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# QUADERNI DEL DIPARTIMENTO DI ECONOMIA POLITICA


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## Price-Level Computation: Illustrations


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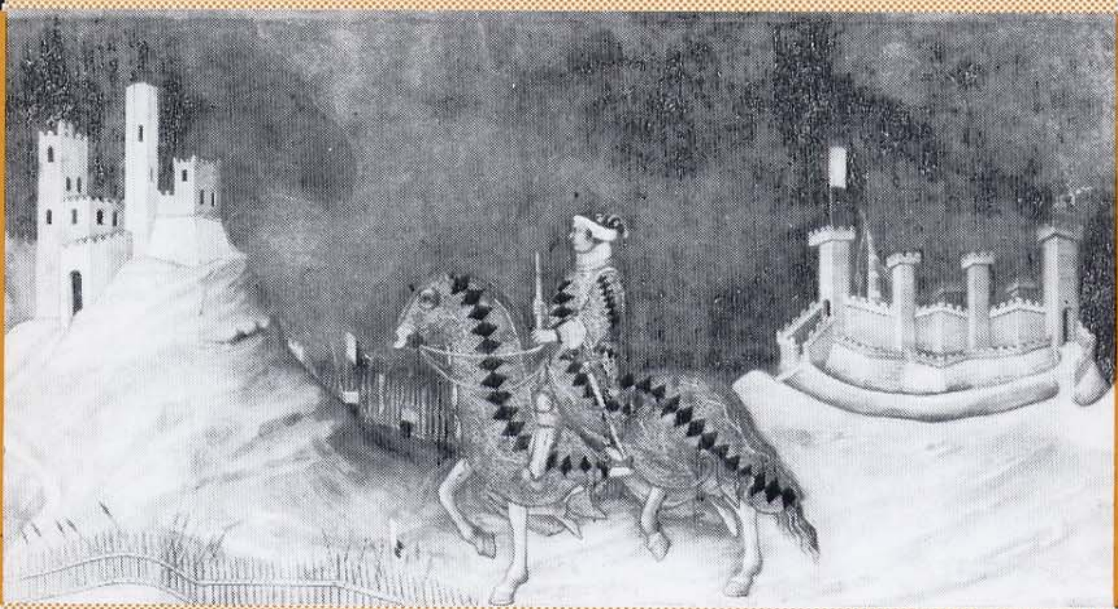

**Abstract -** It has been submitted that, for the very large number of different traditional type formulae to determine price indices associated with a pair of periods, which are joined with the longstanding question of which one to choose, they should all be abandoned. For the method proposed instead, price levels associated with periods are first all computed together, subject to a consistency of the data, and then price indices that are true taken together are determined from their ratios. An approximation method can apply in the case of inconsistency. Here are illustrations of the method.

**Keywords**: index-number problem, inflation, non-parametric, price index, price level, revealed preference.

**Jel Classification**: C43, E31.

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# Introduction

The theory of the Price-Index, proper, starts with the Utility-Cost Factorization Theorem, going back to early1960's. By itself it represents no resolution of the Index-Number Problem, nor had there even been a real idea of what could be meant by such a resolution.

However, the method now proposed does convey some idea of what could be meant by such a resolution. It even represents such a resolution itself.

The method has been available in the main for more than twenty-five years, apart from amplifications made just now. But only recently has it been recognized as a proper resolution of the Index-Number Problem. These first exercises with the arithmetic go to convey the practicality of it.

Encounter with the work of Steve Dowrick and John Quiggin (1997) that shows some awareness of the method and steps towards its application, joined with needs of dealing with the EUKLEMS Project data, have stirred into life that almost forgotten work and exposed its value.

This is the third of three papers by present authors, all currently available with SSRN. The first

"The Super Price-Index: Irving Fisher, and after"

has more to do with history, and

"The Price-Level Computation Method"

is an exposition of the mathematics.

We start with the Laspeyres matrix *L* taken from Dowrick and Quiggin (1997, pp. 50-51, Table 2), who have calculated quantity indexes of per capita GDP in an inter-country comparison based on the International Comparison Project (ICP) data for 1980 published by the United Nations and the Commission of the European Communities (1987).

This source gives prices and quantities for some 38 components of GDP expenditure for 60 countries. In these applications we take the data for various countries, for instance in the first illustration just for US, France, and Italy, to form the matrix *L* (our presentation requires transposition of the matrix given by Dowrick and Quiggin, 1997, pp. 50-51).

Since prices and quantities can be interchanged symmetrically in the method, and the data is in use only to illustrate computational procedure, there is liberty now to say "price" for affinity with the more usual subject even when "quantity" may fit the data source better, but any reader may always for "price" read "quantity".

# I Outline of the Method

## 1 Original data

A price-index formula based on a pair of reference periods has conventionally been *algebraical* and involved data for those periods *alone*. Then there are *inconsistencies* between formulae in the treatment of more than two periods, conflicting with the nature of price indices as such, as gathered by Irving Fisher's "Tests".

Formulae proposed now are of a completely different type, beside being 'non-parametric' rather than conventionally algebraical, are computed *simultaneously for any number* of periods, involving the data for *all* of them, without any of the multi-period consistency problems that go with the conventional formulae. There is either exactness, subject to a condition on the data, or approximation, in the fit to the data of the hypothetical underlying utility, which in any case there is no need to actually construct.

With some $m$ time periods, or countries, or nodes, in any case *references*—perhaps most typically time periods—listed as 1, …, $m$, the initial data has the form of some $m$ demand elements

$$(p_t, x_t) \ \left(t = 1, \ldots, m\right)$$

giving row and column vectors of prices and quantities for some $n$ goods demanded at the prices..

Hence for the initial data scheme:

$m$ number of references
$n$ number of goods
$p$ $m \times n$ price matrix, rows $p_i$
$x$ $n \times m$ quantity matrix, columns $x_j$
$c = px$ $m \times m$ cross-cost matrix, elements $p_i x_j$

The first step is to compute the matrix $L$ of *Laspeyres indices*

$$L_{ij} = p_i x_j / p_j x_j$$

$i$ being index for the *current period* and $j$ for the *base period*. Hence divide column $j$ of $c$ by diagonal element $p_j x_j$ to form the $m \times m$ Laspeyres matrix $L$ with these elements.

The *Paasche indices* are given by

$$K_{ij} = 1 / L_{ji} = p_i x_i / p_j x_i \,,$$

forming the elements of an $m \times m$ matrix $K$, obtained by transposition of $L$ and replacing each element by its reciprocal. The *Laspeyres-Paasche (LP) inequality*

$$K_{ij} \le L_{ij}$$

has significance for Laspeyres and Paasche indices as price-index bounds, and for data consistency.

Another well-known construction that may have comment is the *Fisher index* which is the geometric mean of the Laspeyres and Paasche indices,

$$F_{ij} = \left(L_{ij}K_{ij}\right)^{\frac{1}{2}} = \left(p_i x_j p_i x_i / p_j x_j p_j x_i\right)^{\frac{1}{2}}$$

Central to the proposed method is the system of inequalities

$$(L) \quad L_{ij} \geq P_i / P_j .$$

This serves to determine *price-levels* $P_i$ from which the matrix $P$ of *price-indices*

$$P_{ij} = P_i / P_j$$

is derived, and which enter into the construction of an underlying utility which fits the given demand data and represents all these indices together as *true*.

By the *geometric mean* of two vectors is here meant the vector whose elements correspondingly are geometric means of their elements, and there is a similar understanding about matrices. The same understanding can apply just as well for several vectors, or matrices, also in application of the more general weighted geometric mean.

Any two *price-level solutions* $P_i^a$ and $P_i^b$ have a *geometric mean* with elements which are geometric means

$$P_i^c = \left(P_i^a P_i^b\right)^{\frac{1}{2}}$$

of their elements, which *also is a price-level solution.* For from

$$L_{ij} \geq P_i^a / P_j^a \text{ and } L_{ij} \geq P_i^b / P_j^b$$

follows

$$\begin{aligned} L_{ij}L_{ij} &\geq \left(P_i^a / P_j^a\right)\left(P_i^b / P_j^b\right) \\ &= \left(P_i^a P_i^b\right) / \left(P_j^a P_j^b\right) \\ &= \left(P_i^c\right)^2 / \left(P_j^c\right)^2 \end{aligned}$$

and hence

$$L_{ij} \geq P_i^c / P_j^c .$$

There is a similar conclusion in dealing with the geometric means of several price-level solutions.

It can be added that the *price-index matrix* obtained from the geometric mean of the *price-level solutions*, which is the matrix of ratios of its elements, is the geometric mean *of the price-index matrices* obtained from them.

# 2 Consistency of the data

The solubility of the system ($L$) imposes a condition on the given data, defining its *consistency*, equivalent to the existence of the appropriate underlying utility.

With any *chain* described by a series of periods, or references,

$$s, i, j, \ldots, k, t$$

there is associated the *Laspeyes chain product*

$$L_{sij\ldots kt} = L_{si}L_{ij} \ldots L_{kt}$$

termed the *coefficient* on the chain. Obviously

$$L_{r\ldots s\ldots t} = L_{r\ldots s} L_{s\ldots t}$$

A chain

$$t, i, j, \ldots, k, t$$

whose extremeties are the same defines a *cycle*. It is associated with the *Laspeyres cyclical product*

$$L_{tij\ldots kt} = L_{ti} L_{ij} \ldots L_{kt}$$

which is basis for the important *Laspeyres cyclical product test*, or simply the *cycle test*,

$$L_{t\ldots t} \geq 1 \text{ for all cycles } t\ldots t$$

which is *necessary and sufficient for consistency* of the given data, and is an *extension of the PL-inequality*.

Introducing the *chain Laspeyres* and *Paasche indices*

$$L_{sij\ldots kt} = L_{si} L_{ij} \cdots L_{kt}, \quad K_{sij\ldots kt} = K_{si} K_{ij} \cdots K_{kt},$$

the cycle test $L_{s\ldots t\ldots s} \geq 1$ is equivalently to

(chain *LP*) $K_{s\ldots t} \leq L_{s\ldots t}$

for all possible chains $\ldots$ the two occurrences here being taken separately. Hence, introducing the *derived Laspeyres* and *Paasche indices*

$$M_{st} = \min_{ij\ldots k} L_{si} L_{ij} \cdots L_{kt}, \quad H_{st} = \max_{ij\ldots k} K_{si} K_{ij} \cdots K_{kt},$$

subject to the now to be considered conditions required for their existence, where

$$H_{st} = 1/M_{ts},$$

this condition is equivalent to

(derived *LP*) $H_{st} \leq M_{st}$.

In this case

$$K_{st} \leq H_{st} \leq M_{st} \leq L_{st},$$

showing the relation of bounds for the *LP*-interval and the *narrower bounds* for the derived version that involves more data.

The matrix *M*, and the matrix *H* constructed from it, in exactly the same way as the Paasche matrix *K* is constructed from the Laspeyres matrix *L*, is important in that their columns, currently as a matter of conjecture, provide a complete set of basic solutions of the system of inequalities (*L*), the *canonical price-level solutions*, from which all other solutions may be derived as combinations.

