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# Choice under Markovian Constraints

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**Abstract -** In this paper I provide a descriptive model of choice over time by a population of constrained maximizing agents. Agents' choice sets are markovian in the sense that they depend on previous choices.
The unperturbed dynamics turns out to be trapped into local maxima whatever the length of memory. In the presence of perturbations efficiency is got with a memory of at least two periods. This provides a useful insight for what drives to efficient evolution in this setting: perturbations create variety and a two period long memory allows comparisons and selection.

**JEL classification**: D01; D80; D81

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# 1 Introduction

Quoting Simon,

> "...the task is to replace the global rationality of economic man with a kind of rational behavior that is compatible with the access to information and the computational capacities that are actually possessed by organisms, including man, in the kinds of environments in which such organisms exist." (Simon, 1955)

This paper arises, in accordance with the above-stated Simon's precept, from the recognition that decision problems are often characterized by lack of information, which may be seen as a consequence either of the limited access to information agents have or of the process of reduction of complexity they face when trying to cognitively structure problems (limited computational capacities).

This lack of information is assumed to concern mainly those alternatives that are far from the personal experience of decision-makers. In more detail, in order to evaluate and compare alternatives an agent is supposed to need a deep knowledge of their benefits and costs.[1] Personal experience is assumed to be the channel[2] for the acquisition of that deep information required to assess the value of an alternative. In particular, personal experience is supposed to reveal enough information not only about directly experienced alternatives, but also indirectly about the alternatives which are similar, in that they share most of the information which allows their evaluation. Therefore, previous choices - a synthetic representation of personal experience - affect current choices by providing deep information only about a subset of alternatives. Moreover, if agents are assumed to choose only among the alternatives they are able to correctly evaluate, then past behavior defines the set of possible current behaviors, among which an agent chooses her actual current behavior. By so doing her personal experience can change and, as a consequence, the set of possible future behaviors as well. My aim is to set up a descriptive model of choice which allows to explore the consequences of this kind of assumptions.

A simplified decision theory framework is adopted, where there is no strategic interaction and no uncertainty, and hence choices and outcomes coincide. Preferences are directly defined over the entire set of choices and

[1] Because of the often large numerosity of possible alternatives, global maximization generally presumes that agents have an incredibly large amount of information at their disposal. According to similar arguments many economists, among whom Hayek (1937), Keynes (1937) and Shackle (1973), have criticized the global maximization hypothesis.

[2] Of course, personal experience is a very important channel but not the only one, imitation for instance playing an important role too. Boncinelli (2007b) investigates the case where individual choice is affected both by personal experience and by imitation of others.

they are aimed at representing the true welfare comparison between alternatives for an individual, meaning that an agent would be better off by choosing a preferred alternative.

Given a set of alternatives and a preference relation over it, some peculiar features are introduced to the effect of shaping the model in a substantial way.

- *Constrained maximization.* There exist constraints defining a set of accessible alternatives. Agents pick out their best alternative within such set. Hence, an inferior alternative comes out to be selected when any superior choice is left out of the accessible set. Constraints can be thought of as operating on the preference relation - interpretable as objective preferences - by reducing it to subjective preferences, which are truthful but incomplete.

- *Repetition over time.* Agents choose repeatedly over time, keeping fixed the set of existing alternatives and the preference relation. A time-dimension is here required in order both to represent personal experience as previous choices and to make it possible for the analysis not to be highly dependent on initial conditions.

- *Markovian constraints.* The specification of constraints depends, in the basic version of the model, only upon the alternative selected at the previous time. In other words, the set of alternatives an agent is allowed to choose among is a function of the previous choice. These selectable alternatives should be interpreted as those that share a relevant amount of information with the experienced choice. Afterwards the model will be enriched by endowing agents with memory and letting them choose among alternatives which are accessible from one of those in memory.

These features[3] justify the indexing of the model with respect to time, which is treated as a discrete variable, and the introduction of the notion of accessibility, formally represented by an adjacency matrix showing which alternatives are accessible from every alternative.

Some other elements characterize the model, but their importance is mainly formal.

