y) \to \mathcal{P}(C)$ and $\rightthreetimes_\rho : \mathcal{P}(C) \to \mathcal{P}(Y)$ as follows: for any $Z \subseteq Y$, $B \subseteq C$

$\leftthreetimes_\rho(Z) = \{c \in C : z\rho c \text{ for all } z \in Z\}$ and

$\rightthreetimes_\rho(B) = \{y \in Y : y\rho c \text{ for all } c \in B\}$.

Then, posit

$\mathbf{X} = \{(Z, B) \in \mathcal{P}(Y) \times \mathcal{P}(C) : Z = \rightthreetimes_\rho(B) \text{ and } B = \leftthreetimes_\rho(Z)\}$.

An ordered pair $(Z, B) \in \mathbf{X}$ is said to be a *concept* of context $((Y, C), \rho)$ with *extent* Z and *intent* B.

Thus, the *concept lattice* of (Y, C) is $\mathbb{C}(Y, C) = (\mathbf{X}, \leqslant)$ where for any $(Z_1, B_1), (Z_2, B_2) \in \mathbf{X}$

$(Z_1, B_1) \leqslant (Z_2, B_2)$ iff $Z_1 \subseteq Z_2$ (which is provably equivalent to $B_2 \subseteq B_1$),

and

$(Z_1, B_1) \wedge (Z_2, B_2) = (\rightthreetimes_\rho(\leftthreetimes_\rho(Z_1 \cup Z_2)), B_1 \cap B_2)$

$(Z_1, B_1) \vee (Z_2, B_2) = (Z_1 \cap Z_2, \leftthreetimes_\rho(\rightthreetimes_\rho(B_1 \cup B_2)))$

(where $\wedge$ and $\vee$ denote the greatest lower bound and the least upper bound, respectively).

Moreover, one may define a function $\gamma : Y \to X$ mapping each object $y \in Y$ into the *object concept* of y, namely the $\leqslant$-smallest concept of (Y, C) having y in its extent (see e.g. Carpineto and Romano (2004) for a detailed presentation of concept lattices which emphasizes both their applications in information retrieval and data mining, and related computational aspects)[5]. It should also emphasized that the same construct can be extended to *many-valued contexts*, namely to the case of (finitely) many-valued attributes, by replacing ρ with a suitable ternary relation involving objects, attributes and attribute-values.

If the population of options is described by database $((Y, C), \rho)$ it is quite natural to assess the characteristics of any opportunity $Z \subseteq Y$ by considering the set $y[Z]$ of object concepts of its elements. Thus, we end up with poset $(\gamma[Y], \leqslant_{|\gamma[Y]})$, a subposet of the concept lattice $(\mathbf{X}, \leqslant)$.

[5] See also Vannucci (1999, 2006) for a study of concept lattices of game forms.

3 Width-based rankings: notation, definitions and preliminaries

Let $\mathcal{X} = (X, \leqslant)$ be the universal (finite) poset of options/opportunities i.e. $\leqslant$ is a transitive, reflexive and antisymmetric binary relation on X, and $\mathcal{P}(X)$ the power set of X. An opportunity set is a set $A \subseteq X$ thus $\mathcal{P}(X)$ also denotes the set of all opportunity sets attached to X. A *chain* of $\mathcal{X}$ is a set $C \subseteq X$ such that the restriction $\leqslant_{|C} := \{(x, y) \in C \times C : x \leqslant y\})$ is a *total*, transitive and antisymmetric binary relation on C.A family $\mathbf{C} = \{C_i\}_{i \in I}$ of chains of poset $\mathcal{X} = (X, \leqslant)$ is a *chain decomposition* of $\mathcal{X}$ (w.r.t. set union) if and only if $\mathbf{C}$ is a partition of X, namely $X = \cup_{i \in I} C_i$, and $C_i \cap C_j = \varnothing$ for any $i, j \in I, i \neq j$.

The binary relation $\| \subseteq X \times X$ comprises the set of all $\leqslant$-incomparable ordered pairs: for any $x, y \in X$, $x \| y$ holds if and only if $x \nleqslant y$ and $y \nleqslant x$.An *antichain* of $(X, \leqslant)$ is a set $A \subseteq X$ such that $x \| y$ for any $x, y \in X$ such that $x \neq y$: $\mathcal{A}_{\mathcal{X}}$ will denote the set of all antichains of $\mathcal{X}$. The *size* of an antichain A is given by its cardinality $\# A$. For any pair of ordered sets $(A, \leqslant'), (B, \leqslant'')$ an *order-isomorphism* from $(A, \leqslant')$ to $(B, \leqslant'')$ is a surjective function $f : A \rightarrow B$ such that for any $x, y \in A$, $x \leqslant' y$ if and only if $f(x) \leqslant'' f(y)$[6]. Subsets $A, B \subseteq X$ are *isomorphic* in $\mathcal{X}$ if and only if there exists an order-isomorphism from $(A, \leqslant_{|A})$ to $(B, \leqslant_{|B})$. The following notation will also be used: for any $A \subseteq X$,

$D_{\mathcal{X}}[A] := \{x \in X : \text{there exists } y \in A \text{ such that } y < x\}$, and

$U_{\mathcal{X}}[A] := \{x \in X : \text{there exists } y \in A \text{ such that } x < y\}$.

The *width* function $w_{\mathcal{X}} : \mathcal{P}(X) \rightarrow \mathbb{Z}_+$of $\mathcal{X}$ attaches to each set $Y \subseteq X$ the size of any antichain $A \in \mathcal{A}_{\mathcal{X}}$ of maximum size amongst antichains of $\mathcal{X}$ included in Y, namely $w_{\mathcal{X}}(Y) = \# A$, where i) $A \subseteq Y$, ii) $A \in \mathcal{A}_{\mathcal{X}}$, and iii) $\# A \geq \# B$ for any B which also satisfies clauses i) and ii) above.

Remark 1 *It should be noticed that, by definition, $w_{\mathcal{X}}$ is subposet-invariant i.e. for any $A \subseteq Y \subseteq X, w_{\mathcal{X}}(A) = w_{\mathcal{X}|Y}(A)$ where $\mathcal{X}|Y := (Y, \leqslant_{|Y})$.*

The following classic result due to Dilworth (see e.g. Anderson (1987), chpt.2, or Trotter (1992), chpt.1) will be used in the ensuing analysis:

Theorem 2 *(Dilworth). Let $\mathcal{Y} = (Y, \leqslant)$ be a finite poset. Then $w_{\mathcal{Y}}(Y) = \min \{\# \mathbf{C} : \mathbf{C}$ is a chain decomposition of $\mathcal{Y}\}$.*

---

[6]It is easily checked that, by definition, an order-isomorphism f has to be injective as well. An order-isomorphism from $(A, \leqslant)$ to itself is an *order-automorphism* of $(X, \leqslant)$.