# 3 Price-Quantity duality

With any determination of price levels $P_t$, there is an associated determination of quantity levels $X_t$, where

$$P_t X_t = p_t x_t \quad (t = 1, \ldots, m).$$

While for price levels,

$$p_t x_s / p_s x_s \geq P_t / P_s ,$$

for quantity levels equivalently in a dual fashion,

$$p_t x_s / p_t x_t \geq X_s / X_t ,$$

and one could just as well have solved for the quantity levels first, by the same method as for price levels, and then determined the price levels from these. Whichever way,

$$P_s X_t \leq p_s x_t \quad (s, t = 1, \ldots, m),$$

with equality for $s = t$.

The introduction of cost-efficiency up to a level e, where $0 \leq e \leq 1$, would require

$$P_t X_t \geq e p_t x_t \quad (t = 1, \ldots, m).$$

good also for any lower level, and highest level 1 imposing the equality.

# 4 The Power Algorithm

For the main step in the proposed method, matrix $L$ is raised to the *m*th power in the modified arithmetic where + means *min*, to determine

$$M = L^m .$$

Diagonal elements $M_{ii} = 1$ tell the consistency of the system of inequalities ($L$) for the determination of price-levels $P_i$, and provide the *first* and *second canonical price-level solutions*, with any *t* as *base*, given by

$$P_i = M_{it} ,$$

and

$$P_i = H_{it} ,$$

that is, by *columns of the matrices M and H.* From these are derived the two systems of *canonical price indices*

$$P_{ij} = P_i / P_j .$$

The price indices in either system, with any base, will all be true together in respect to a utility that fits the data by criterion of cost-efficiency of demand in each period *i*, so the cost $p_i x_i$ is the minimum cost, at the prices $p_i$, of the utility of $x_i$.

# 5 Cost-efficiency and approximation

Diagonal elements $M_{ii} < 1$ tell the *inconsistency* of the system, and enable determination of a *critical cost efficiency* $e^*$ so that the system

$$(L/e) \quad L_{ij} / e \geq P_i / P_j \; (i \neq j)$$

is consistent if and only if $e \le e^*$. Then with

$$L^*_{ij} = L_{ij} / e^* \left(i \neq j\right)$$

as the elements of the *ajusted Laspeyres matrix*, the system

$$\left(L^*\right) \quad L^*_{ij} \ge P_i / P_j$$

is consistent, and with

$$M^* = \left(L^*\right)^m$$

there may be obtained canonical price levels and price indices from $M^*$, as before from *M*. Now, instead, the price levels of a canonical system are together true in respect to a utility that fits the data not exactly, but approximately in the sense of partial cost efficiency at the level $e^*$ in each period, meaning that the fraction $e^*$ of the cost, in the period, is at most the minimum cost at the prices of gaining at least the utility. Hence in the case $e^*$ = 1 that goes with ordinary consistency, the fit would be exact as before.

For any element $M_{ii} < 1$ determine the number $d_i$ of nodes in the path $i\ldots i$ and

$$e_i = (M_{ii})^{\frac{1}{d_i}}$$

giving this the value 1 in case $M_{ii} \ge 1$ and then

$$e^* = \min_i e_i$$

is the *critical cost- efficiency*.

Consistency requires $M_{ii} = 1$, in this case compute the *2m* canonical price-level solutions $P_r = M_{rt}$ and $P_r = H_{rt}$ a pair determined for every node *t* and compute the canonical mean price-level solution $\overline{P}_r$ and with this the matrix of canonical mean price-indices $\overline{P}_{rs} = \overline{P}_r / \overline{P}_s$. In the other case, of inconsistency, with the critical cost-efficiency $e^*$ form the *ajusted Laspeyres matrix* and proceed exactly as before with this in place of original *L*.

An alternative procedure for the critical cost-efficiency is available, especially if the path *i…i* for elements $M_{ii} < 1$ is not known:

**Critical cost-efficiency**
**crude approximation method**

TEST e: if L/e consistent then YES

0 HIGH = 1 LOW = 0 D = 1/n (for n steps, eg 10)

1 e = (HIGH + LOW)/2 TEST e

2 if YES then LOW = e else HIGH = e

3 if HIGH - LOW < D then $e^*$ = LOW end else 1

It should be reminded that the following illustrations are not intended for communications of any kind of actual economic information. They are the first calculations made following the method, just to assist understanding of it and show the shape of its arithmetic, beside being stimulus for the software development.

# II Illustrations

## 1 Three references with consistency, and graphics

We start with the Laspeyres matrix *L* taken from Dowrick and Quiggin (1997, pp. 50-51, Table 2), who have calculated bilateral quantity indexes of per capita GDP in an inter-country comparison based on the International Comparison Project (ICP) data for 1980 published by the United Nations and the Commission of the European Communities (1987).

This source gives prices and quantities for some 38 components of GDP expenditure for 60 countries. In the following application, we take the data for the US, France, and Italy to form the matrix *L* (our presentation requires transposition of the matrix given by Dowrick and Quiggin, 1997, pp. 50-51).

By raising the matrix *L* to powers in a modified arithmetic where + means *min*, we have

**Illustration 1**

**L Laspeyres**

| | | |
|---|---|---|
| 1 | 1.182937 | 1.500803 |
| 0.913018 | 1 | 1.266174 |
| 0.747516 | 0.813833 | 1 |

**L power 2**

| | | |
|---|---|---|
| 1 | 1.182937 | 1.49780407 |
| 0.913018 | 1 | 1.266174 |
| 0.743044178 | 0.813833 | 1 |

**L power 3 = M derived Laspeyres**

| | | |
|---|---|---|
| 1 | 1.182937 | 1.49780407 |
| 0.913018 | 1 | 1.266174 |
| 0.743044178 | 0.813833 | 1 |

**Paths**

| | | |
|---|---|---|
| 1,1,1,1 | 1,1,1,2 | 1,2,2,3 |
| 2,1,1,1 | 2,2,2,2 | 2,2,2,3 |
| 3,2,1,1 | 3,2,2,2 | 3,3,3,3 |

**Consistency case: all diagonal elements = 1**

Note that

$$L \geq L^2 = L^3 .$$

and at this point one could add " = ..." because after one equality only others can follow.

Now we have the *derived Laspeyres matrix*

$$M = L^3$$

The *Paasche matrix K* is

| | | |
|---|---|---|
| 1 | 1.0952687 | 1.3377640 |
| 0.8453536 | 1 | 1.2287533 |
| 0.6663100 | 0.7897809 | 1 |

and the *derived Paasche matrix* is

**H derived Paasche**

| | | |
|---|---|---|
| 1 | 1.09526866 | 1.345815 |
| 0.845353557 | 1 | 1.22875332 |
| 0.667644065 | 0.789780867 | 1 |

Note that

$$K_{st} \leq H_{st} \leq M_{st} \leq L_{st} ,$$

showing the relation of the original *LP*-interval and the narrower bounds that involve more data.

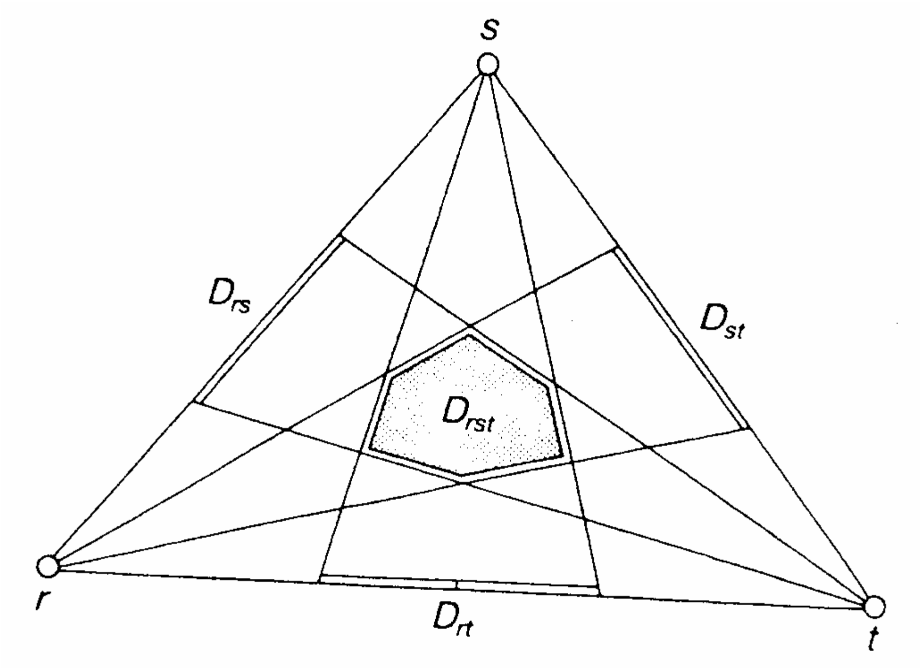
**The 6 canonical price-level systems - the 6 columns of M and H**

The geometric mean of the matrices *H* and *M*, element by element, is the matrix *F*, whose columns coincide with the geometric means of their corresponding columns:

**F derived Fisher - mean of derived Laspeyres M and derived Paasche H**

| | | |
|---|---|---|
| 1 | 1.13825912 | 1.41977716 |
| 0.878534583 | 1 | 1.24732334 |
| 0.704335883 | 0.801716741 | 1 |

The columns of *M* and *H* are all solutions of system (*L*). These are the 6 canonical price-level solutions, from which all other solutions can be derived, being the 6 vertices of the convex hexagonal region described by solutions normalized to sum 1 each determining a point in the simplex of reference. The columns of *F* are geometric means of opposite pairs of vertices of the hexagon.




The 6 canonical solutions as vertices for the solution set

The 6 canonical solutions, a basis for all solutions, are given by columns of *M* and *H*, and canonical geometric mean solution has elements given by the geometric means of their columns, or of columns of the matrix *F*, so it is

P mean canonical price-level system - mean of columns of F

1.17351085

1.03096986

0.826545798

The matrix of canonical mean price-indices obtained from this, by taking ratios of the elements, is

**P/P mean canonical price-index system**

| | | |
|---|---|---|
| 1 | 1.13825912 | 1.41977716 |
| 0.878534583 | 1 | 1.24732334 |
| 0.704335883 | 0.801716741 | 1 |

and coincides with the mean of individual canonical price-index matrices.

Notice that this matrix *P/P* coincides with the matrix *F* used to obtain it, and see End-note No. 2.