- *Finite alternatives.* The set of alternatives is assumed to be finite. The reason is mainly technical, since the mathematical techniques used for the derivation of results have been developed for a finite state space. Moreover, reality is likely to be finite; therefore the hypothesis of finiteness seems easier to be accepted than the hypothesis of infiniteness.

[3] From a more general point of view, the methodological framework which this paper belongs to is dealt with in Boncinelli (2007a).

- *Infinite population.* The variables under analysis in the model are the fractions of population choosing each alternative. Since the preference relation over alternatives does not depend on what others do, the behavior of a single agent could be analyzed. However, population is supposed to be infinite in order to interpret transition probabilities as actual frequencies.

- *Identical agents.* Agents are assumed to have identical preferences. Since no interaction among agents is allowed, there is no need to consider hetherogeneity in preferences, this last case being decomposable in several models with homogeneous preferences.

As regards the relation to the existing literature, it is worth mentioning the stream based on landscapes. It makes an extensive use of local maximization techniques and shares some of the principles underlying the present paper, even if it is rather different from a technical point of view. It is a cross-disciplines literature, born in population biology (Wright, 1949; Kauffman, 1993) and now spreading in economics too (Kauffman et al., 2000; Friedman and Yellin, 1997). The techniques used in this paper are standard ones for dynamic models based on finite Markov chains. When perturbations are inserted the analysis resorts to the stochastically stable distribution, as developed in economics by Foster and Young (1990), Young (1993), Kandori et al. (1993), Ellison (2000).

The outline of the paper is as follows. Section 1 has been presenting the paper. Section 2 introduces the basic model. A population of agents chooses repeatedly over time within a set of possible alternatives, ranked according to a common order relation. At each time every agent can select only among the alternatives which are accessible (because similar) from the one she has selected the previous time. Section 3 modifies the model allowing agents to store in their memory a sequence of past choices, hence potentially enlarging the set of accessible alternatives. In both cases the behavior of the unperturbed dynamics and of the perturbed one is analyzed from an aggregate point of view. The dependence of results on perturbations and memory provides a natural interpretation for their roles in the evolution of the system towards the efficiency. Section 4 briefly explores possible extensions for the model. The final section 5 deals with some concluding remarks, summarizing and interpreting assumptions and results. Proofs are collected in appendix A.

## 2 The model

**Definitions.** Consider an infinite population of agents who at any discrete point in time have to choose within a finite set of alternatives $\mathcal{C}$ of cardinality $n$. Let $x$ be a vector describing, from an aggregate point of view, how population is shared out among alternatives; every component $x_a$ represents the fraction of population choosing alternative $a \in \mathcal{C}$.

All agents have preferences defined on $\mathcal{C}$. These preferences are assumed to be identical across agents and representable by means of a linear order[4] relation $\succ$. The possibility of ties is ruled out, because this allows a simpler analysis without modifying the substance of results.

Finally, I introduce a notion of accessibility between alternatives, formally represented by a $(n \times n)$-adjacency matrix $A$ where $A_{ab} = 1$ ($A_{ab} = 0$) means that alternative $b$ is (is not) accessible from $a$. In order to correctly represent the idea of similarity - that is sharing a large amount of the information required to be evaluated - such a matrix is supposed to be reflexive, $A_{aa} = 1$ for all $a \in \mathcal{C}$, and symmetric, $A_{ab} = 1 \Rightarrow A_{ba} = 1$ for all $a, b \in \mathcal{C}$.

With the purpose of simplifying the following exposition, define matrix $B$ such that $B_{ab} = 1$ if and only if $b$ is the best alternative according to $\succ$ among those accessible from $a$, and $B_{ab} = 0$ otherwise. With abuse of notation, I will refer to $A(a)$ as the choice set available from $a$, formally $A(a) \equiv \{b \in \mathcal{C} : A_{ab} = 1\}$, and to $B(a)$ as the best alternative accessible from $a$, formally $B(a) \equiv b$ such that $B_{ab} = 1$. In the terminology of Mas-Colell et al. (1995, pp. 9-11), you can think of $\{A(a) : a \in \mathcal{C}\}$ as a family of budget sets and of $B(\cdot)$ as a choice rule. Notice that $B(\cdot)$ is a well-defined function because ties among alternatives have been ruled out by the assumption of linear order preferences.