We are mainly interested in two simple *width-based* rankings of opportunity sets as defined below

Definition 3 *The* width-ranking *induced by* $\mathcal{X} = (X, \leqslant)$ *on* $\mathcal{P}(X)$ *is the total and transitive binary relation* $\succcurlyeq^*_{w_\mathcal{X}}$ *defined by the following rule: for any* $A, B \subseteq X$, $A \succcurlyeq_{w_\mathcal{X}} B$ *iff* $w_\mathcal{X}(A) \geq w_\mathcal{X}(B)$.

Definition 4 *The* undominated width-ranking *induced by* $\mathcal{X} = (X, \leqslant)$ *on* $\mathcal{P}(X)$ *is the total and transitive binary relation* $\succcurlyeq^*_{w_\mathcal{X}}$ *defined by the following rule:*

for any $A, B \subseteq X$, $A \succcurlyeq^*_{w_\mathcal{X}} B$ *iff* $w_\mathcal{X}(\max_{\leqslant_{|A}}) \geq w_\mathcal{X}(\max_{\leqslant_{|B}})$ *iff* $\#(\max_{\leqslant_{|A}}) \geq \#(\max_{\leqslant_{|B}})$.

Clearly enough, the width-ranking decrees that an opportunity set to be more diverse than another if and only if it includes a larger antichain. Similarly, the undominated width-ranking declares an opportunity set to be more diverse than another if and only if the antichain of its (locally) undominated options is larger.[7]

We shall now provide a characterization of those width-based rankings through the following axioms

Definition 5 ***Indifference between Isomorphic Sets*** *(IIS) A binary relational system* $(\mathcal{P}(X), \succcurlyeq)$ *satisfies* Indifference between Isomorphic Sets *with respect to poset* $\mathcal{X} = (X, \leqslant)$ *iff for any* $A, B \subseteq X$, *if* A *and* B *are order-isomorphic in* $\mathcal{X}$ *then* $A \succcurlyeq B$.

In words, IIS simply requires that two order-isomorphic sets be equally ranked in terms of diversity. It amounts to a strengthened, and adapted, version of the standard notion of Indifference between No Choice Situations i.e. between singletons (see e.g. Barberà, Bossert and Pattanaik (2004)).

Definition 6 ***Weak Monotonicity*** *(WMON) A binary relational system* $(\mathcal{P}(X), \succcurlyeq)$ *satisfies* Weak Monotonicity *iff* $A \succcurlyeq B$ *for any* $A, B \subseteq X$ *such that* $B \subseteq A$.

[7] Counterparts of the undominated width-ranking in multi-preferential settings are introduced and characterized by Pattanaik and Xu (1998) and Peragine and Romero-Medina (2006).

Definition 7 ***Strict Monotonicity for Antichains*** *(SMONA) A binary relational system* $(\mathcal{P}(X), \succcurlyeq)$ *satisfies* Strict Monotonicity for Antichains *with respect to poset* $\mathcal{X} = (X, \leqslant)$ *iff for any* $A, B \in \mathcal{A}_{\mathcal{X}}$ *,* $B \subset A$ *entails* $A \succ B$*.*[8]

Of course, WMON amounts to requiring that the diversity preorder preserves the set-inclusion preorder. SMONA is the restriction of the strict version of set-inclusion monotonicity to antichains.

Definition 8 ***Antichain Restricted Irrelevance of Connected Opportunities*** *(ARICO) A binary relational system* $(\mathcal{P}(X), \succcurlyeq)$ *satisfies* Antichain Restricted Irrelevance of Connected Opportunities *with respect to poset* $\mathcal{X} = (X, \leqslant)$ *iff* $A \succcurlyeq A \cup \{x\}$ *for any antichain* A *of* $\mathcal{X}$ *and any* $x \in D_{\mathcal{X}}[A] \cup U_{\mathcal{X}}[A]$.

Thus, ARICO is a restricted independence condition that requires the addition to an antichain of alternatives that are comparable to some options of the former to be diversity-irrelevant.

Definition 9 ***Irrelevance of Dominated Opportunities*** *(IDO) A binary relational system* $(\mathcal{P}(X), \succcurlyeq)$ *satisfies* Irrelevance of Dominated Opportunities *with respect to poset* $\mathcal{X} = (X, \leqslant)$ *iff for any* $A \subseteq X$ *and any* $x \in X$*, if* $x \in D_{\mathcal{X}}[A]$ *then* $A \sim A \cup \{x\}$ *.*

IDO is also an independence condition that requires the addition to any set of alternatives that are dominated by some options of that set to be diversity-irrelevant.

4 Width-based rankings: characterizations

We are now ready to state and prove our characterization of the width-ranking, namely

Theorem 10 *Let* $\mathcal{X} = (X, \leqslant)$ *be a poset, and* $\succcurlyeq$ *a preorder i.e. a reflexive and transitive binary relation on* $\mathcal{P}(X)$*. Then* $(\mathcal{P}(X), \succcurlyeq)$ *satisfies IIS, WMON, SMONA and ARICO if and only if* $\succcurlyeq = \succcurlyeq_{w_{\mathcal{X}}}$*.*

Proof. $\Longleftarrow$:*It is immediately checked that* $(\mathcal{P}(X), \succcurlyeq_{w_{\mathcal{X}}})$ *is a (totally) preordered set and satisfies WMON. Also, if* $A, B \in \mathcal{A}_{\mathcal{X}}$ *and* $B \subset A$ *then by*

[8] Of course, $\succ$ and $\sim$ denote, respectively, the asymmetric and symmetric components of $\succcurlyeq$.

definition $w_{\mathcal{X}}(A) = \#A > \#B = w(B)$, *i.e.* $A \succ_{w_{\mathcal{X}}} B$, *hence SMONA is also satisfied.*

To check that IIS holds, notice that if A, $B \subseteq X$ *are order-isomorphic w.r.t.* $\mathcal{X}$, *then* $\#A = \#B$ *and for any* $x, y \in A$, $x \leqslant y$ *iff* $f(x) \leqslant f(y)$ *and* $x \| y$ *iff* $f(x) \| f(y)$, *where* f *is an order-automorphism of* $\mathcal{X}$ *such that* $f[A] = B$. *It follows that for any antichain* B' *of* $\mathcal{X}|B$, $f^{-1}[B']$ *is an antichain of* $\mathcal{X}|A$. *In particular, let* $B' \subseteq B$ *be an antichain of* $\mathcal{X}$ *of maximum size, i.e.* $w(B) = \#B'$. *Then,* $f^{-1}[B] = A'$ *is an antichain of* A, *hence* $w(A) \geq w(B)$ *i.e.* $A \succcurlyeq_{w_{\mathcal{X}}} B$. *Thus, IIS is satisfied.*

To check that $(\mathcal{P}(X), \succcurlyeq_{w_{\mathcal{X}}})$ *satisfies ARICO as well, take any antichain* $A \subseteq X$ *and any* $x \in D_{\mathcal{X}}[A] \cup U_{\mathcal{X}}[A]$. *Clearly,* $w_{\mathcal{X}}(A \cup \{x\}) = w_{\mathcal{X}}(A)$ *hence in particular* $A \succcurlyeq_{w_{\mathcal{X}}} A \cup \{x\}$.