## 2 Four references, with consistency

The start with the given Laspeyres matrix *L* (Dowrick and Quiggin, 1997) and raising it to powers in a modified arithmetic where + means *min*, using CM's FORTRAN program, we have

$$L \equiv \begin{bmatrix} 1.000 & 1.122 & \mathbf{1.183} & \mathbf{1.501} \\ 0.898 & 1.000 & 1.042 & \mathbf{1.350} \\ \mathbf{0.913} & 0.979 & 1.000 & 1.266 \\ \mathbf{0.747} & \mathbf{0.812} & 0.814 & 1.000 \end{bmatrix}$$

(elements that change are in bold).

and then

$$L^4 \equiv \begin{bmatrix} L_{11} & L_{12} & L_{123} & L_{1234} \\ L_{21} & L_{22} & L_{23} & L_{234} \\ L_{321} & L_{32} & L_{33} & L_{34} \\ L_{4321} & L_{432} & L_{43} & L_{44} \end{bmatrix}$$

where

$$L_{rij\ldots ks} = L_{ri} L_{ij} \ldots L_{ks} .$$

With

$$M = L^4 ,$$

therefore

$$M = \begin{bmatrix} 1.000 & 1.122 & 1.169 & 1.480 \\ 0.898 & 1.000 & 1.042 & 1.319 \\ 0.879 & 0.979 & 1.000 & 1.266 \\ 0.715 & 0.797 & 0.814 & 1.000 \end{bmatrix}$$

Note the triangle inequality $M_{rs} M_{st} \geq M_{rt}$

From the matrix $L$, the Paasche matrix *K* is derive by $K_{ij} = 1/L_{ji}$, so that

$$K = \begin{bmatrix} 1.000 & 1.114 & 1.095 & 1.338 \\ 0.891 & 1.000 & 1.021 & 1.231 \\ 0.845 & 0.960 & 1.000 & 1.229 \\ 0.666 & 0.741 & 0.790 & 1.000 \end{bmatrix}$$

from which, by similar procedure, $H_{ij} = 1/M_{ji}$, we have

$$H \equiv \begin{bmatrix} 1.000 & 1.114 & 1.138 & 1.398 \\ 0.891 & 1.000 & 1.021 & 1.255 \\ 0.856 & 0.960 & 1.000 & 1.229 \\ 0.676 & 0.758 & 0.790 & 1.000 \end{bmatrix}$$

Alternatively, just as $M = L^m$ so similarly $H = K^m$ where the arithmetic for powers now has + meaning *max* instead of *min*.

Note $K \le L$ for original bounds, and moreover

$$K \le H \le M \le L$$

showing tighter bounds obtained with additional data.

With any $P_i$ which are a price-level solution being such that

$$L_{ij} \ge P_i / P_j$$

there is associated a price-index matrix with elements

$$P_{ij} = P_i / P_j$$

The 8 canonical solutions, a basis for all solutions, are given by columns of *M* and *H*, and canonical geometric mean, which has elements given by the geometric means of their elements, is also a solution. It is

[1.167 1.044 1.012 0.811].

The matrix of canonical mean price-indices obtained from this, by taking ratios of the canonical mean price-levels, is

$$\begin{bmatrix} 1.000 & 1.118 & 1.153 & 1.438 \\ 0.894 & 1.000 & 1.031 & 1.287 \\ 0.867 & 0.969 & 1.000 & 1.247 \\ 0.695 & 0.777 & 0.802 & 1.000 \end{bmatrix}$$

and coincides with the mean of individual canonical price-index matrices, derived from the individual canonical price-level solution elements..

By taking weighted geometric means instead of the simple geometric mean, it is possible to arrive at all possible price-level solutions, and consequently all possible systems of true price-indices, without any guidance for choosing just one from among them. Here we have, for want of that guidance and to that extent arbitrarily, adopted one, with weights all equal and no reason for making them different, as a standard, in order to eliminate that residual indecision.

Following the above report done using FORTRAN, we include the routine output from another program using BBC BASIC for Windows[3] This deals with two text files kept in folder c:\0\ as here indicated:

```
REM input from C:\0\?-input.txt output to C:\0\?-output.txt
REM change the ?
```

[3] We acknowledge with thanks the guidance received from Richard Russell, longtime developer of this rendering of BASIC, http://www.compulink.co.uk/~rrussell/bbcwin/bbcwin.html, http://www.rtrussell.co.uk/, info@rtrussell.co.uk.
.

```
*SPOOL "C:\0\2-output.txt"
F%=OPENIN "C:\0\2-input.txt"
```

For instance 2-input.txt looks like

```
4, 4
1.0000000, 1.1218730, 1.1829370, 1.5008030
0.8976280, 1.0000000, 1.0418520, 1.3498590
0.9130180, 0.9792190, 1.0000000, 1.2661740
0.7475160, 0.8122070, 0.8138330, 1.0000000
```

which tells it is a 4 x 4 matrix and then tells the elements, the comma "," being the delimiter.

As for 2-output.txt, which need not even exist initially and if it does any contents will be overwritten, it recieves the output when the program is run, which with this input is as follows—showing reassuring agreement with earlier figures.

When this program is compiled and so not available for alteration, it will refer to two files called input.txt and output.txt always with these names though they can have different applications. This compiled version will be supplied to anyone wishing to use it.

**Illustration 2**

**L Laspeyres**

| | | | |
|---|---|---|---|
| 1 | 1.121873 | **1.182937** | **1.500803** |
| 0.897628 | 1 | 1.041852 | **1.349859** |
| **0.913018** | 0.979219 | 1 | 1.266174 |
| **0.747516** | **0.812207** | 0.813833 | 1 |

**L power 2**

| | | | |
|---|---|---|---|
| 1 | 1.121873 | 1.16882563 | 1.47993662 |
| 0.897628 | 1 | 1.041852 | 1.31916592 |
| 0.878974393 | 0.979219 | 1 | 1.266174 |
| **0.729059745** | 0.796920736 | 0.813833 | 1 |

**L power 3**

| | | | |
|---|---|---|---|
| 1 | 1.121873 | 1.16882563 | 1.47993662 |
| 0.897628 | 1 | 1.041852 | 1.31916592 |
| 0.878974393 | 0.979219 | 1 | 1.266174 |
| 0.715338367 | 0.796920736 | 0.813833 | 1 |

(Elements that change are in bold. This is after the last power that changes so generation of powers could have stopped here. Note that

$$L \geq L^2 \geq L^3 = L^4 .$$

and at this point one could add " **= ...**" because after one equality only others can follow.)

**L power 4 = M derived Laspeyres**

| | | | |
|---|---|---|---|
| 1 | 1.121873 | 1.16882563 | 1.47993662 |
| 0.897628 | 1 | 1.041852 | 1.31916592 |

| | | | |
|---|---|---|---|
| 0.878974393 | 0.979219 | 1 | 1.266174 |
| 0.715338367 | 0.796920736 | 0.813833 | 1 |

**Paths**

| | | | |
|---|---|---|---|
| 1,1,1,1,1 | 1,1,1,1,2 | 1,2,2,2,3 | 1,3,3,3,4 |
| 2,1,1,1,1 | 2,2,2,2,2 | 2,2,2,2,3 | 2,3,3,3,4 |
| 3,2,1,1,1 | 3,2,2,2,2 | 3,3,3,3,3 | 3,3,3,3,4 |
| 4,2,2,1,1 | 4,3,2,2,2 | 4,3,3,3,3 | 4,4,4,4,4 |

**Consistency case: all diagonal elements = 1**

**H derived Paasche**

| | | | |
|---|---|---|---|
| 1 | 1.11404725 | 1.13768957 | 1.39793984 |
| 0.891366491 | 1 | 1.02122202 | 1.25482994 |
| 0.855559611 | 0.959829227 | 1 | 1.22875332 |
| 0.675704611 | 0.758054759 | 0.789780867 | 1 |

**The 8 canonical price-level systems - the 8 columns of M and H**

**F derived Fisher - mean of derived Laspeyres M and derived Paasche H**

| | | | |
|---|---|---|---|
| 1 | 1.11795328 | 1.15315252 | 1.43835405 |
| 0.894491767 | 1 | 1.03148543 | 1.28659585 |
| 0.867187978 | 0.96947564 | 1 | 1.24732334 |
| 0.695239119 | 0.77724485 | 0.801716741 | 1 |

**P mean canonical price-level - mean of columns of F**

| |
|---|
| 1.16692797 |
| 1.04380747 |
| 1.01194591 |
| 0.811293977 |

**P/P mean canonical price-index**

| | | | |
|---|---|---|---|
| 1 | 1.11795328 | 1.15315252 | 1.43835405 |
| 0.894491767 | 1 | 1.03148543 | 1.28659585 |
| 0.867187978 | 0.96947564 | 1 | 1.24732334 |
| 0.69523912 | 0.77724485 | 0.801716741 | 1 |

As with Illustration No. 1, notice that this matrix *P* coincides with the matrix *F* used to obtain it, see end-note No. 2.

# 3 Case of inconsistency and approximation

Starting with the Laspeyres matrix $L$ for the countries Canada, U.S., Norway, Luxembourg, Germany in the year 1980 taken from Dowrick and Quiggin (1997, pp. 50-51), and raising it to powers in the (+ = *min*)-arithmetic using the FORTRAN program:

**L POWER 1**

1.0000000 1.0171450 1.1252440 1.2008140 1.1537290
0.9139310 1.0000000 1.1274960 1.1411080 1.1218730
0.9685060 1.0171450 1.0000000 1.1207520 1.0650260
0.9389430 0.9398820 1.0345840 1.0000000 1.0222430
0.8886960 0.8976270 1.0030040 1.0387310 1.0000000

**… … … … … … … … …**

**M = L POWER 5**

0.8641568 0.8789728 0.9723873 1.0376916 0.9970028
0.7897797 0.8641568 0.9559967 0.9860962 0.9694742
0.8122053 0.8789728 0.9723873 1.0140962 0.9970021
0.7795783 0.8122054 0.8985241 0.9588679 0.9212698
0.7626154 0.7756905 0.8581284 0.9157593 0.8798515

Inconsistency case since some diagonal elements < 1

diagonal elements < 1 (in this case all)
associated cost-efficiencies $e_i$
critical cost efficiency is minimum of these

| $i$ | $M_{ii}$ | path | $d_i$ | $e_i = (M_{ii})^{\frac{1}{d_i}}$ |
|---|---|---|---|---|
| 1 | 0.8641568 | 12121 | 3 | 0.952498 |
| 2 | 0.8641568 | 21212 | 3 | 0.952498 |
| 3 | 0.9723873 | 321213 | 4 | 0.990710 |
| 4 | 0.9588679 | 421214 | 4 | 0.986097 |
| 5 | 0.8798515 | 521215 | 4 | 0.968506 |

**critical cost-efficiency** $e^* = \min_i e_i = e_1 = e_2 =$ 0.952498
used to determine the ***adjusted Laspeyres matrix*** = *L**

Being near to the value 1, associated with the consistency case where fit of data to the hypothetical underlying utility is exact, this represents a high level of cost-efficiency, and a closeness of fit for the approximating utility.

**Note**: By computer error the degree $d_i$ associated with a path is 1 less than the correct count. The effect is to make the cost-efficiency less than critical, resulting in allowance of a looser fit for the approximate utility. A revision could provide the correction, and moreover a redevelopment of the approach to cost-efficiency where the critical uniform bound is replaced by discrimination, but for the time being the error does not damage, even enhances, the value of the illustration.