An alternative $a$ is a *local maximum* when $a = B(a)$. The *global maximum* is that alternative which is first-ranked according to $\succ$; obviously, it is a local maximum too.

The following representation will turn out to be useful in explaining and understanding the main results throughout the paper. Consider alternatives as nodes (I will often refer to alternatives as nodes when dealing with this kind of graph representation), with a directed edge from $a$ to $b$ if and only if $b \neq a$ and $B_{ab} = 1$. A particular example of such a representation is depicted in figure 1, where every edge is to be interpreted as directed from bottom to top.

In this example there are two trees and, in general, any matrix $B$ can be represented in this way as a collection of trees, as a consequence of the uniqueness of the best accessible alternative and of the impossibility of having cycles (by transitivity of $\succ$). The number of trees forming the graph is determined by the number of local maxima. A tree can naturally be interpreted as the basin of attraction of its root.

Finally, let $S^r(a)$ be the set of nodes from which $a$ is reached with $r$ steps following an ascending path in the graph representation,[5] with $S^0(a) = \{a\}$. Let $d_b(a) \equiv r$ denote the distance from $b$ to $a$ when $b \in S^r(a)$, and let $d_b$ be

[4]A linear order is a binary relation which is antisymmetric, transitive and total.

[5]Formally, for $r \geq 1$, $b \in S^r(a) \Leftrightarrow \big(B^r(b) = a \wedge B^{r-1}(b) \neq a\big)$.

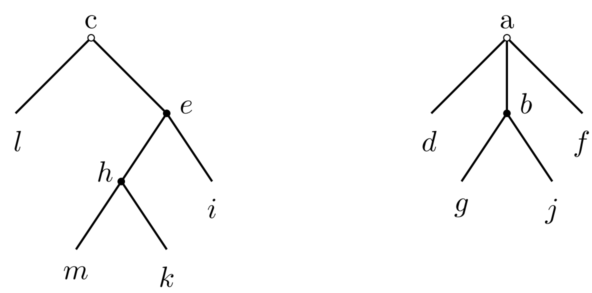



Figure 1: Basins of attraction.

the maximum over $a$ of $d_b(a)$, that is the number of iterations of $B$ taking $b$ to a local maximum. Let $\tilde{S}(a) \equiv \bigcup_r S^r(a)$ be the set of nodes containing $a$ itself and all the nodes from which $a$ is reachable by some number of steps, and let $l(a)$ be the cardinality of such a set. When $a$ is a local maximum I will refer to $l(a)$ as the size of its basin of attraction.

**Dynamics.** At any point in time agents select the best available alternative. What is available is determined by their previous choice on the basis of the accessibility matrix $A$. What is best is determined by the preference relation $\succ$. The overall information of best accessibility is summarized by matrix $B$, as explained in the previous paragraph. From an aggregate point of view such a choice rule produces the following dynamics in population choices:

$$x^{t+1} = x^t B \tag{1}$$

Lemma 1 provides a description of the behavior of dynamic system (1).

**Lemma 1** (Equilibria)**.** *Let $x^0$ be the initial population state.*

1. *The system will reach an equilibrium population state $\tilde{x}$ such that for all $a \in \mathcal{C}$,*

$$\tilde{x}_a = \begin{cases} 0 & \text{if } a \notin B(a) \\ \\ \sum_{b \in \tilde{S}(a)} x_b^0 & \text{if } a \in B(a) \end{cases} \tag{2}$$

2. *The time $d(x^0)$ required to reach $\tilde{x}$ is equal to the maximum distance to be covered by some initially existing fraction of population,*

$$d(x^0) = \max_{\substack{a \in \mathcal{C}: \\ x_a^0 > 0}} d_a$$

**Perturbations.** By lemma 1 it is known that if the global maximum is the only local maximum of the system, then there exists a unique equilibrium with the whole population concentrated in the global maximum. However, if the best accessibility matrix is such that more local maxima exist, then the dynamic system has infinitely many equilibria - all the ways population can be shared out among local maxima - and it will converge to a particular one depending on the initial condition.