$\Longrightarrow$:*Conversely, let* $(\mathcal{P}(X), \succcurlyeq)$ *be a preordered set which satisfies IIS,WMON, SMONA, and ARICO.*

First, suppose that $A \succcurlyeq B$. *By Dilworth's Theorem as mentioned above,* $B = \cup_{i=1}^{w_{\mathcal{X}}(B)} C'_i$, $A = \cup_{i=1}^{w_{\mathcal{X}}(A)} C_i$ *where* $\{C'_i\}_{i \in \{1,..,w_{\mathcal{X}}(B)\}}$, $\{C_i\}_{i \in \{1,..,w_{\mathcal{X}}(A)\}}$ *are chain decompositions of minimum cardinality of* $(B, \leqslant_{|B})$ *and* $(A, \leqslant_{|A})$, *respectively. Thus, there exist* $B' = \left\{c'_1, .., c'_{w_{\mathcal{X}}(B)}\right\}$ *with* $c'_i \in C'_i$, $i = 1, .., w_{\mathcal{X}}(B)$, *and* $A' = \left\{c_1, .., c_{w_{\mathcal{X}}(A)}\right\}$ *with* $c_i \in C_i$, $i = 1, .., w(A)$ *such that* B' *is an antichain of maximum size in* $(B, \leqslant_{|B})$, *and* A' *is an antichain of maximum size in* $(A, \leqslant_{|A})$. *Now, notice that, by construction, for any* $x \in A \backslash A'$ *and any* $y \in B \backslash B'$, $x \in D_{\mathcal{X}}[A'] \cup U_{\mathcal{X}}[A']$ *and* $y \in D_{\mathcal{X}}[B'] \cup U_{\mathcal{X}}[B']$. *Thus, by suitably repeated applications of ARICO it follows that* $B' \succcurlyeq B$ *and* $A' \succcurlyeq A$, *while by WMON,* $A \succcurlyeq A'$ *and* $B \succcurlyeq B'$. *Therefore,* $A \sim A'$ *and* $B' \sim B$, *whence* $A' \succcurlyeq B'$. *Let us now assume that* $w_{\mathcal{X}}(B) > w_{\mathcal{X}}(A)$ *i.e.* $\#B' > \#A'$. *Therefore there exists* $B'' \subset B'$ *such that* $\#B'' = \#A'$. *Since both* $A' \in \mathcal{A}_{\mathcal{X}}$ *and* $B'' \in \mathcal{A}_{\mathcal{X}}$ *it follows that* A' *and* B'' *are order-isomorphic in* $\mathcal{X}$ *hence by IIS* $A' \sim B''$. *However,* $B' \succ B''$ *by SMONA. Thus, by transitivity,* $B' \succ A'$, *a contradiction. Therefore, it must be case that* $w_{\mathcal{X}}(A) \geq w_{\mathcal{X}}(B)$ *i.e.* $A \succcurlyeq_{w_{\mathcal{X}}} B$.

Next, suppose that $A \succcurlyeq_{w_{\mathcal{X}}} B$ *i.e.* $w_{\mathcal{X}}(A) \geq w_{\mathcal{X}}(B)$. *Let* A', B' *be antichains of maximum size of* $(A, \leqslant_{|A})$ *and* $(B, \leqslant_{|B})$, *respectively, as defined in the paragraphs above: clearly* $\#A' \geq \#B'$. *Also, notice that, again, by ARICO and WMON,* $A \sim A'$ *and* $B \sim B'$. *Then, observe that if* $\#A' = \#B'$ *then since both* A' *and* B' *are antichains of* $\mathcal{X}$, *they are also order-isomorphic in* $\mathcal{X}$ *whence in particular* $A \sim A' \succcurlyeq B' \sim B$, *by IIS. Moreover, if* $\#A' > \#B'$ *then there exists an antichain* $A'' \subset A'$ *such that*

$\#A'' = \#B'$. Again, A'' and B' are order-isomorphic in $\mathcal{X}$ (since they are both antichains), hence by IIS $A'' \sim B'$. But $A' \succ A''$ by SMONA, hence $A \sim A' \succ B' \sim B$ by transitivity.

In any case, $A \succcurlyeq B$ and the proof of the thesis is completed. ■

The foregoing characterization result is tight. Indeed, to check the independence of the axioms employed let us consider the following list of examples.