**L*   POWER  1 — adjusted L**

| | | | | |
|---|---|---|---|---|
| 1.0000000 | 1.0678707 | 1.1813607 | 1.2606994 | 1.2112663 |
| 0.9595094 | 1.0000000 | 1.1837250 | 1.1980159 | 1.1778216 |
| 1.0168061 | 1.0678707 | 1.0000000 | 1.1766447 | 1.1181396 |
| 0.9857687 | 0.9867546 | 1.0861794 | 1.0000000 | 1.0732230 |
| 0.9330159 | 0.9423923 | 1.0530245 | 1.0905332 | 1.0000000 |

**POWER  2**

| | | | | |
|---|---|---|---|---|
| 1.0000000 | 1.0678707 | 1.1813607 | 1.2606994 | 1.2112663 |
| 0.9595094 | 1.0000000 | 1.1335267 | 1.1980159 | 1.1622213 |
| 1.0168061 | 1.0537261 | 1.0000000 | 1.1766447 | 1.1181396 |
| 0.9468003 | 0.9867546 | 1.0861794 | 1.0000000 | 1.0732230 |
| 0.9042342 | 0.9423923 | 1.0530245 | 1.0905332 | 1.0000000 |

**POWER  3 (no change after this power)**

… … … … … … … … …

**M* = L*  POWER  5**

| | | | | |
|---|---|---|---|---|
| 1.0000000 | 1.0678707 | 1.1813607 | 1.2606994 | 1.2112663 |
| 0.9595094 | 1.0000000 | 1.1335267 | 1.1980159 | 1.1622213 |
| 1.0110601 | 1.0537261 | 1.0000000 | 1.1766447 | 1.1181396 |
| 0.9468003 | 0.9867546 | 1.0861794 | 1.0000000 | 1.0732230 |
| 0.9042342 | 0.9423923 | 1.0530245 | 1.0905332 | 1.0000000 |

**Consistent,  all diagonal elements = 1**

From *derived Laspeyres M** determine
by transposition and element inversion

**the derived Paasche H***

| | | | | |
|---|---|---|---|---|
| 1.0000000 | 1.0421993 | 0.9890609 | 1.0561890 | 1.1059081 |
| 0.9364429 | 1.0000000 | 0.9490132 | 1.0134232 | 1.0611292 |
| 0.8464815 | 0.8822024 | 1.0000000 | 0.9206582 | 0.9496455 |
| 0.7932105 | 0.8347135 | 0.8498742 | 1.0000000 | 0.9169826 |
| 0.8255823 | 0.8604213 | 0.8943427 | 0.9317728 | 1.0000000 |

(Alternatively, just as $M = L^m$ so similarly $H = K^m$ where now the arithmetic for powers has + meaning *max* instead of *min*, and same here for adjusted *-versions.)

The columns of M* and H* provide the 10 *canonical price-level solutions* in 5 *opposite pairs*. Then determine

***F**** **the matrix geometric mean of** ***M**** **and** ***H****

| | | | | |
|---|---|---|---|---|
| 1.0000000 | 1.0549569 | 1.0809430 | 1.1539224 | 1.1573890 |
| 0.9479060 | 1.0000000 | 1.0371749 | 1.1018607 | 1.1105256 |
| 0.9251182 | 0.9641575 | 1.0000000 | 1.0408110 | 1.0304544 |
| 0.8666094 | 0.9075557 | 0.9607892 | 1.0000000 | 0.9920316 |
| 0.8640138 | 0.9004745 | 0.9704457 | 1.0080324 | 1.0000000 |

The columns are *geometric means of opposite pairs of canonical solutions*.
Now determine the geometric mean of the columns of F*

**Mean canonical price-level solution**

```
1.087774
1.037659
0.991172
0.943996
0.946864
```

Coincides with *the geometric mean of all 10 of the canonical price-level solutions*. Finally form the mean canonical price-index matrix *P*, given by ratios of elements of the mean canonical price-level solution.

**Mean canonical price-index matrix**

| | | | | |
|---|---|---|---|---|
| 1.00000 | 1.05496 | 1.08094 | 1.15392 | 1.15739 |
| 0.94791 | 1.00000 | 1.03717 | 1.10186 | 1.11053 |
| 0.92512 | 0.96416 | 1.00000 | 1.04081 | 1.03045 |
| 0.86661 | 0.90756 | 0.96079 | 1.00000 | 0.99203 |
| 0.86401 | 0.90047 | 0.97045 | 1.00803 | 1.00000 |

Coincides with *the mean of the 10 canonical price-index matrices obtained from the 10 individual canonical price-level solutions.*

The BASIC program produces the following.

**Illustration 3**

**L Laspeyres**

| | | | | |
|---|---|---|---|---|
| 1 | 1.017145 | 1.125244 | 1.200814 | 1.153729 |
| 0.913931 | 1 | 1.127496 | 1.141108 | 1.121873 |
| 0.968506 | 1.017145 | 1 | 1.120752 | 1.065026 |
| 0.938943 | 0.939882 | 1.034584 | 1 | 1.022243 |
| 0.888696 | 0.897627 | 1.003004 | 1.038731 | 1 |

**L power 2**

| | | | | |
|---|---|---|---|---|
| 0.929600347 | 0.945538345 | 1.04602721 | 1.07896137 | 1.06077394 |
| 0.913931 | 0.929600347 | 1.02839537 | 1.06077439 | 1.04289353 |
| 0.929600347 | 0.945538345 | 1 | 1.07896137 | 1.06077394 |
| 0.858987296 | 0.873714633 | 0.966570301 | 0.997002758 | 0.980196857 |
| 0.820369142 | 0.834434371 | 0.923115455 | 0.952179736 | 0.936129391 |

**L power 3**

| | | | | |
|---|---|---|---|---|
| 0.864156805 | 0.878972774 | 0.97238726 | 1.00300286 | 0.986095823 |
| 0.849590575 | 0.864156805 | 0.955996697 | 0.986096244 | 0.969474188 |
| 0.864156805 | 0.878972774 | 0.97238726 | 1.00300286 | 0.986095823 |
| 0.798514889 | 0.812205427 | 0.898524088 | 0.92681411 | 0.911191339 |
| 0.762615439 | 0.775690481 | 0.858128447 | 0.885146613 | 0.870226207 |

**L power 4**

| | | | | |
|---|---|---|---|---|
| 0.803320466 | 0.817093396 | 0.903931535 | 0.932391811 | 0.916675019 |
| 0.789779693 | 0.803320466 | 0.888694861 | 0.91667541 | 0.901223541 |
| 0.803320466 | 0.817093396 | 0.903931535 | 0.932391811 | 0.916675019 |
| 0.742299718 | 0.755026446 | 0.835268304 | 0.861566718 | 0.847043784 |
| 0.708927577 | 0.72108214 | 0.797716502 | 0.822832599 | 0.808962584 |

**L power 5 = M derived Laspeyres**

| | | | | |
|---|---|---|---|---|
| 0.746766984 | 0.759570304 | 0.840295068 | 0.866751751 | 0.852141416 |
| 0.734179477 | 0.746766984 | 0.826131051 | 0.85214178 | 0.837777717 |
| 0.746766984 | 0.759570304 | 0.840295068 | 0.866751751 | 0.852141416 |
| 0.690042075 | 0.701872846 | 0.776465705 | 0.80091272 | 0.787412196 |
| 0.659019321 | 0.670318208 | 0.741557537 | 0.764905469 | 0.752011899 |

**Paths**

| | | | | |
|---|---|---|---|---|
| 1,2,2,2,2,1 | 1,1,1,1,1,2 | 1,1,1,1,1,3 | 1,2,2,2,2,4 | 1,2,2,2,2,5 |
| 2,1,2,2,2,1 | 2,1,1,1,1,2 | 2,1,1,1,1,3 | 2,2,2,2,2,4 | 2,2,2,2,2,5 |
| 3,2,2,2,2,1 | 3,1,1,1,1,2 | 3,3,1,1,1,3 | 3,2,2,2,2,4 | 3,2,2,2,2,5 |
| 4,2,2,2,2,1 | 4,1,1,1,1,2 | 4,1,1,1,1,3 | 4,2,2,2,2,4 | 4,2,2,2,2,5 |
| 5,2,2,2,2,1 | 5,1,1,1,1,2 | 5,1,1,1,1,3 | 5,2,2,2,2,4 | 5,2,2,2,2,5 |

**Inconsistency case: some diagonal elements < 1**

| | |
|---|---|
| 1 | 0.746766984 |
| 2 | 0.746766984 |
| 3 | 0.840295068 |
| 4 | 0.80091272 |
| 5 | 0.752011899 |

**Effective paths - Factor counts - Efficiencies**

| | | | |
|---|---|---|---|
| 1 | 1,2,1 | 2 | 0.864156805 |
| 2 | 2,1,2 | 2 | 0.864156805 |
| 3 | 3,1,3 | 2 | 0.916676098 |
| 4 | 4,2,4 | 2 | 0.894937272 |
| 5 | 5,2,5 | 2 | 0.867186196 |

**Critical cost-efficiency  0.864156805**

**L adjusted Laspeyres to replace original L**

| | | | | |
|---|---|---|---|---|
| 1 | 1.17703754 | 1.30212942 | 1.38957883 | 1.33509219 |
| 1.05759857 | 1 | 1.30473543 | 1.3204872 | 1.29822851 |
| 1.12075262 | 1.17703754 | 1 | 1.29693129 | 1.23244531 |
| 1.08654239 | 1.087629 | 1.19721791 | 1 | 1.18293693 |
| 1.02839669 | 1.03873162 | 1.16067361 | 1.2020168 | 1 |

**L power 2 … …  … L power 4**

**L power 5 = M derived Laspeyres**

| | | | | |
|---|---|---|---|---|
| 1 | 1.17703754 | 1.30212942 | 1.38957883 | 1.33509219 |
| 1.05759857 | 1 | 1.30473543 | 1.3204872 | 1.29822851 |
| 1.12075262 | 1.17703754 | 1 | 1.29693129 | 1.23244531 |
| 1.08654239 | 1.087629 | 1.19721791 | 1 | 1.18293693 |
| 1.02839669 | 1.03873162 | 1.16067361 | 1.2020168 | 1 |

**Paths**

| | | | | |
|---|---|---|---|---|
| 1,1,1,1,1,1 | 1,1,1,1,1,2 | 1,1,1,1,1,3 | 1,1,1,1,1,4 | 1,1,1,1,1,5 |
| 2,1,1,1,1,1 | 2,2,2,2,2,2 | 2,2,2,2,2,3 | 2,2,2,2,2,4 | 2,2,2,2,2,5 |
| 3,1,1,1,1,1 | 3,2,2,2,2,2 | 3,3,3,3,3,3 | 3,3,3,3,3,4 | 3,3,3,3,3,5 |
| 4,1,1,1,1,1 | 4,2,2,2,2,2 | 4,3,3,3,3,3 | 4,4,4,4,4,4 | 4,4,4,4,4,5 |
| 5,1,1,1,1,1 | 5,2,2,2,2,2 | 5,3,3,3,3,3 | 5,4,4,4,4,4 | 5,5,5,5,5,5 |

**Consistency case: all diagonal elements = 1**

**H derived Paasche**

| | | | | |
|---|---|---|---|---|
| 1 | 0.945538345 | 0.892257565 | 0.920350655 | 0.972387414 |
| 0.849590575 | 1 | 0.849590575 | 0.919431168 | 0.96271258 |
| 0.767972818 | 0.76643891 | 1 | 0.835269834 | 0.861568653 |
| 0.719642514 | 0.757296246 | 0.771050871 | 1 | 0.831935126 |
| 0.749011948 | 0.77028042 | 0.811395032 | 0.845353605 | 1 |