With the aim of selecting among equilibria, perturbations are inserted in the above-described dynamics. Consider matrix $B^\epsilon$ such that $B^\epsilon_{ab} = \epsilon$ when $B_{ab} = 0$ and $B^\epsilon_{ab} = 1 - (n-1)\epsilon$ when $B_{ab} = 1$. The resulting Markov chain is irreducible and aperiodic.[6] Hence it is known that the population state will converge to the solution of $\hat{x} = \hat{x}B^\epsilon$. The next objective is to provide a characterization of the invariant distribution $\hat{x}$; this is what proposition 1 performs.

While a proof is given in the appendix, an intuitive explanation of the results is here provided. Since $\hat{x} = \hat{x}B^\epsilon$, then $\hat{x}_a = \sum_{b \in \mathcal{C}} B^\epsilon_{ba}\hat{x}_b$, that is the long run frequency of an alternative $a$ is equal to the sum over all alternatives of their long run frequency multiplied by their transition probabilities to $a$. Consider an alternative $a$ which is never a best accessible alternative. Therefore it is reachable only by mistake, $\hat{x}_a = \epsilon \sum_{b \in \mathcal{C}} \hat{x}_b$, and since $\sum_{b \in \mathcal{C}} \hat{x}_b = 1$, it comes out that $\hat{x}_a = \epsilon$. In a graph representation alternatives like the one just described are those at the bottom of an inverted tree; in figure 1 their collection is $\{l, m, k, i, d, g, j, f\}$. Consider now an alternative which is one step up in a tree, being the best accessible one from some terminal nodes only, for instance alternative $h$ in figure 1. Its long run frequency is

$$\hat{x}_h = (\hat{x}_m + \hat{x}_k)(1 - (n-1)\epsilon) + (1 - \hat{x}_m - \hat{x}_k)\epsilon$$

Hence, $\hat{x}_h = \epsilon(3 - 2n\epsilon)$. Notice that $l(h) = 3$. This can be interpreted as follows. Any given alternative is adopted with probability $\epsilon$ as a consequence of a perturbation. Alternative $h$ can be reached either directly by a perturbation or by the dynamic mechanism of best accessibility starting from a node in $\tilde{S}(h) \setminus \{h\}$ previously reached through a perturbation. This

[6] In this footnote a state is meant in Markovian sense and does not refer to a population state. In particular, a state corresponds here to an alternative $c \in \mathcal{C}$. A Markov chain is said irreducible when there is a positive probability of moving from any state to any other state in a finite number of periods, and it is said aperiodic when for every state $s$ unity is the greatest common divisor of the set of all the integers $r$ such that there is a positive probability of moving from $s$ to $s$ in exactly $r$ periods.

explains $l(a)\epsilon$. The negative higher order terms in $\epsilon$ measure the probability that going from a node in $\tilde{S}(h) \setminus \{h\}$ towards $h$ something goes wrong, that is some perturbation moves agents out of the best accessibility path. Generalizing this result for every node, the result in the upper part of (3) is obtained. For the long run frequency of local maxima an adjustment is required in order to take into account that they are left only by mistake; this has the main effect of lowering the degree of the polynomial.

The results in (4) are got simply by taking the limit as $\epsilon$ goes to zero of (3).

Finally, (5) provides a bound to the distance between the actual system after a certain number of periods and the invariant distribution. The usefulness of this result relies on the long run type of the analysis so far done; previous results are asymptotically relevant as time tends to infinity. The triangular distance is used in (5) instead of standard euclidean distance, since it is reputed more appropriate in this framework. In fact, for instance, the distance between a population state where all agents select alternative $a$ and another where all agents select alternative $b$ is for my concern equal to the distance between a population state where agents are shared out among alternatives $a$ and $b$ and another where agents are shared out among alternatives $c$ and $d$. The triangular distance in (5) expresses the amount of population out of the invariant distribution. Notice that convergence to the latter may be faster than established by the bound - exploiting the flow through the basins of attraction during the initial periods - but obviously never slower.