Example 11 *The independence of IIS can be shown by considering a poset $\mathcal{X} = (X, \leqslant)$ with at least two distinct totally disconnected elements $y, z \in X$ (i.e. $y \| x$ for any $x \in X \setminus \{y\}$ and $z \| x$ for any $x \in X \setminus \{z\}$), and taking $(\mathcal{P}(X), \succcurlyeq^z_{w_\mathcal{X}})$ where $\succcurlyeq^z_{w_\mathcal{X}}$ is the 'refinement' of $\succcurlyeq_{w_\mathcal{X}}$ defined as follows: for any $A, B \subseteq X$, $A \succcurlyeq^z_{w_\mathcal{X}} B$ iff either $w_\mathcal{X}(A) > w_\mathcal{X}(B)$ or $w_\mathcal{X}(A) = w_\mathcal{X}(B)$ and $z \notin B$, or else $w_\mathcal{X}(A) = w_\mathcal{X}(B)$ and either $z \in A \cap B$ or $z \notin A \cup B$. Notice that $\succcurlyeq^z_{w_\mathcal{X}}$ is indeed a preorder: reflexivity is obvious, and transitivity also holds (to see this, assume $A \succcurlyeq^z_{w_\mathcal{X}} B$ and $B \succcurlyeq^z_{w_\mathcal{X}} C$: then (i) $w_\mathcal{X}(A) > w_\mathcal{X}(B)$ and $w_\mathcal{X}(B) > w_\mathcal{X}(C)$ or $w_\mathcal{X}(B) = w_\mathcal{X}(C)$ imply $w_\mathcal{X}(A) > w_\mathcal{X}(C)$, and similarly $w_\mathcal{X}(A) = w_\mathcal{X}(B)$ and $w_\mathcal{X}(B) > w_\mathcal{X}(C)$ imply $w_\mathcal{X}(A) > w_\mathcal{X}(C)$ whence $A \succcurlyeq^z_{w_\mathcal{X}} C$;(ii) if $w_\mathcal{X}(A) = w_\mathcal{X}(B) = w_\mathcal{X}(C)$ and $z \notin B$ then $B \succcurlyeq^z_{w_\mathcal{X}} C$ entails $z \notin C$ (indeed $z \notin B \cup C$) whence again $A \succcurlyeq^z_{w_\mathcal{X}} C$; (iii) if $w_\mathcal{X}(A) = w_\mathcal{X}(B) = w_\mathcal{X}(C)$ and $z \notin C$ then $A \succcurlyeq^z_{w_\mathcal{X}} C$ by definition; iv) if $w_\mathcal{X}(A) = w_\mathcal{X}(B) = w_\mathcal{X}(C)$ and $z \in A \cap B$ then it cannot be the case that $z \notin B \cup C$: thus $B \succcurlyeq^z_{w_\mathcal{X}} C$ entails $z \in B \cap C$, hence $z \in A \cap C$, and therefore $A \succcurlyeq^z_{w_\mathcal{X}} C$). Moreover, if $B \subseteq A$ then by definition $w_\mathcal{X}(A) \geq w_\mathcal{X}(B)$ and either $z \notin B$ or $z \in A \cap B$. Thus, $A \succcurlyeq^z_{w_\mathcal{X}} B$ holds in any case, and $(\mathcal{P}(X), \succcurlyeq^z_{w_\mathcal{X}})$ satisfies WMON. If A, B are antichains and $B \subset A$ then $w_\mathcal{X}(A) > w_\mathcal{X}(B)$ whence both $A \succcurlyeq^z_{w_\mathcal{X}} B$ and* not *$B \succcurlyeq^z_{w_\mathcal{X}} A$: it follows that $(\mathcal{P}(X), \succcurlyeq^z_{w_\mathcal{X}})$ satisfies SMONA. Now, let us consider an antichain A and $x \in D_\mathcal{X}[A] \cup U_\mathcal{X}[A]$. Clearly, $w_\mathcal{X}(A) = w_\mathcal{X}(A \cup \{x\})$ and $x \neq z$, by definition of z. It follows that either $z \notin A \cup \{x\}$ or $z \in A$, hence in any case $A \succcurlyeq^z_{w_\mathcal{X}} A \cup \{x\}$, and ARICO is also satisfied by $(\mathcal{P}(X), \succcurlyeq^z_{w_\mathcal{X}})$. However, it is immediately checked that $\{z\} \succ^z_{w_\mathcal{X}} \{y\}$: thus, IIS is violated by $(\mathcal{P}(X), \succcurlyeq^z_{w_\mathcal{X}})$.*

Example 12 *Independence of WMON from the other axioms can be shown by considering the undominated width ranking $(\mathcal{P}(X), \succcurlyeq^*_{w_\mathcal{X}})$ as defined above. Indeed, it is easily checked that $(\mathcal{P}(X), \succcurlyeq^*_{w_\mathcal{X}})$ is a preordered set that satisfies SMONA, and IIS (see the proof of the next characterization result below).*

Moreover, $(\mathcal{P}(X), \succcurlyeq^*_{w_\mathcal{X}})$ *satisfies ARICO: for any antichain* A *of* $\mathcal{X}$ *and any* $x \in D_\mathcal{X}[A] \cup U_\mathcal{X}[A]$, $A \cup \{x\} \notin \mathcal{A}_\mathcal{X}$ *hence* $\#(\max_{\leqslant|A}) = \#(\max_{\leqslant|B})$.*However, in general* $(\mathcal{P}(X), \succcurlyeq^*_{w_\mathcal{X}})$ *does* not *satisfy WMON. To see this, take* $\mathcal{X} = (\{x, y, z\}, \leqslant)$ *with* $\leqslant = \{(z, x), (z, y)\}$. *Clearly,* $\max_{\leqslant|\{x,y\}} = \{x, y\}$, *while* $\max_{\leqslant|\{x,y,z\}} = \{z\}$, *hence* $\{x, y\} \succ^*_{w_\mathcal{X}} \{x, y, z\}$.

Example 13 *The independence of SMONA is immediately verified by considering the universal binary relation* $\succcurlyeq^U = \mathcal{P}(X) \times \mathcal{P}(X)$. *Clearly,* $(\mathcal{P}(X), \succcurlyeq^U)$ *is a (totally) preordered set and satisfies IIS, WMON and ARICO, but violates SMONA.*

Example 14 *The independence of ARICO can be shown by considering the binary relational system* $(\mathcal{P}(X), \succcurlyeq^\#_{\|(\mathcal{X})})$ *where* $\succcurlyeq^\#_{\|(\mathcal{X})}$ *is defined by the following rule: for any* $A, B \subseteq X$, $A \succcurlyeq^\#_{\|(\mathcal{X})} B$ *iff* $\#\{(x, y) : (x, y) \in A \times A \text{ and } x\|y\} \geq \#\{(x, y) : (x, y) \in B \times B \text{ and } x\|y\}$. *Clearly,* $\succcurlyeq^\#_{\|(\mathcal{X})}$ *is a (total) preorder. IIS, WMON, and SMONA are also obviously satisfied. However, if* $A = \{y, z\}$ *is an antichain,* $x \notin A$, $x \in D_\mathcal{X}[A]$ *and* $\leqslant_{|A\cup\{x\}} = \{(y, x)\}$ *then* $\{x, y, z\} \succ^\#_{\|(\mathcal{X})} \{y, z\}$. *Thus,* $(\mathcal{P}(X), \succcurlyeq^\#_{\|(\mathcal{X})})$ *does* not *satisfy ARICO.*

Let us now turn to our characterization of the undominated width ranking.

Theorem 15 *Let* $\mathcal{X} = (X, \leqslant)$ *be a poset, and* $\succcurlyeq$ *a preorder i.e. a reflexive and transitive binary relation on* $\mathcal{P}(X)$. *Then* $(\mathcal{P}(X), \succcurlyeq)$ *satisfies IIS, SMONA and IDO if and only if* $\succcurlyeq = \succcurlyeq^*_{w_\mathcal{X}}$.