**The 10 canonical price-level systems - columns of M and H**

**F derived Fisher - mean of derived Laspeyres M and derived Paasche H**

| | | | | |
|---|---|---|---|---|
| 1 | 1.05495693 | 1.07788442 | 1.13088451 | 1.13939758 |
| 0.947905995 | 1 | 1.05284896 | 1.10186074 | 1.1179539 |
| 0.927743253 | 0.949803858 | 1 | 1.04081102 | 1.03045439 |
| 0.88426359 | 0.907555705 | 0.960789211 | 1 | 0.992031646 |
| 0.877656772 | 0.894491269 | 0.970445672 | 1.00803236 | 1 |

**P mean canonical price-level - mean of columns of F**

| |
|---|
| 1.07939424 |
| 1.04216477 |
| 0.988763618 |
| 0.94781098 |
| 0.94856982 |

**P/P mean canonical price-index**

| | | | | |
|---|---|---|---|---|
| 1 | 1.0357232 | 1.09166055 | 1.13882858 | 1.13791754 |
| 0.965508928 | 1 | 1.05400801 | 1.09954917 | 1.09866955 |
| 0.916035665 | 0.948759394 | 1 | 1.0432076 | 1.04237305 |
| 0.878095276 | 0.909463653 | 0.958581974 | 1 | 0.999200017 |
| 0.8787983 | 0.910191791 | 0.959349437 | 1.00080062 | 1 |

# 4 Inconsistency and approximation again

## CAN, US, NOR, LUX, GER, DEN, FRA, BEL, NED, AUT, JPN, UK, ITA, SPN, IRL, GRC, PRT (17 COUNTRIES)

Again taken from Dowrick and Quiggin (1997).

### Illustration 4

### L Laspeyres

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.017145 | 1.125244 | 1.200814 | 1.153729 | 1.193631 | 1.204422 | 1.228753 | 1.24982 | 1.382647 | 1.496306 | 1.541876 | 1.55893 | 1.853359 | 2.325651 | 2.637944 | 3.625528 |
| 0.913931 | 1 | 1.127496 | 1.141108 | 1.121873 | 1.172337 | 1.182936 | 1.208041 | 1.232445 | 1.367521 | 1.460823 | 1.463747 | 1.500802 | 1.823941 | 2.241174 | 2.585709 | 3.45907 |
| 0.968506 | 1.017145 | 1 | 1.120752 | 1.065026 | 1.081122 | 1.111821 | 1.124119 | 1.159512 | 1.294338 | 1.388189 | 1.420487 | 1.474029 | 1.73846 | 2.192406 | 2.464527 | 3.293661 |
| 0.938943 | 0.939882 | 1.034584 | 1 | 1.022243 | 1.069295 | 1.070365 | 1.088717 | 1.136553 | 1.261119 | 1.368889 | 1.392359 | 1.37163 | 1.670294 | 2.083397 | 2.391689 | 3.326763 |
| 0.888696 | 0.897627 | 1.003004 | 1.038731 | 1 | 1.043937 | 1.041852 | 1.059715 | 1.087628 | 1.208041 | 1.30604 | 1.321807 | 1.349858 | 1.604801 | 2.027898 | 2.30712 | 3.183559 |
| 0.969475 | 0.963676 | 1.006018 | 1.05654 | 1.012072 | 1 | 1.021222 | 1.042894 | 1.082204 | 1.227525 | 1.30604 | 1.239861 | 1.247323 | 1.617691 | 1.873859 | 2.190215 | 2.857651 |
| 0.887807 | 0.913017 | 0.97824 | 1.00904 | 0.979219 | 1.00904 | 1 | 1.02429 | 1.063962 | 1.177036 | 1.286596 | 1.243587 | 1.266174 | 1.569881 | 1.902178 | 2.166255 | 2.880604 |
| 0.878095 | 0.887807 | 0.959829 | 0.98906 | 0.960789 | 0.986097 | 0.984127 | 1 | 1.041852 | 1.16649 | 1.263644 | 1.226298 | 1.234912 | 1.543418 | 1.847807 | 2.104336 | 2.903741 |
| 0.848742 | 0.875465 | 0.928671 | 0.963676 | 0.935195 | 0.965605 | 0.969475 | 0.976285 | 1 | 1.135417 | 1.218962 | 1.18649 | 1.233678 | 1.508325 | 1.847807 | 2.100131 | 2.869104 |
| 0.73565 | 0.773368 | 0.837779 | 0.880733 | 0.847046 | 0.876341 | 0.875465 | 0.895834 | 0.917594 | 1 | 1.11071 | 1.102962 | 1.09308 | 1.355269 | 1.647073 | 1.845961 | 2.567672 |
| 0.729059 | 0.768741 | 0.802518 | 0.939882 | 0.842821 | 0.840296 | 0.855559 | 0.877217 | 1.121873 | 1.030454 | 1 | 1.091988 | 1.072508 | 1.321807 | 1.574598 | 1.782469 | 2.539583 |

0.79692  0.799315 0.826959 0.88692  0.838618 0.842821 0.854704 0.865887 0.889585
1.004008 1.055484 1  1.041852 1.341783 1.551155 1.829421 2.325651

0.758054 0.747515 0.805735 0.869358 0.812207 0.825306 0.813833 0.834435 0.858988
0.960789 1.078962 0.998002 1  1.273794 1.506817 1.709156 2.37976

0.563831 0.562704 0.631915 0.67977  0.631283 0.662324 0.64856  0.66365  0.673006
0.754273 0.863294 0.812207 0.827786 1  1.231213 1.375751 1.950332

0.493121 0.50258  0.547167 0.551011 0.547715 0.562704 0.55046  0.564395 0.593926
0.664978 0.718923 0.675028 0.67368  0.875465 1  1.133148 1.630684

0.478547 0.476637 0.523614 0.695586 0.555992 0.528876 0.52782  0.537944 0.544438
0.635082 0.71177  0.653769 0.619402 0.824482 0.979219 1  1.55426

0.370093 0.374186 0.396531 0.439551 0.39812  0.39812  0.39812  0.408199 0.409016
0.47001  0.54335  0.461626 0.462088 0.62688  0.689354 0.80493  1

**L power 2  ……  L power 16**

**L power 17 = M derived Laspeyres**

0.310985711 0.316317561 0.349934805 0.360952499 0.354868131 0.370459972 0.369720072
0.376059081 0.385964515 0.428695252 0.462083968 0.457943038 0.462083244 0.569492731
0.694189753 0.783480195 1.05864531

0.305743734 0.310985711 0.344036303 0.354868282 0.348886472 0.364215497 0.363488069
0.369720228 0.379458696 0.421469163 0.454295079 0.450223948 0.454294367 0.55989336
0.682488487 0.770273849 1.04080078

0.310985711 0.316317561 0.349934805 0.360952499 0.354868131 0.370459972 0.369720072
0.376059081 0.385964515 0.428695252 0.462083968 0.457943038 0.462083244 0.569492731
0.694189753 0.783480195 1.05864531

0.287363033 0.292289872 0.323353528 0.333534311 0.327912116 0.34231959  0.341635893
0.347493387 0.356646398 0.39613128  0.426983767 0.423157385 0.426983098 0.526233691
0.641458645 0.723966526 0.978229919

0.274443831 0.279149171 0.308816275 0.318539352 0.313169918 0.326929665 0.326276705
0.331870859 0.340612371 0.378322101 0.407787529 0.404133173 0.40778689  0.502575397
0.612620094 0.691418606 0.934250883

0.294637899 0.299689466 0.331539528 0.341978047 0.33621352  0.350985734 0.350284729
0.356290511 0.365675239 0.406159717 0.437793265 0.433870014 0.437792578 0.539555794
0.657697776 0.742294423 1.00299473

0.279149227 0.283935241 0.314110993 0.324000775 0.31853928  0.332534941 0.331870786
0.337560854 0.34645224  0.384808511 0.41477913  0.411062119 0.41477848  0.511192156
0.623123591 0.703273119 0.950268806

0.271441428 0.276095291 0.305437838 0.315054545 0.309743853 0.323353068 0.322707252
0.328240207 0.336886087 0.374183273 0.403326351 0.399711973 0.403325719 0.497077244
0.605918057 0.683854516 0.924030218

0.267667939 0.272257105 0.301191742 0.310674761 0.305437896 0.31885792  0.318221082
0.32367712  0.332202808 0.368981501 0.397719441 0.394155309 0.397718818 0.49016704
0.597494784 0.674347796 0.911184655

0.23645242  0.240506397 0.266066667 0.274443774 0.269817633 0.281672611 0.281110041
0.285929793 0.293461213 0.325950764 0.351337276 0.348188795 0.351336726 0.433003608
0.527814756 0.595705146 0.804922017

0.235037744 0.239067466 0.264474811 0.272801798 0.268203336 0.279987386 0.279428182
0.284219098 0.291705458 0.324000626 0.349235253 0.346105609 0.349234706 0.430412981
0.524656883 0.59214109  0.800106233

0.244385553 0.248575544 0.274993378 0.283651542 0.278870191 0.291122911 0.290541466
0.295522924 0.303307028 0.336886625 0.363124871 0.359870756 0.363124302 0.447531161
0.545523286 0.615691443 0.831927677

0.228548028 0.232466483 0.257172297 0.265269364 0.260797871 0.272256547 0.271712784
0.276371416 0.283651067 0.315054521 0.339592386 0.336549155 0.339591853 0.418528684
0.510170381 0.575791256 0.778014195

0.172043222 0.174992903 0.193590604 0.199685802 0.196319813 0.204945517 0.20453619
0.208043051 0.213522926 0.237162384 0.255633658 0.253342817 0.255633257 0.315054233

0.384039002 0.433436176 0.585662762

0.153660686 0.156295198 0.172905765 0.178349701 0.175343363 0.183047425 0.182681834
0.185813992 0.190708352 0.211821972 0.228319621 0.226273552 0.228319263 0.281391205
0.343005064 0.387124231 0.523085656

0.145728776 0.148227296 0.163980431 0.169143354 0.166292202 0.173598582 0.173251863
0.17622234 0.180864055 0.200887798 0.216533844 0.214593392 0.216533504 0.266865891
0.325299265 0.367141017 0.496084161

0.113607209 0.115555005 0.127835831 0.13186074 0.12963804 0.135333946 0.135063651
0.137379375 0.140997962 0.156608067 0.168805409 0.167292672 0.168805144 0.208043256
0.253596733 0.286215717 0.386737187

## Inconsistency case: some diagonal elements < 1

1 0.310985711
2 0.310985711
3 0.349934805
4 0.333534311
5 0.313169918
6 0.350985734
7 0.331870786
8 0.328240207
9 0.332202808
10 0.325950764
11 0.349235253
12 0.359870756
13 0.339591853
14 0.315054233
15 0.343005064
16 0.367141017
17 0.386737187