**Proposition 1** (Equilibrium Distribution)**.** *Let $\hat{x}$ be the solution of $\hat{x} = \hat{x}B^{\epsilon}$. Then:*

1. *for all $a \in \mathcal{C}$,*

$$\hat{x}_a(\epsilon) = \begin{cases} \epsilon\left(l(a) - \sum_{b\in\tilde{S}(a)}\left(1-(1-n\epsilon)^{d_b(a)}\right)\right) & \text{if } a \notin B(a) \\ \dfrac{l(a) - \sum_{b\in\tilde{S}(a)}\left(1-(1-n\epsilon)^{d_b(a)}\right)}{n} & \text{if } a \in B(a) \end{cases} \tag{3}$$

2. *for all $a \in \mathcal{C}$,*

$$\lim_{\epsilon\longrightarrow 0} \hat{x}_a(\epsilon) = \begin{cases} 0 & \text{if } a \notin B(a) \\ \dfrac{l(a)}{n} & \text{if } a \in B(a) \end{cases} \tag{4}$$

3. *for any initial population state $x^0$,*

$$\sum_{a\in\mathcal{C}} |x_a^t - \hat{x}_a| \leq (1-n\epsilon)^t \sum_{a\in\mathcal{C}} |x_a^0 - \hat{x}_a| \tag{5}$$

## 3 The Addition of Memory

In the last paragraph of the previous section perturbations have been introduced into the dynamics of (1). With a probability $n\epsilon$ agents who are selecting the global maximum find themselves choosing another alternative, which is surely inferior and possibly belonging to another basin of attraction. Being interested in preserving a cognitive interpretation for local maximization, one might reasonably expect those agents to jump back to their previous superior alternative. This kind of intuition relies on the idea that agents have *memory* and they can actually choose alternatives accessible from any remembered choice. In order to introduce agents with memory and to analyze the effects of such an introduction, an enrichment of the model is required.

The natural way to proceed is to enlarge the population state representation from the aggregate description of population choices to the aggregate description of population sequences of choices of length equal to the length of memory. Let $k$ be the length of memory, that is the number of periods agents can store information about. A population state is now a $n^k$-dimensional vector $y$ such that its component $y_{a_1 a_2 \dots a_k}$ represents the fraction of agents whose choices have been $a_1$ in time $t$, $a_2$ in time $t-1$, ..., $a_k$ in time $t-k+1$.

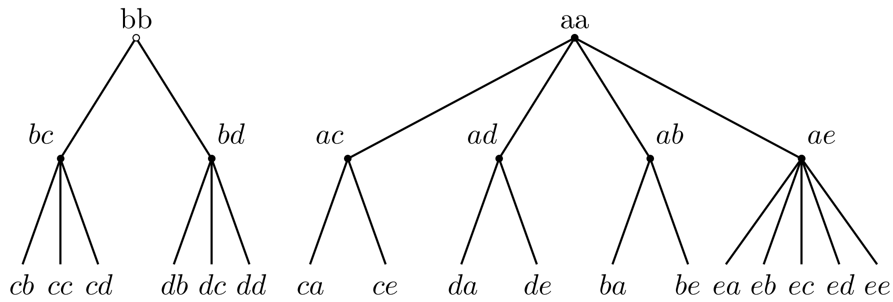
Relying on intuition, the notion of accessibility between sequences of choices is defined as follows: a sequence is accessible from another one if it is obtainable from the latter by deleting the oldest choice, moving any other choice one step to the right and adding in the first position an alternative accessible from at least one of the choices in memory. Using the same notation employed for the $k$-th power of matrix $A$, I denote by $A^k$ the adjacency matrix representing accessibility when the length of memory is $k$, being confident that the context always clarifies the meaning of $A^k$. Formally, rows of $A$ are such that $A^k_{(a_1 a_2 \dots a_k)(b_1 b_2 \dots b_k)} = 1$ if and only if $b_i = a_{i-1}$ for $i = 2, \dots, k$ and $b_1 \in \bigcup_{i=1}^k A(a_i)$.

Similarly to the case without memory (or, with the notation used in this section, when $k = 1$), and with the same abuse of notation as for $A^k$, let $B^k$ be the matrix such that $B^k_{(a_1 a_2 \dots a_k)(b_1 b_2 \dots b_k)} = 1$ if and only if $(b_1 b_2 \dots b_k)$ is accessible from $(a_1 a_2 \dots a_k)$ and $b_1$ is the alternative which is highest ranked in preference within the set $\bigcup_{i=1}^k A(a_i)$ of possible choices.