Proof. $\Longleftarrow$:*It is immediately checked that* $\succcurlyeq^*_{w_\mathcal{X}}$ *is indeed a (total) preorder. Moreover, if* $A, B \in \mathcal{A}_\mathcal{X}$ *and* $B \subset A$ *then by definition* $w_\mathcal{X}(\max \leqslant_{|A}) = \#A > \#B = w(\max_{\leqslant|B})$, *i.e.* $A \succ^*_{w_\mathcal{X}} B$, *hence SMONA is also satisfied by* $(\mathcal{P}(X), \succcurlyeq^*_{w_X})$.

To check that $(\mathcal{P}(X), \succcurlyeq^*_{w_X})$ *satisfies IDO as well, take any* $A \subseteq X$ *and* $y \in X$ *such that* $y < x$ *for some* $x \in A$, *and consider* $\mathcal{A}' = (A, \leqslant_{|A})$, $\mathcal{A}'' = (A \cup \{y\}, \leqslant_{|A\cup\{y\}})$. *Clearly,* $\max_{\leqslant|A} = \max_{\leqslant|A\cup\{y\}}$ *hence in particular* $A \succcurlyeq^*_{w_\mathcal{X}} A \cup \{y\}$ *as required.*

Finally, if A, $B \subseteq X$ *are order-isomorphic w.r.t.* $\mathcal{X}$, *then* $\#A = \#B$ *and for any* $x, y \in A$, $x \leqslant y$ *iff* $f(x) \leqslant f(y)$ *and* $x\|y$ *iff* $f(x)\|f(y)$ *where* f *is a suitable order-automorphism of* $\mathcal{X}$. *It follows that for any antichain* C

of $\mathcal{X}|B$, $f^{-1}[B]$ is an antichain of $\mathcal{X}|A$. Moreover, $f^{-1}[\max_{\leqslant|B}] = \max_{\leqslant|A}$. Thus, $A \succcurlyeq^*_{w\mathcal{X}} B$ and IIS is satisfied.

$\Longrightarrow$:*Conversely, let $(\mathcal{P}(X), \succcurlyeq)$ be a preordered set which satisfies IIS, SMONA and IDO.*

First, suppose that $A \succcurlyeq B$. By suitably repeated applications of IDO it follows that $\max_{\leqslant|A} \sim A$ and $\max_{\leqslant|B} \sim B$, whence $\max_{\leqslant|A} \succcurlyeq \max_{\leqslant|B}$. Let us now assume that $\#(\max_{\leqslant|B}) > \#(\max_{\leqslant|A})$. Then, there exists an antichain $B' \subset \max_{\leqslant|B}$ such that $\#B' = \#(\max_{\leqslant|A})$. Since both $\max_{\leqslant|A} \in \mathcal{A}_{\mathcal{X}}$ and $B' \in \mathcal{A}_{\mathcal{X}}$ it follows that $\max_{\leqslant|A}$ and B' are order-isomorphic in $\mathcal{X}$ hence by IIS $\max_{\leqslant|A} \sim B'$. Since, by construction, $\max_{\leqslant|B} \in \mathcal{A}_{\mathcal{X}}$ as well it follows by SMONA that $\max_{\leqslant|B} \succ B'$. Thus, by transitivity, $\max_{\leqslant|B} \succ \max_{\leqslant|A}$, a contradiction. Therefore, it must be case that $\#(\max_{\leqslant|A}) \geq \#(\max_{\leqslant|B})$ i.e. $A \succcurlyeq^*_{w\mathcal{X}} B$.

Next, suppose that $A \succcurlyeq^*_{w\mathcal{X}} B$ i.e. $\#(\max_{\leqslant|A}) \geq \#(\max_{\leqslant|B})$. First, notice that, by IDO, $A \sim \max_{\leqslant|A}$ and $B \sim \max_{\leqslant|B}$. Also, observe that if $\#(\max_{\leqslant|A}) = \#(\max_{\leqslant|B})$ then since both $\max_{\leqslant|A}$ and $\max_{\leqslant|B}$ are antichains of $\mathcal{X}$, $\max_{\leqslant|A}$ and $\max_{\leqslant|B}$ are also order-isomorphic in $\mathcal{X}$ whence $\max_{\leqslant|A} \succcurlyeq \max_{\leqslant|B}$, by IIS. Moreover, if $(\max_{\leqslant|A}) > \#(\max_{\leqslant|B})$ then there exists an antichain $A' \subset \max_{\leqslant|A}$ such that $\#A' = \#(\max_{\leqslant|B})$. Again, A' and $\max_{\leqslant|B}$ are order-isomorphic in $\mathcal{X}$ (since they are both antichains), hence by IIS $A' \sim \max_{\leqslant|B}$ .But $\max_{\leqslant|A} \succ A'$ by SMONA, hence $\max_{\leqslant|A} \succ \max_{\leqslant|B}$ by transitivity.

Thus, in any case $A \succcurlyeq B$ as required. ■

The foregoing characterization of the undominated width ranking is also tight, as shown by the following examples.

Example 16 *The independence of IIS can be shown by considering a poset $\mathcal{X} = (X, \leqslant)$ with at least two distinct totally disconnected elements $y, z \in X$ (i.e. $y \| x$ for any $x \in X \setminus \{y\}$ and $z \| x$ for any $x \in X \setminus \{z\}$). Then, take $(\mathcal{P}(X), \succcurlyeq^{*z}_{w\mathcal{X}})$ where $\succcurlyeq^{*z}_{w\mathcal{X}}$ is the 'refinement' of $\succcurlyeq^*_{w\mathcal{X}}$ defined as follows: for any $A, B \subseteq X$, $A \succcurlyeq^{*z}_{w\mathcal{X}} B$ iff either $\#(\max_{\leqslant|A}) > \#(\max_{\leqslant|B})$ or $\#(\max_{\leqslant|A}) = \#(\max_{\leqslant|B})$ and $z \notin B$, or else $\#(\max_{\leqslant|A}) = \#(\max_{\leqslant|B})$ and either $z \in A \cap B$ or $z \notin A \cup B$. It is easily checked that $\succcurlyeq^{*z}_{w\mathcal{X}}$ is a preorder: reflexivity is obvious, and transitivity also holds. Indeed, assume $A \succcurlyeq^{*z}_{w\mathcal{X}} B$ and $B \succcurlyeq^{*z}_{w\mathcal{X}} C$: then (i) $\#(\max_{\leqslant|A}) > \#(\max_{\leqslant|B})$ and $\#(\max_{\leqslant|B}) > \#(\max_{\leqslant|C})$ or $\#(\max_{\leqslant|B}) = \#(\max_{\leqslant|C})$ imply $\#(\max_{\leqslant|A}) > \#(\max_{\leqslant|C})$, and similarly $\#(\max_{\leqslant|A}) = \#(\max_{\leqslant|B})$ and $\#(\max_{\leqslant|B}) > \#(\max_{\leqslant|C})$ entail $\#(\max_{\leqslant|A}) > \#(\max_{\leqslant|C})$;*