## Effective paths - Factor counts - Efficiencies

1 1,2,1 2 0.557660928
2 2,1,2 2 0.557660928
3 3,1,3 2 0.591552876
4 4,2,4 2 0.577524295
5 5,2,5 2 0.559615866
6 6,5,6 2 0.59244049
7 7,5,7 2 0.576082274
8 8,5,8 2 0.572922514
9 9,5,9 2 0.576370374
10 10,5,10 2 0.570920978
11 11,2,11 2 0.590961296
12 12,9,12 2 0.599892287
13 13,6,13 2 0.582745102
14 14,5,14 2 0.56129692
15 15,6,15 2 0.585666342
16 16,14,16 2 0.605921626
17 17,6,17 2 0.621881972

## Critical cost-efficiency 0.557660928

## L adjusted Laspeyres to replace original L

1 1.82394884 2.01779243 2.15330488 2.06887186 2.1404243
2.15977477 2.20340522 2.24118266 2.47936861 2.68318242 2.76489875
2.79548005 3.32345142 4.17036748 4.73037265 6.50131257

1.63886504 1 2.02183073 2.04623983 2.01174754 2.1022398
2.12124598 2.16626437 2.21002573 2.4522446 2.61955415 2.62479748
2.69124467 3.27069893 4.01888296 4.63670462 6.20281936

1.73672917 1.82394884 1 2.00973736 1.90980925 1.93867267
1.99372225 2.01577508 2.07924196 2.32101253 2.48930655 2.54722346
2.64323521 3.11741403 3.93143197 4.41940053 5.90620722

1.68371667 1.68540049 1.85522053 1 1.83309059 1.91746444
1.91938317 1.95229206 2.03807178 2.26144407 2.45469771 2.49678421

| | | | | | |
|---|---|---|---|---|---|
| 2.45961288 | 2.99517846 | 3.73595656 | 4.28878711 | 5.96556587 | |

| | | | | | |
|---|---|---|---|---|---|
| 1.59361353 | 1.60962864 | 1.79859113 | 1.86265694 | 1 | 1.87199237 |
| 1.86825354 | 1.90028554 | 1.95033926 | 2.16626437 | 2.34199661 | 2.37027006 |
| 2.42057123 | 2.87773613 | 3.63643551 | 4.13713761 | 5.70877184 | |

| | | | | | |
|---|---|---|---|---|---|
| 1.73846678 | 1.72806799 | 1.80399585 | 1.89459212 | 1.81485191 | 1 |
| 1.83125973 | 1.87012205 | 1.94061292 | 2.20120317 | 2.34199661 | 2.22332413 |
| 2.23670503 | 2.90085053 | 3.3602121 | 3.92750306 | 5.12435219 | |

| | | | | | |
|---|---|---|---|---|---|
| 1.59201937 | 1.63722605 | 1.75418422 | 1.80941491 | 1.75593977 | 1.80941491 |
| 1 | 1.83676128 | 1.90790128 | 2.11066607 | 2.30712954 | 2.23000561 |
| 2.27050872 | 2.81511743 | 3.41099386 | 3.88453788 | 5.16551161 | |

| | | | | | |
|---|---|---|---|---|---|
| 1.57460377 | 1.59201937 | 1.72116954 | 1.77358669 | 1.72289101 | 1.76827343 |
| 1.76474081 | 1 | 1.86825354 | 2.09175494 | 2.26597191 | 2.1990029 |
| 2.21444957 | 2.76766387 | 3.31349554 | 3.77350446 | 5.20700098 | |

| | | | | | |
|---|---|---|---|---|---|
| 1.52196784 | 1.56988764 | 1.66529687 | 1.72806799 | 1.67699574 | 1.73152708 |
| 1.73846678 | 1.75067851 | 1 | 2.0360347 | 2.18584796 | 2.12761903 |
| 2.21223675 | 2.70473495 | 3.31349554 | 3.76596404 | 5.14488976 | |

| | | | | | |
|---|---|---|---|---|---|
| 1.31917078 | 1.38680686 | 1.50230894 | 1.57933424 | 1.51892657 | 1.57145849 |
| 1.56988764 | 1.60641342 | 1.64543355 | 1 | 1.99173 | 1.97783625 |
| 1.96011581 | 2.43027426 | 2.95353846 | 3.31018529 | 4.60436059 | |

| | | | | | |
|---|---|---|---|---|---|
| 1.30735177 | 1.3785097 | 1.43907877 | 1.68540049 | 1.51135028 | 1.50682244 |
| 1.53419212 | 1.57302934 | 2.01174754 | 1.84781459 | 1 | 1.95815763 |
| 1.92322601 | 2.37027006 | 2.82357598 | 3.19633116 | 4.55399128 | |

| | | | | | |
|---|---|---|---|---|---|
| 1.42904041 | 1.43333513 | 1.48290647 | 1.5904288 | 1.50381344 | 1.51135028 |
| 1.53265893 | 1.55271233 | 1.59520769 | 1.80039151 | 1.8926985 | 1 |
| 1.86825354 | 2.40609111 | 2.78153789 | 3.28052569 | 4.17036748 | |

| | | | | | |
|---|---|---|---|---|---|
| 1.35934573 | 1.34044715 | 1.4448475 | 1.55893655 | 1.45645312 | 1.47994231 |
| 1.45936887 | 1.49631247 | 1.54034102 | 1.72289101 | 1.93479935 | 1.78962152 |
| 1 | 2.28417294 | 2.70203079 | 3.06486597 | 4.26739598 | |

| | | | | | |
|---|---|---|---|---|---|
| 1.0110642 | 1.00904326 | 1.13315273 | 1.21896652 | 1.13201942 | 1.18768228 |
| 1.16300061 | 1.19006007 | 1.20683729 | 1.35256562 | 1.54806255 | 1.45645312 |
| 1.48438945 | 1 | 2.2078165 | 2.46700267 | 3.49734382 | |

| | | | | | |
|---|---|---|---|---|---|
| 0.884266721 | 0.901228641 | 0.981182243 | 0.988075321 | 0.982164919 | 1.00904326 |
| 0.987087265 | 1.01207557 | 1.06503069 | 1.19244144 | 1.28917585 | 1.21046314 |
| 1.2080459 | 1.56988764 | 1 | 2.03196592 | 2.92414964 | |

| | | | | | |
|---|---|---|---|---|---|
| 0.858132561 | 0.85470754 | 0.938946901 | 1.24732784 | 0.997007271 | 0.948382742 |
| 0.946489118 | 0.964643519 | 0.976288588 | 1.1388318 | 1.27634906 | 1.17234141 |
| 1.11071436 | 1.47846471 | 1.75593977 | 1 | 2.78710579 | |

| | | | | | |
|---|---|---|---|---|---|
| 0.663652376 | 0.670991962 | 0.711061113 | 0.788204764 | 0.713910515 | 0.713910515 |
| 0.713910515 | 0.731984221 | 0.733449269 | 0.842823975 | 0.974337582 | 0.82778975 |
| 0.82861821 | 1.12412394 | 1.23615259 | 1.44340398 | 1 | |

**L power 2 …… L power 16**

**L power 17 = M derived Laspeyres**

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1.82394884 | 2.01779243 | 2.15330488 | 2.06887186 | 2.1404243 |
| 2.15977477 | 2.20340522 | 2.24118266 | 2.47936861 | 2.68318242 | 2.76489875 |
| 2.79548005 | 3.32345142 | 4.17036748 | 4.73037265 | 6.50131257 | |

| | | | | | |
|---|---|---|---|---|---|
| 1.63886504 | 1 | 2.02183073 | 2.04623983 | 2.01174754 | 2.1022398 |
| 2.12124598 | 2.16626437 | 2.21002573 | 2.4522446 | 2.61955415 | 2.62479748 |
| 2.69124467 | 3.27069893 | 4.01888296 | 4.63670462 | 6.20281936 | |

| | | | | | |
|---|---|---|---|---|---|
| 1.73672917 | 1.82394884 | 1 | 2.00973736 | 1.90980925 | 1.93867267 |
| 1.99372225 | 2.01577508 | 2.07924196 | 2.32101253 | 2.48930655 | 2.54722346 |
| 2.64323521 | 3.11741403 | 3.93143197 | 4.41940053 | 5.90620722 | |

| | | | | | |
|---|---|---|---|---|---|
| 1.68371667 | 1.68540049 | 1.85522053 | 1 | 1.83309059 | 1.91746444 |
| 1.91938317 | 1.95229206 | 2.03807178 | 2.26144407 | 2.45469771 | 2.49678421 |
| 2.45961288 | 2.99517846 | 3.73595656 | 4.28878711 | 5.96556587 | |

| | | | | | |
|---|---|---|---|---|---|
| 1.59361353 | 1.60962864 | 1.79859113 | 1.86265694 | 1 | 1.87199237 |
| 1.86825354 | 1.90028554 | 1.95033926 | 2.16626437 | 2.34199661 | 2.37027006 |
| 2.42057123 | 2.87773613 | 3.63643551 | 4.13713761 | 5.70877184 | |
| | | | | | |
| 1.73846678 | 1.72806799 | 1.80399585 | 1.89459212 | 1.81485191 | 1 |
| 1.83125973 | 1.87012205 | 1.94061292 | 2.20120317 | 2.34199661 | 2.22332413 |
| 2.23670503 | 2.90085053 | 3.3602121 | 3.92750306 | 5.12435219 | |
| | | | | | |
| 1.59201937 | 1.63722605 | 1.75418422 | 1.80941491 | 1.75593977 | 1.80941491 |
| 1 | 1.83676128 | 1.90790128 | 2.11066607 | 2.30712954 | 2.23000561 |
| 2.27050872 | 2.81511743 | 3.41099386 | 3.88453788 | 5.16551161 | |
| | | | | | |
| 1.57460377 | 1.59201937 | 1.72116954 | 1.77358669 | 1.72289101 | 1.76827343 |
| 1.76474081 | 1 | 1.86825354 | 2.09175494 | 2.26597191 | 2.1990029 |
| 2.21444957 | 2.76766387 | 3.31349554 | 3.77350446 | 5.20700098 | |
| | | | | | |
| 1.52196784 | 1.56988764 | 1.66529687 | 1.72806799 | 1.67699574 | 1.73152708 |
| 1.73846678 | 1.75067851 | 1 | 2.0360347 | 2.18584796 | 2.12761903 |
| 2.21223675 | 2.70473495 | 3.31349554 | 3.76596404 | 5.14488976 | |
| | | | | | |
| 1.31917078 | 1.38680686 | 1.50230894 | 1.57933424 | 1.51892657 | 1.57145849 |
| 1.56988764 | 1.60641342 | 1.64543355 | 1 | 1.99173 | 1.97783625 |
| 1.96011581 | 2.43027426 | 2.95353846 | 3.31018529 | 4.60436059 | |
| | | | | | |
| 1.30735177 | 1.3785097 | 1.43907877 | 1.68540049 | 1.51135028 | 1.50682244 |
| 1.53419212 | 1.57302934 | 2.01174754 | 1.84781459 | 1 | 1.95815763 |
| 1.92322601 | 2.37027006 | 2.82357598 | 3.19633116 | 4.55399128 | |
| | | | | | |
| 1.42904041 | 1.43333513 | 1.48290647 | 1.5904288 | 1.50381344 | 1.51135028 |
| 1.53265893 | 1.55271233 | 1.59520769 | 1.80039151 | 1.8926985 | 1 |
| 1.86825354 | 2.40609111 | 2.78153789 | 3.28052569 | 4.17036748 | |
| | | | | | |
| 1.35934573 | 1.34044715 | 1.4448475 | 1.55893655 | 1.45645312 | 1.47994231 |
| 1.45936887 | 1.49631247 | 1.54034102 | 1.72289101 | 1.93479935 | 1.78962152 |
| 1 | 2.28417294 | 2.70203079 | 3.06486597 | 4.26739598 | |
| | | | | | |
| 1.0110642 | 1.00904326 | 1.13315273 | 1.21896652 | 1.13201942 | 1.18768228 |
| 1.16300061 | 1.19006007 | 1.20683729 | 1.35256562 | 1.54806255 | 1.45645312 |
| 1.48438945 | 1 | 2.2078165 | 2.46700267 | 3.49734382 | |
| | | | | | |
| 0.884266721 | 0.901228641 | 0.981182243 | 0.988075321 | 0.982164919 | 1.00904326 |
| 0.987087265 | 1.01207557 | 1.06503069 | 1.19244144 | 1.28917585 | 1.21046314 |
| 1.2080459 | 1.56988764 | 1 | 2.03196592 | 2.92414964 | |
| | | | | | |
| 0.858132561 | 0.85470754 | 0.938946901 | 1.24732784 | 0.997007271 | 0.948382742 |
| 0.946489118 | 0.964643519 | 0.976288588 | 1.1388318 | 1.27634906 | 1.17234141 |
| 1.11071436 | 1.47846471 | 1.75593977 | 1 | 2.78710579 | |
| | | | | | |
| 0.663652376 | 0.670991962 | 0.711061113 | 0.788204764 | 0.713910515 | 0.713910515 |
| 0.713910515 | 0.731984221 | 0.733449269 | 0.842823975 | 0.974337582 | 0.82778975 |
| 0.82861821 | 1.12412394 | 1.23615259 | 1.44340398 | 1 | |