Local maxima of matrix $B^k$, i.e. sequences for which $B^k(a_1 a_2 \dots a_k) = (a_1 a_2 \dots a_k)$, consist of $k$ occurrences in a row of the same local maximum of matrix $B$, hence occupying all positions in memory, $a_i = a_{i+1} = a = B(a)$ for $i = 1, \dots, k-1$. Therefore, the number of local maxima does not change with the length of memory. Analogously, the global maximum of matrix $B^k$ is the sequence with all positions held by the global maximum of matrix $B$.

Figure 2 shows a couple of graph representations for the same underlying model with a change only in the length of agents' memory, which in the upper part is one period long while in the lower part is two period long. Notice that in the latter case nodes are no longer alternatives but sequences of alternatives.[7] The underlying preference order is such that $a \succ b \succ c \succ d \succ e$. In the basic case, when $k = 1$, two local maxima exist, the first accessible from two alternatives while the second, which is the global one, accessible from only one alternative.


Case $k = 1$




Case $k = 2$

Figure 2: Graph representation with memory.

Clearly, by changing the length of memory the relative numerosity of the basins of attraction changes. Here the basin of attraction of the global maximum becomes larger relatively to the basin of attraction of the local maximum. However, this is not a general result. Take for instance the example in figure 3. Notice that without memory $l(b)/l(a) = 2/5$. Preferences are such that $a \succ b \succ c \succ d \succ e \succ f \succ g$. Now consider the case when $k = 2$. In the basin of attraction of $aa$ there are $aa$, $dd$, $da$ and $ad$, all the ways $a$ and $d$ are preceded or followed by any other alternative, which are $2 \times 2 \times 5$ in number, and all the dispositions with repetition of $e$, $f$ and $g$, which amount to $3 \times 3$. In the basin of attraction of $bb$ there are $bb$, $cc$, $bc$ and $cb$, and all the ways $b$ and $c$ are preceded or followed by $e$ or $f$ or $g$, which

[7]The example I have chosen is extremely simple in order not to get an extremely involved figure, a very likely occurrence when $k > 1$.

are $2 \times 2 \times 3$ in number. Therefore, $l(bb)/l(aa) = 16/33 > 2/5 = l(b)/l(a)$.


Figure 3: Changes in the relative size of the basins of attraction.

**Dynamics.** The preliminary work in the previous paragraph has produced a representation of the model with memory which is formally equivalent to the one of the model without memory. The unperturbed dynamics of such a system is described by the following equation.

$$y^{t+1} = y^t B^k \tag{6}$$

The dynamics in (6) is analogous to that in (1). Hence, one may refer to lemma 1 to have a characterization of its behavior.

**Perturbations.** The same drawback encountered in the version of the model without memory - possibly infinite equilibria - suggests again to resort to perturbations with purpose of selection.

First suppose that perturbations occur in the same way as in the model without memory. Let $B^{k,\epsilon}$ be such that starting from any sequence $(a_1 \dots a_k)$ with probability $1-(n^k-1)\epsilon$ agents select the best accessible sequence, and with probability $\epsilon$ a perturbation occurs moving agents to any other sequence. The invariant distribution and the stochastically stable distribution of the irreducible and aperiodic Markov chain $y^{t+1} = y^t B^{k,\epsilon}$ are characterized as in proposition 1. Here the introduction of memory has the only effect to modify the relative long run frequency of local maxima by means of the changes it has caused on the relative size of the basins of attraction, as already mentioned.

However, from an interpretative point of view, some problems arise when thinking of perturbations which may lead an agent from a sequence $(a_1 \dots a_k)$ to a sequence $(b_1 \dots b_k)$ such that it is not true that $b_i = a_{i-1}$ for $i = 2, \dots, k$. The only conceivable interpretation is obtained if one imagines that when a perturbation happens, an agent is replaced by a new one whose sequence in memory is somehow randomly selected.