(ii) if $\#(\max_{\leqslant|A}) = \#(\max_{\leqslant|B}) = \#(\max_{\leqslant|C})$ *and* $z \notin B$ *then by* $B \succcurlyeq^{*z}_{w\mathcal{X}} C$ *it must be the case that* $z \notin C$*; (iii) if* $\#(\max_{\leqslant|A}) = \#(\max_{\leqslant|B}) = \#(\max_{\leqslant|C})$ *and* $z \notin C$ *then* $A \succcurlyeq^{*z}_{w\mathcal{X}} C$ *by definition; finally, if* $z \in B \cap C$ *then* $A \succcurlyeq^{*z}_{w\mathcal{X}} B$ *entails* $z \in A \cap B$, *hence* $z \in A \cap C$ *and thus* $A \succcurlyeq^{*z}_{w\mathcal{X}} C$, *by definition. In any case,* $A \succcurlyeq^{*z}_{w\mathcal{X}} C$ *and transitivity follows. If* A, B *are antichains and* $B \subset A$ *then* $\#(\max_{\leqslant|A}) > \#(\max_{\leqslant|B})$ *whence both* $A \succcurlyeq^{*z}_{w\mathcal{X}} B$ *and* not $B \succcurlyeq^{*z}_{w\mathcal{X}} A$*: it follows that* $(\mathcal{P}(X), \succcurlyeq^{*z}_{w\mathcal{X}})$ *satisfies SMONA. Now, let us consider an antichain* A *and* $x \in D_{\mathcal{X}}[A]$. *Clearly,* $\max_{\leqslant|A\cup\{x\}} = \max_{\leqslant|A}$ *hence* $A \cup \{x\} \sim^{*z}_{w\mathcal{X}} A$. *Thus,* $(\mathcal{P}(X), \succcurlyeq^{*z}_{w\mathcal{X}})$ *satisfies IDO. However, it is immediately checked that* $\{z\} \succ^{*z}_{w\mathcal{X}} \{y\}$*: thus, IIS is violated by* $(\mathcal{P}(X), \succcurlyeq^{*z}_{w\mathcal{X}})$.

Example 17 *Consider the universal binary relation* $\succcurlyeq^U = \mathcal{P}(X) \times \mathcal{P}(X)$. *Of course* $(\mathcal{P}(X), \succcurlyeq^U)$ *is a (totally) preordered set and satisfies IIS and IDO but -as observed above- it fails to satisfy SMONA.*

Example 18 *As observed above, the width-ranking* $(\mathcal{P}(X), \succcurlyeq_{w_{\mathcal{X}}})$ *is a preordered set and satisfies both IIS and SMONA. However, let* $\mathcal{X} = (X, \leqslant)$, $A = \{y, z, u, v\} \subseteq X$, $\leqslant_{|A} = \{(y, u), (z, v)\}$, $x \in D_{\mathcal{X}}[\{z\}] \subseteq D_{\mathcal{X}}[A]$, *and* $x \notin D_{\mathcal{X}}[A \setminus \{z\}] \cup U_{\mathcal{X}}[A \setminus \{z\}]$. *Then, by definition* $w_{\mathcal{X}}(A \cup \{x\}) = 3$ *while* $w_{\mathcal{X}}(A) = 2$. *Hence* $A \cup \{x\} \succ_{w_{\mathcal{X}}} A$ *and IDO is violated.*

5 An illustration: the Noah's ark problem and the space voyager problem as knapsack problems with partially undefined parameters

The Noah's ark problem was evoked by Weitzman (1998) as a paradigm of biodiversity preservation issues: precisely as mythical Noah, the actual environment-conscious decision-maker has to maximize diversity under a certain constraint (though most typically a budget, not a capacity constraint[9]).

[9] The text of Genesis is quite specific about the ark's total capacity: " (15) And this is the fashion which thou shalt make it of: The length of the ark shall be three hundred cubits, the breadth of it fifty cubits, and the height of it thirty cubits. (16) A window shalt thou make to the ark, and in a cubit shalt thou finish it above; and the door of the ark shalt thou set in the side thereof; with lower, second, and third stories shalt thou make it. "

Notice that the original version of Noah's ark problem is in fact a 'pure'-diversity constrained maximization problem: population sizes are entirely disregarded, sustainability is apparently not an issue. Of course, the required solution of the Noah's ark problem is simply the best strategy to preserve biological information as embodied in living species. The latter are indeed depositories of genetic information content hence may regarded as libraries or collections of different books. According to that apt metaphor suggested by Weitzman, the ark may accommodate a collection of libraries and Noah has to maximize the number of different books taken in (or more precisely the total diversity value of the overall library embarked).

In order to solve the Noah's ark problem, Weitzman introduces a suitable diversity function: the marginal diversity or distinctiveness of a species, namely the diversity loss if it get extincts, amounts to an extended point-to-set-distance induced by dissimilarities (see Weitzman(1998) p. 1292)). As for the diversity function itself, it is a monotonic aggregate index indeed a sum of distances/dissimilarities between options in an evolutionary tree.

Addressing a 'local' version of the Noah's ark problem within their own binary multi-attribute model of diversity, Nehring and Puppe (2002) also introduce a dissimilarity pseudometric[10]: the dissimilarity attached to any ordered pair of population units is given by the total weights of the attributes possessed by the former but not by the latter. Their work addresses in a quite general way the problem of establishing conditions under which diversity may be regarded as aggregate dissimilarity, hence essentially computable from binary dissimilarity comparisons. Nehring and Puppe's results imply that this is in fact the case when the relevant attributes are the clades of an evolutionary tree[11], or the clades of a philogenetic tree[12](see Nehring and

[10]Some terminology is in order here. A *premetric* defined on set X is a function $d : X \times X \to \mathbb{R}$ such that for any $x, y \in X$, $d(x,y) \geq 0$ (non-negativity) and $d(x,x) = 0$ ('indiscernibility of identicals') hold. A premetric is *symmetric* if for any $x, y \in X$, $d(x,y) = d(y,x)$ (symmetry) holds. A *pseudometric* is a non-negative symmetric function $d : X \times X \to \mathbb{R}$ such that for any $x, y, z \in X$, $d(x,z) \leq d(x,y) + d(y,z)$ (subadditivity) holds. A *semimetric* is a symmetric premetric such that for any $x, y \in X$, $d(x,y) = 0$ entails $x = y$ (the 'identity of indiscernibles' principle). A *metric* is a subadditive semimetric.