**Consistency case: all diagonal elements = 1**

**Hence immediately the wanted final answer:**

**P mean canonical price-index**

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1.02510404 | 1.0858391 | 1.14268694 | 1.12972093 | 1.14936063 |
| 1.16241544 | 1.1847125 | 1.2219059 | 1.35642562 | 1.41731595 | 1.40970785 |
| 1.43721337 | 1.80377676 | 2.14583724 | 2.32196113 | 3.16721761 | |
| | | | | | |
| 0.975510742 | 1 | 1.0592477 | 1.11470339 | 1.1020549 | 1.12121364 |
| 1.13394875 | 1.15569977 | 1.19198233 | 1.32320776 | 1.38260694 | 1.37518515 |
| 1.40201709 | 1.7596036 | 2.09328728 | 2.26509802 | 3.0896548 | |
| | | | | | |
| 0.920946763 | 0.944066245 | 1 | 1.05235384 | 1.04041283 | 1.05849995 |
| 1.07052274 | 1.09105714 | 1.12531028 | 1.24919578 | 1.30527254 | 1.29826588 |
| 1.32359701 | 1.66118237 | 1.97620186 | 2.13840258 | 2.91683881 | |
| | | | | | |
| 0.87513033 | 0.897099634 | 0.950250725 | 1 | 0.988653048 | 1.00584035 |
| 1.01726501 | 1.03677784 | 1.06932691 | 1.1870492 | 1.24033618 | 1.2336781 |
| 1.25774901 | 1.57853975 | 1.87788725 | 2.03201861 | 2.77172819 | |

| | | | | | |
|---|---|---|---|---|---|
| 0.885174361 | 0.907395811 | 0.961156927 | 1.01147718 | 1 | 1.01738456 |
| 1.02894035 | 1.04867713 | 1.08159977 | 1.20067318 | 1.25457174 | 1.24783725 |
| 1.27218443 | 1.59665694 | 1.89944011 | 2.05534046 | 2.80353982 | |
| 0.870048942 | 0.891890682 | 0.944733155 | 0.994193564 | 0.982912497 | 1 |
| 1.01135833 | 1.03075786 | 1.06311793 | 1.18015667 | 1.23313424 | 1.22651483 |
| 1.25044598 | 1.56937406 | 1.86698342 | 2.02021982 | 2.75563433 | |
| 0.860277629 | 0.88187407 | 0.934123082 | 0.983028013 | 0.971873641 | 0.988769238 |
| 1 | 1.01918166 | 1.05117831 | 1.16690262 | 1.21928521 | 1.21274013 |
| 1.23640251 | 1.55174879 | 1.84601577 | 1.99753121 | 2.72468646 | |
| 0.844086644 | 0.865276626 | 0.916542278 | 0.964526786 | 0.953582347 | 0.970159958 |
| 0.98117935 | 1 | 1.03139445 | 1.14494075 | 1.19633747 | 1.18991557 |
| 1.21313261 | 1.52254387 | 1.81127255 | 1.95993638 | 2.67340608 | |
| 0.818393628 | 0.838938612 | 0.888643797 | 0.935167713 | 0.924556409 | 0.940629417 |
| 0.951313391 | 0.969561163 | 1 | 1.11009008 | 1.15992234 | 1.15369592 |
| 1.17620627 | 1.4761994 | 1.75613952 | 1.90027819 | 2.59203071 | |
| 0.737231726 | 0.755739218 | 0.80051503 | 0.842425066 | 0.832866109 | 0.847345121 |
| 0.856969542 | 0.873407642 | 0.900827793 | 1 | 1.04489028 | 1.03928135 |
| 1.0595593 | 1.32980145 | 1.58197929 | 1.71182341 | 2.33497331 | |
| 0.705558983 | 0.723271361 | 0.766123527 | 0.806233036 | 0.797084748 | 0.810941717 |
| 0.820152657 | 0.835884548 | 0.862126681 | 0.95703828 | 1 | 0.994632038 |
| 1.01403881 | 1.27267089 | 1.51401474 | 1.63828053 | 2.23465884 | |
| 0.709366837 | 0.727174808 | 0.770258244 | 0.810584221 | 0.801386561 | 0.815318315 |
| 0.824578965 | 0.84039576 | 0.866779521 | 0.962203351 | 1.00539693 | 1 |
| 1.01951151 | 1.27953941 | 1.52218577 | 1.64712222 | 2.24671914 | |
| 0.695790909 | 0.71325807 | 0.755516971 | 0.795071186 | 0.786049551 | 0.799714678 |
| 0.808798097 | 0.824312189 | 0.850191015 | 0.943788615 | 0.986155554 | 0.980861908 |
| 1 | 1.25505147 | 1.49305404 | 1.61559944 | 2.20372122 | |
| 0.554392331 | 0.568309816 | 0.601980867 | 0.633496877 | 0.626308618 | 0.637196719 |
| 0.644434206 | 0.656795525 | 0.677415258 | 0.751991961 | 0.785749094 | 0.781531222 |
| 0.796780072 | 1 | 1.18963571 | 1.28727744 | 1.75588115 | |
| 0.46601857 | 0.477717517 | 0.506021182 | 0.532513334 | 0.526470931 | 0.535623397 |
| 0.541707182 | 0.552098026 | 0.569430838 | 0.632119527 | 0.660495552 | 0.656950037 |
| 0.669768121 | 0.840593464 | 1 | 1.082077 | 1.47598222 | |
| 0.430670431 | 0.441481998 | 0.467638792 | 0.492121478 | 0.486537399 | 0.494995639 |
| 0.50061796 | 0.510220645 | 0.52623874 | 0.584172407 | 0.610396072 | 0.607119489 |
| 0.618965304 | 0.776833314 | 0.924148649 | 1 | 1.36402698 | |
| 0.315734542 | 0.323660753 | 0.342836909 | 0.360785738 | 0.356691919 | 0.362892852 |
| 0.367014707 | 0.374054659 | 0.385797898 | 0.428270421 | 0.447495602 | 0.445093462 |
| 0.453777906 | 0.569514627 | 0.677514936 | 0.733123332 | 1 | |

# 5 EUKLEMS data

Household Consumption in Italy 1992-2004, from ISTAT's tables.

Data collected for the EUKLEMS database concerning Italy. This includes only Italy because at this level of detail the EUKLEMS project does not provide the data collected from the national statistical institutes.

Initial treatment just for the five years 1999-2004. Other years dealt with elsewhere together with inputs of production concerning other countries.

Illustration 5: 1999-2004

L Laspeyres

| | | | | |
|---|---|---|---|---|
| 1 | 0.9741 | 0.94679 | 0.9206 | 0.89744 |
| 1.02659 | 1 | 0.97171 | 0.94472 | 0.92068 |
| 1.05674 | 1.02939 | 1 | 0.97205 | 0.94724 |
| 1.08751 | 1.0593 | 1.02895 | 1 | 0.97412 |
| 1.1173 | 1.08823 | 1.0572 | 1.02735 | 1 |

L power 2

| | | | | |
|---|---|---|---|---|
| 1 | 0.9741 | 0.946542711 | 0.920086842 | 0.896274995 |
| 1.02659 | 1 | 0.97171 | 0.944550706 | 0.920105733 |
| 1.05674 | 1.02937043 | 1 | 0.97205 | 0.946893346 |
| 1.08733262 | 1.05917071 | 1.02895 | 1 | 0.97412 |
| 1.11716604 | 1.08823 | 1.05709178 | 1.02735 | 1 |

L power 3 – final power followed by repetitions

| | | | | |
|---|---|---|---|---|
| 1 | 0.9741 | 0.946542711 | 0.920086842 | 0.896274995 |
| 1.02659 | 1 | 0.97171 | 0.944550706 | 0.920105733 |
| 1.05674 | 1.02937043 | 1 | 0.97205 | 0.946893346 |
| 1.08733262 | 1.05917071 | 1.02895 | 1 | 0.97412 |
| 1.11707117 | 1.08813903 | 1.05709178 | 1.02735 | 1 |

L power 4

| | | | | |
|---|---|---|---|---|
| 1 | 0.9741 | 0.946542711 | 0.920086842 | 0.896274995 |
| 1.02659 | 1 | 0.97171 | 0.944550706 | 0.920105733 |
| 1.05674 | 1.02937043 | 1 | 0.97205 | 0.946893346 |
| 1.08733262 | 1.05917071 | 1.02895 | 1 | 0.97412 |
| 1.11707117 | 1.08813903 | 1.05709178 | 1.02735 | 1 |

L power 5 = M derived Laspeyres

| | | | | |
|---|---|---|---|---|
| 1 | 0.9741 | 0.946542711 | 0.920086842 | 0.896274995 |
| 1.02659 | 1 | 0.97171 | 0.944550706 | 0.920105733 |
| 1.05674 | 1.02937043 | 1 | 0.97205 | 0.946893346 |
| 1.08733262 | 1.05917071 | 1.02895 | 1 | 0.97412 |
| 1.11707117 | 1.08813903 | 1.05709178 | 1.02735 | 1 |