It seems to me more convincing to represent perturbations as something (mistakes, experimentation, ...) which affects only agents' choices at the current time. According to this idea, matrix $B^{k,\epsilon}$ of perturbed transition probabilities should be such that from any sequence $(a_1 \dots a_k)$ with probability $1-(n-1)\epsilon$ the best accessible sequence is selected, with probability $\epsilon$ any other sequence $(b_1 \dots b_k)$ such that $b_i = a_{i-1}$ for $i = 2, \dots, k$ is got by perturbation, while any sequence which is not accessible receives zero probability. Notice that the just defined Markov chain still remains irreducible and aperiodic, because the composition for $k$ times of matrix $B^{k,\epsilon}$ has all positive entries. Hence the system will converge to the solution of $\hat{y} = \hat{y}B^{k,\epsilon}$. Since the invariant distribution $\hat{y}(\epsilon)$ consists of a large variety of terms hard to be interpreted, proposition 2 only provides a description of the limit of $\hat{y}$ when the perturbation goes to zero - that is the stochastically stable distribution - and a couple of bounds to the time of convergence.

According to (7), even with a length of memory of only two periods agents are able to converge to the global maximum, hence obtaining efficiency. The result is intuitively understandable with the following sketched mutation counting argument. Take the case of two period long memory. If $a$ and $b$ are the global maximum and a local maximum respectively, then consider the sequences $(a, a)$ and $(b, b)$. In order to go from $(a, a)$ to $(b, b)$ two perturbations are required to occur, since once $(b, a)$ is reached by a single mutation memory allows to go back to the superior choice. Only one perturbation instead is needed to move from $(b, b)$ to $(a, a)$.

This result allows to reflect on the evolutive meaning of perturbations (or mistakes) in this framework. To say that in biological terms, when a mutation has occurred some kind of mechanism which allows a comparison between the mutant and the wild type and hence the selection of the best one is needed to push evolution towards efficiency. Memory makes room for such a comparison.

Inequality in (8) is obtained similarly to the corresponding inequality in (5) and plays the same role of bound to the time of convergence to the invariant distribution. However, the greater $k$ becomes the higher is the over-estimation in (8).[8] Another bound, which proves to be tighter for sufficiently small $\epsilon$, is provided by (9), where the following consideration is exploited: starting from two sequences of choices, after a trial of the global

[8] A strictly better bound would be:

$$\sum_{a_1 \dots a_k \in \mathcal{C}^k} |y^t_{a_1 \dots a_k} - \hat{y}_{a_1 \dots a_k}| \leq (1 - Pr(t, n\epsilon, k)) \sum_{a_1 \dots a_k \in \mathcal{C}^k} |y^0_{a_1 \dots a_k} - \hat{y}_{a_1 \dots a_k}|$$

where $Pr(t, n\epsilon, k)$ is the probability to have at least a string of $k$ consecutive perturbations in a sequence of $t$ trials, computable thanks to the recurrence $Pr(t, n\epsilon, k) = Pr(t-1, n\epsilon, k) + (1 - n\epsilon)n^k\epsilon^k(1 - Pr(t-k-1, n\epsilon, k))$, with $Pr(j, n\epsilon, k)$ for $j = 1, \dots, k-1$ and $Pr(k, n\epsilon, k) = n^k\epsilon^k$. However the application of this bound would require computations case by case because of the recursion.

maximum - which happens with at least probability $\epsilon$ - then the probability distribution of the choice at the next time is identical for the two sequences. That leads to the same sequence after $k-1$ periods.

Notice that when $k = 1$ the current model is exactly the same as the model without memory. Therefore in that case proposition 1 applies.