[11]A clade of an evolutionary tree is the set of all points that share a common ancestor, with the latter included.

[12]Starting from an evolutionary tree, a philogenetic tree is a refinement of the former. Such refinement is obtained through intersection between 'evolutionary' clades and the taxa or similarity classes resulting from a certain taxonomic classification, under the as-

Puppe (2002), pp. 1179-1182).

Thus, as a matter of fact, both Weitzman's and Nehring-Puppe's approaches to the Noah's problem make an essential use of evolutionary trees, which in turn embody a considerable amount of knowledge on the evolutionary biology of planet Earth's life forms.

But then, what if the decision maker has no reliable information whatsoever on the evolutionary history of the relevant lineages? Indeed, precisely that would be the likely predicament of, say, a robotic space voyager designed to explore a remote solar system in search of alien forms of life. In the event of a fateful arrival on a planet teeming with an exceedingly rich variety of absolutely alien living entities, how is the robot to select an optimally diverse subpopulation to single out for analysis given its limited resources? And, for that matter, how would mythical Noah himself proceed to select an optimally diverse subpopulation of earthlings to embark on his ship, under the not unreasonable assumption he is a creationist, totally unaware of evolutionary biology including the notion itself of an evolutionary tree?

Presumably, both the robotic space voyager and creationist Noah would have to rely on some practical binary or many-valued ordinal taxonomic criteria. Therefore, the resulting data structure would be amenable to diversity assessment via widths of the resulting dominance poset and subposets.

In fact, under the foregoing stipulations the space voyager's problem can be modelled as a partially unspecified instance of the classic *knapsack problem*, namely

(KP) $$\max \left\{ \mathbf{d} \cdot \mathbf{x} | \mathbf{a} \cdot \mathbf{x} \leq k, \mathbf{x} \in 2^{2^P} \right\}$$

where $(P, \leqslant)$ is the relevant population poset, $\mathbf{d} \in \mathbb{Q}_+^{2^P}$ denotes the unspecified non-negative rational vector of diversity weights of subpopulations, $\mathbf{a} \in \mathbb{Q}_+^{2^P}$ is the non-negative rational vector of capacity requirements for subpopulations, $k \in \mathbb{Q}_+$ is the non-negative rational scalar denoting available capacity (see e.g. Schrijver (1986), chpt. 16). Reliance on width parameters would enable the decision-maker to specify the diversity weights $\mathbf{d}$, and reduce the original space voyager problem to a well-defined instance of KP.

sumption that the latter is consistent with the evolutionary tree under consideration.

6 Related literature and concluding remarks

As mentioned in the Introduction, in recent years the general issue of diversity measurement has been addressed in a number of distinguished contributions.

Weitzman (1992) introduces a 'pure' diversity function which relies on a dissimilarity metric, and is uniquely defined up to an additive constant of integration: the larger pairwise dissimilarities between options in the opportunity set, the larger the diversity of the latter. If the dissimilarity metric is in fact an ultrametric[13] which in a sense corresponds to a 'perfect' taxonomy, then the diversity value of an opportunity set is the length of the associated taxonomic tree. If not, the diversity value can be anyway regarded as "the tightest or most parsimonious feasible reconstruction, in the sense of being the minimal number of character-state changes required to account for diversity of a set" (Weitzman (1992), p.378).

A characterization of the ranking induced by Weitzman's diversity function is provided by Bossert, Pattanaik and Xu (2001) within a 'dissimilarity' semimetric space (X, d): they first characterize the class of rankings consistent with a certain 'lexicographic distance' induced by d, and then show that the Weitzman's diversity ranking is precisely the only ranking of that class which satisfies two d-restricted monotonicity and independence conditions, plus a d-related 'indifference for link elements' property requiring that certain elements do not contribute to the diversity rank of certain opportunity sets. Thus, the Weitzman ranking is essentially the lexicographic-distance-based total preorder which is singled out by the requirement that the aggregation of the distances involved be additive (see Bossert, Pattanaik and Xu (2001)).

As mentioned before, the instrumental role of option-diversity assessments in the evaluation of opportunity sets in terms of freedom of choice is another major source of the growing concern for diversity rankings (see Barberà, Bossert and Xu (2004) for an extensive survey of the literature on rankings of opportunity sets, including freedom-of-choice rankings). Pattanaik and Xu (2000) is an early contribution to the literature on diversity that is mainly motivated by the concern for such freedom-rankings of opportunity sets. That work relies on a *simple* i.e. two-valued 'dissimilarity' semimetric which induces in a obvious way a binary similarity relation on the set of options. Then, Pattanaik and Xu provide a characterization of

[13] A metric $\delta : X \times X \to \mathbb{R}_+$ is an *ultrametric* if $\delta(x,z) \leq \max\{\delta(x,y), \delta(y,z)\}$ for all $x, y, z \in X$.

the total preorder that ranks opportunity sets according to the sizes of their smallest i.e. coarsest similarity-based partitions.

In a subsequent work of theirs, the same authors consider a weaker (finite) dissimilarity space (namely a symmetric premetric space, where the 'identity of indiscernibles' principle may not hold), and offer a characterization of the *partial* preorder of opportunity sets dictated by a dominance ranking induced by the underlying premetric (see Pattanaik and Xu (2006)).

Working in the same vein, but within a much richer environment which combines a binary similarity relation with a variable profile of preferences chosen from a given reference set of total preorders on the set of options, Peragine and Romero-Medina (2006) characterize two distinct total preorders which rank opportunity sets according to the number of 'dissimilar' unilateral optima (of unilateral 'dissimilar' optima, respectively) they include.

Van Hees (2004) also starts from a 'dissimilarity' metric space of options, and considers several extended point-to-set distances relying on that 'dissimilarity' metric. He shows that several combinations of distance-respecting independence properties for diversity rankings are inconsistent with the requirement of 'equi-diversity' for singletons when the underlying 'dissimilarity' space of options includes linear sequences of three or four options such that: a) adjacent options are at equal distances and b) the distance between the extremal options is given by the sum of intermediate distances (see Van Hees (2004)).

Bossert, Pattanaik and Xu (2003) discuss the axiomatic foundations of several (pre)metric-oriented diversity rankings proposed in the extant literature from a quite general perspective, distinguishing between ordinal and ratio-scale (pre)metrics. By definition, the former -as opposed to the latter- do not attach any significant role to comparisons concerning differences between distances. For instance, the Pattanaik-Xu similarity-based ranking and the premetric-based dominance ranking mentioned above all rely on ordinal (pre)metrics, while Weitzman's diversity ranking and some of the diversity rankings considered by Van Hees (2004) rely on a ratio-scale metric.