Paths

| | | | | |
|---|---|---|---|---|
| 1,1,1,1,1,1 | 1,1,1,1,1,2 | 1,2,2,2,2,3 | 1,3,3,3,3,4 | 1,4,4,4,4,5 |
| 2,1,1,1,1,1 | 2,2,2,2,2,2 | 2,2,2,2,2,3 | 2,3,3,3,3,4 | 2,4,4,4,4,5 |
| 3,1,1,1,1,1 | 3,1,1,1,1,2 | 3,3,3,3,3,3 | 3,3,3,3,3,4 | 3,4,4,4,4,5 |
| 4,3,1,1,1,1 | 4,1,1,1,1,2 | 4,3,3,3,3,3 | 4,4,4,4,4,4 | 4,4,4,4,4,5 |
| 5,2,3,1,1,1 | 5,2,1,1,1,2 | 5,4,3,3,3,3 | 5,4,4,4,4,4 | 5,5,5,5,5,5 |

Consistency case: all diagonal elements = 1

H derived Paasche

| | | | | |
|---|---|---|---|---|
| 1 | 0.974098715 | 0.946306566 | 0.919681778 | 0.89519811 |
| 1.02658865 | 1 | 0.971467576 | 0.944134871 | 0.919000215 |
| 1.05647636 | 1.02911363 | 1 | 0.971864522 | 0.94599165 |
| 1.08685393 | 1.05870441 | 1.02875367 | 1 | 0.973378109 |
| 1.115729 | 1.08683162 | 1.05608515 | 1.02656757 | 1 |

The 10 canonical price-level systems - the 10 columns of M and H

F derived Fisher - mean of derived Laspeyres M and derived Paasche H

| | | | | |
|---|---|---|---|---|
| 1 | 0.974099358 | 0.946424631 | 0.919884288 | 0.89573639 |
| 1.02658932 | 1 | 0.97158878 | 0.944342766 | 0.919552808 |
| 1.05660817 | 1.02924202 | 1 | 0.971957257 | 0.946442391 |
| 1.08709325 | 1.05893754 | 1.02885183 | 1 | 0.973748984 |
| 1.11639988 | 1.08748513 | 1.05658835 | 1.02695871 | 1 |

P mean canonical price-level - mean of columns of F

0.946499834
0.971666239
1.00007959
1.02893372
1.05667244

**P/P mean canonical price-index**

| | | | | |
|---|---|---|---|---|
| 1 | 0.974099743 | 0.946424506 | 0.919884167 | 0.895736272 |
| 1.02658892 | 1 | 0.971588908 | 0.94434289 | 0.91955293 |
| 1.05660831 | 1.02924189 | 1 | 0.971957257 | 0.946442391 |
| 1.08709339 | 1.0589374 | 1.02885183 | 1 | 0.973748983 |
| 1.11640003 | 1.08748498 | 1.05658835 | 1.02695871 | 1 |

**X mean canonical quantity-level: PX = px**

```
768307.584
772120.065
771211.618
775995.565
780996.047
```

**X/X mean canonical quantity-index**

| | | | | |
|---|---|---|---|---|
| 1 | 0.99506232 | 0.996234453 | 0.990092751 | 0.983753485 |
| 1.00496218 | 1 | 1.00117795 | 0.995005771 | 0.988635049 |
| 1.00377978 | 0.998823437 | 1 | 0.993835084 | 0.987471858 |
| 1.01000639 | 1.0050193 | 1.00620316 | 1 | 0.993597301 |
| 1.01651482 | 1.0114956 | 1.01268709 | 1.00644396 | 1 |

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## NOTES

Note 1

**Price-levels**

Starting with the Laspeyres matrix

$$L = \begin{bmatrix} \mathbf{1} & L_{12} & L_{13} & L_{14} \\ L_{21} & \mathbf{1} & L_{23} & L_{24} \\ L_{31} & L_{32} & \mathbf{1} & L_{34} \\ L_{41} & L_{42} & L_{43} & \mathbf{1} \end{bmatrix}.$$

in the case of 4 periods, we can always redefine the order of periods so that raising the $L$ matrix to power $m$ = 4 in the arithmetic where + means *min* yields, in the case of consistency of the data,

$$M = \begin{bmatrix} \mathbf{1} & L_{12} & L_{123} & L_{1234} \\ L_{21} & \mathbf{1} & L_{23} & L_{234} \\ L_{321} & L_{32} & \mathbf{1} & L_{34} \\ L_{4321} & L_{432} & L_{43} & \mathbf{1} \end{bmatrix}$$

where $L_{rij\ldots ks} = L_{ri} L_{ij} \ldots L_{ks}$.

By normalising each column of the $M$ matrix by the respective first element, we obtain

$$A = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ L_{21} & K_{21} & K_{21} & K_{21} \\ L_{321} & L_{32}K_{21} & K_{321} & K_{321} \\ L_{4321} & L_{432}K_{21} & L_{43}K_{321} & K_{4321} \end{bmatrix}$$

where $K_{ij} = \mathbf{1}/L_{ji}$ is the Paasche index of $i$ over $j$.

In the matrix $A$, the Afriat upper bound for the price-level solution is given by the first column so that

$$P_1 = \mathbf{1} \quad P_2 = L_{21} \quad P_3 = L_{321} \quad P_4 = L_{4321},$$

which is also equal to the last column of the matrix $B$ defined below.

In the case of consistency of the data, raising the Paasche matrix

$$K \equiv \begin{bmatrix} 1 & K_{12} & K_{13} & K_{14} \\ K_{21} & 1 & K_{23} & K_{24} \\ K_{31} & K_{32} & 1 & K_{34} \\ K_{41} & K_{42} & K_{43} & 1 \end{bmatrix}$$

to the power $m = 4$ in the arithmetic where + now means *max* yields

$$H \equiv \begin{bmatrix} \mathbf{1} & K_{\mathbf{12}} & K_{\mathbf{123}} & K_{\mathbf{1234}} \\ K_{\mathbf{21}} & \mathbf{1} & K_{\mathbf{23}} & K_{\mathbf{234}} \\ K_{\mathbf{321}} & K_{\mathbf{32}} & \mathbf{1} & K_{\mathbf{43}} \\ K_{\mathbf{4321}} & K_{\mathbf{432}} & K_{\mathbf{34}} & \mathbf{1} \end{bmatrix}$$

By normalising each column of the $H$ matrix by the respective first element, we obtain

$$B = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ K_{\mathbf{21}} & L_{\mathbf{21}} & L_{\mathbf{21}} & L_{\mathbf{21}} \\ K_{\mathbf{321}} & K_{\mathbf{32}}L_{\mathbf{21}} & L_{\mathbf{321}} & L_{\mathbf{321}} \\ K_{\mathbf{4321}} & K_{\mathbf{432}}L_{\mathbf{21}} & K_{\mathbf{43}}L_{\mathbf{321}} & L_{\mathbf{4321}} \end{bmatrix}$$

In the matrix *B*, the Afriat lower bound for the price-level solution is given by the first column so that

$$P_{\mathbf{1}} = \mathbf{1} \quad P_{\mathbf{2}} = K_{\mathbf{21}} \quad P_{\mathbf{3}} = K_{\mathbf{321}} \quad P_{\mathbf{4}} = K_{\mathbf{4321}} \, ,$$

which coincides with the last column of the matrix *A*.

Note 2

**The Triangle Equality**

From the $m \times m$ Laspeyres matrix $L$ we obtain the derived Laspeyres matrix $M = L^m$ (where + means min), in the consistency case necessarily having the *triangle inequality* property

$$M_{ij} M_{jk} \geq M_{ik} \, .$$

But this is without need for the *triangle equality* property

$$M_{ij} M_{jk} = M_{ik}$$

which corresponds to Fisher's price-index *Chain Test*, equivalent to the elements having the form of ratios of some numbers, for instance, for any k,

$$M_{ij} = M_{ik} / M_{jk}$$

for all i, j.

We then obtain the derived Paasche matrix *H* where

$$H_{ij} = \mathbf{1}/M_{ji},$$

and then *F* (for Fisher), the geometric mean of *M* and *H*, with elements

$$\begin{aligned} F_{ij} &= \left(H_{ij}M_{ij}\right)^{\frac{1}{2}} \\ &= \left(M_{ij}/M_{ji}\right)^{\frac{1}{2}} \end{aligned}.$$

The geometric mean of the columns of *F* has elements

$$F_i = \left(\Pi_k F_{ik}\right)^{\frac{1}{m}}$$

with ratios

$$\begin{aligned} F_i/F_j &= \left(\Pi_k F_{ik}\right)^{\frac{1}{m}} / \left(\Pi_k F_{jk}\right)^{\frac{1}{m}} \\ &= \left(\Pi_k \left(F_{ik}/F_{jk}\right)\right)^{\frac{1}{m}} \\ &= \left(\Pi_k \left(\left(M_{ik}/M_{ki}\right)^{\frac{1}{2}} / \left(M_{jk}/M_{kj}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{m}} \\ &= \left(\Pi_k \left(\left(M_{ik}M_{kj}\right)/\left(M_{jk}M_{ki}\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{m}} \end{aligned}$$

Hence subject to the triangle equality,

$$\begin{aligned} F_i/F_j &= \left(\Pi_k \left(M_{ij}/M_{ji}\right)^{\frac{1}{2}}\right)^{\frac{1}{m}} \\ &= \left(M_{ij}/M_{ji}\right)^{\frac{1}{2}} \\ &= F_{ij} \end{aligned}.$$

This shows how, subject to the triangle equality for *M*, from *F* we obtain the price-levels $F_i$ representing the geometric mean of the columns of *F* and hence of all the columns of *M* and *H*, and then from the price-indices which are their ratios we just get back to *F*. Our Illustrations nos. 1 and 2 are examples.

Note 3
**Ratio Matrix**

A matrix *M* is a *ratio matrix* if its elements $M_{ij}$ have the form of ratios of some numbers $X_i$,

(i) $M_{ij} = X_i / X_j$

forming the elements of a vector *X*, the *base vector* from which it is *derived*.

A test for *M* being a ratio matrix is the *triangle equality*

(ii) $M_{ij} M_{jk} = M_{ik}$ .

Were $M$ a matrix of price-indices this would correspond to Fisher's Chain Test. In case $M$ is a *derived Laspeyres matrix* one would just have the *triangle inequality*

(iii) $M_{ij} M_{jk} \geq M_{ik}$ ,

with ordinary data without the further imposition.

From (i) obviously (ii). And from (ii), for any $k$, we have

(iv) $M_{ij} = M_{ik} / M_{jk}$

for all $i$, $j$ which exhibits (i) with

$$X_i = M_{ik}$$

making base vector $X$ coincide with column $k$ of $M$. Hence:

*For a given matrix to be a ratio matrix the triangle equality is necessary and sufficient and then any column of it is a base vector from which it can be derived.*

Hence given that $M$ is a ratio matrix its columns, though different, would all be base vectors for the same ratio matrix, coinciding with the matrix $M$ itself.

The test for the triangle inequality is that the matrix be idempotent, or reproduced when multiplied by itself (with + = min). The test for the triangle equality is that the matrix coincide with the ratio matrix derived from any of its columns.