Some recent papers by Nehring and Puppe (2002, 2003) contribute a quite different (pre)metric-free approach to diversity measurement. Indeed, Nehring and Puppe (2002) introduce a binary multi-attribute representation of opportunity sets. Next, starting from a submodular non-negative diversity function they show via conjugate Möbius inversion that choosing that function amounts to selecting the set of relevant binary attributes and their (positive) weights. Moreover, they show that if i) the option set is suitably embedded in a mixture space of lotteries over opportunity sets, and ii) such a mixture space is endowed with a total preorder (to be interpreted as preference for diversity) obeying the standard set of von Neumann-Morgenstern axioms as supplemented with a mild positivity requirement, then the foregoing diversity function may be regarded as an expected diversity function representing those preferences (see Nehring and Puppe (2002)). Then, as mentioned in Section 5, Nehring and Puppe focus on the problem of establishing some conditions on the structure of relevant binary attributes ensuring that such a diversity function only depend on binary dissimilarity information, with special emphasis on the case of *monotonic* dependence (see Nehring and Puppe (2002,2003)).

To the best of our knowledge, width-based diversity rankings for opportunity posets were first suggested and discussed in Basili and Vannucci (2000) (but see also Vannucci (1999), that includes a short discussion of width in concept lattices of game forms as a key parameter when contrasting distributed and hierarchical decision mechanisms). Subsequently, the width-based ranking was suggested as a suitable complexity index for some environmental systems as represented by the majorization posets of certain integer partitions attached to them, namely their Young diagram lattices (see Seitz (2006)). As for the undominated width-ranking, its counterpart in a multi-preferential setting was first introduced and characterized in Pattanaik and Xu (1998). Moreover, the criterion of 'undominated diversity' for allocations as strongly advocated by Van Parijs (1995) amounts to selecting allocations whose individual components form an antichain with respect to the unanimity partial (pre)order and are therefore of maximum width for the relevant population of agents. However, the characterizations of width-based rankings for general posetic populations of options provided in Section 4 are apparently a distinctive contribution of the present paper.

We wish to emphasize that our focussing on posets should not be construed as an attempt to downplay the role and relevance of (pre)metric spaces and other data structures in the measurement of diversity. In our view, the main message of the present work is rather that a) distinct notions of diversity may conveniently apply to distinct data structures and b) even very common and parsimonious data structures as posets may support several interesting diversity rankings.

Finally, a short comment on the relationship between diversity and dissimilarity in a posetic setting is in order here. Of course, the notion itself of width rests on (binary) incomparability: thus, in a sense diversity as width depends by definition on (binary) dissimilarities and nothing else. On the other hand, it is easily checked that neither the width-ranking nor the undominated width-ranking are monotonic in dissimilarities. Clearly enough, incomparability-monotonic diversity rankings might also be considered and analyzed. This is however best left as a topic for further research.

References

[1] Anderson I.: *Combinatorics of Finite Sets.* Oxford University Press, Oxford 1987.

[2] Barberà S., W.Bossert, P.K. Pattanaik: Ranking Sets of Objects, in S. Barberà, P.C. Hammond, C. Seidl (eds.): *Handbook of Utility Theory, vol.II, Extensions*, Kluwer Academic Publishers, Dordrecht 2004, pp.893-977.

[3] Basili M., S. Vannucci: Patterns of Freedom and Flexibility. Working Paper DEP 283, Siena 2000.

[4] Baumgärtner S.: Why the Measurement of Species Diversity Requires Prior Value Judgments. Mimeographed, Department of Economics, University of Heidelberg, Heidelberg 2006.

[5] Bossert W., P.K. Pattanaik, Y. Xu: The Measurement of Diversity. Working Paper CRDE 17-2001, Montreal 2001.

[6] Bossert W., P.K. Pattanaik, Y. Xu: Similarity of Options and the Measurement of Diversity, *Journal of Theoretical Politics,* 15 (2003), 405-421.

[7] Brüggemann R., L. Carlsen (eds.): *Partial Order in Environmental Sciences and Chemistry.* Springer, Berlin 2006.

[8] Carpineto C., G. Romano: *Concept Data Analysis. Theory and Applications.* Wiley, Chichester 2004.

[9] Nehring K., C. Puppe: A Theory of Diversity, *Econometrica,* 70 (2002), 1155-1198.

[10] Nehring K., C. Puppe: Diversity and Dissimilarity in Lines and Hierarchies, *Mathematical Social Sciences,* 45 (2003), 167-183.

[11] Pattanaik P.K., Y. Xu: On Preference and Freedom, *Theory and Decision,* 48 (1998), 173-198.

[12] Pattanaik P.K., Y. Xu: On Diversity and Freedom of Choice, *Mathematical Social Sciences,* 40 (2000), 123-130.

[13] Pattanaik P.K., Y. Xu: Ordinal Distance, Dominance, and the Measurement of Diversity. Working Paper 06-57, GSU Department of Economics, Atlanta 2006.

[14] Peragine V., A. Romero-Medina: On Preference, Freedom and Diversity, *Social Choice and Welfare,* 27 (2006), 29-40.

[15] Schrijver A.: *Theory of Linear and Integer Programming.* Wiley, New York 1986.

[16] Seitz W.: Partial Orders and Complexity: The Young Diagram Lattice, in R.Brüggemann, L. Carlsen (eds.): *Partial Order in Environmental Sciences and Chemistry.* Springer, Berlin 2006.

[17] Trotter W.T.: *Combinatorics and Partially Ordered Sets. Dimension Theory.* John Hopkins University Press, Baltimore 1992.

[18] Van Hees M.: Freedom of Choice and Diversity of Options: Some Difficulties, *Social Choice and Welfare,* 22 (2004), 253-266.

[19] Vannucci S. : On a Lattice-Theoretic Representation of Coalitional Power in Game Correspondences, in H. De Swart(ed.): *Logic, Game Theory, and Social Choice.* Tilburg University Press, Tilburg 1999.

[20] Vannucci S.: On Concept Lattices of Efficiently Solvable Voting Protocols, in S. Ben Yahia, E.M. Nguifo (eds.): *CLA06, Proceedings of the Fourth International Conference on Concept Lattices and Their Applications.* Publications de la Faculté de Sciences de Tunis, Tunis 2006.

[21] Van Parjis P.: *Real Freedom for All.* Oxford University Press, Oxford 1995.

[22] Weitzman M.: On Diversity, *Quarterly Journal of Economics,* 107 (1992), 363-405.

[23] Weitzman M.: The Noah's Ark Problem, *Econometrica,* 66 (1998), 1279-1298.