**Proposition 2** (Equilibrium Distribution with Memory)**.** *Let $\hat{y}$ be the solution of $\hat{y} = \hat{y}B^{k,\epsilon}$. If $k \geq 2$, then*

*1.*

$$\lim_{\epsilon \to 0} \hat{y}_{a_1 \ldots a_k} = \begin{cases} 1 & \text{if } a_i \text{ is the global maximum, for } i = 1, \ldots, k \\ \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

*2. for any initial population state $y^0$,*

$$\sum_{a_1 \ldots a_k \in \mathcal{C}^k} |y^t_{a_1 \ldots a_k} - \hat{y}_{a_1 \ldots a_k}| \leq (1 - n^k \epsilon^k)^{[t/k]} \sum_{a_1 \ldots a_k \in \mathcal{C}^k} |y^0_{a_1 \ldots a_k} - \hat{y}_{a_1 \ldots a_k}| \tag{8}$$

*with $[t/k]$ the integer part of the ratio $t/k$;*

*3. for any initial population state $y^0$,*

$$\sum_{a_1 \ldots a_k \in \mathcal{C}^k} |y^t_{a_1 \ldots a_k} - \hat{y}_{a_1 \ldots a_k}| \leq (1-\epsilon)^{t-k+1} \sum_{a_1 \ldots a_k \in \mathcal{C}^k} |y^0_{a_1 \ldots a_k} - \hat{y}_{a_1 \ldots a_k}| \tag{9}$$

# 4 Extensions

With the aim to understand the robustness of the result of convergence to the global maximum stated in proposition 2, some extensions of the previous framework might be considered. For instance, population may be thought as divided into several groups characterized by different lengths of memory. In this case the resulting model can simply be decomposed into as many different models - each of the type discussed in this paper - as existing lengths of memory. Alternatively, instead of being fixed, the length of memory may be assumed to be variable according to a certain distribution. Even if its formal representation would be rather complicated, intuition suggests that, as long as the probability to have no memory (or $k = 1$ in the previously used notation) is bounded away from zero, the system is bounded away from the population state where all agents select the global maximum, since a non negligible fraction of population will be taken by loss of memory and will not come back after a perturbation. Therefore, as far as forgetful people exist (or as far as people may be forgetful with a positive probability in the version with variable memory) efficiency is not got.

Other possible extensions might concern the variability of the preference relation and/or of the matrix of accessibility. Some care must be given to that, since their constancy over time plays the role of constant environment in an evolutive theory, hence being crucial for the long run type of analysis in this paper. Anyhow, some interesting research may be done for particular cases of environmental changes.

As regards the case of variable preference relation, one may consider a game theoretic framework of the model, where others' choices affect individual preferences. Think for instance of alternatives as investment opportunities whose yields depend on aggregate investment decisions. One might analyze how different matrices of accessibility and different stylized rules for the determination of yields affect results in terms of efficiency or equality of returns among investment opportunities.

Coming to possible extensions regarding the variability of the accessibility matrix, it should be first noticed that such matrix is meant to represent the existing informative similarities between alternatives. Therefore it should be considered inherently determined by exogenous factors. However, one may reasonably think of an external agent (a public authority or a consumer association) having the power to modify to some extent the structure of accessibility between alternatives by paying a cost. An analysis of the optimality of interventions in this framework might be usefully carried out.

Finally, the temporal evolution of the system might be investigated in deeper details. Bounds to the rate of convergence have been calculated for the perturbed dynamics without and with memory, because results hold asymptotically. However those bounds might be refined and the dependence of the rate of convergence on the length of memory might be more carefully analyzed, even by exploiting computer simulations when analytical techniques become ineffective.

## 5 Conclusive Discussion

In this paper a descriptive model of choice has been presented. An infinite population of agents chooses repeatedly over time within a finite set of alternatives. Choices are taken on the basis of a common preference relation in the presence of markovian constraints, this meaning that available choices at a certain time depend upon previous choices.

An unperturbed dynamics selects, as equilibria which the system converges towards, those population states where all agents are shared out only among local maxima; however, which particular equilibrium is selected depends on the initial condition.

A perturbed dynamics is then analyzed and its unique prediction is fully characterized: population tends to be shared out only among local maxima, as in the unperturbed dynamics, and the fraction choosing a local maximum tends to the relative size of its basin of attraction as the perturbation tends to zero. Being such a perturbed analysis relevant in the long run, a bound to the rate of convergence to the equilibrium population state is provided.

The possibility to choose back a previous alternative when a perturbation has brought to an inferior alternative is then considered. Agents are given the possibility to store in their memory a sequence of choices of a certain length and to choose among all the alternatives which are accessible from at least one of those in memory. Without perturbations the system converges to a population state where only local maxima are chosen. No significant differences arise with respect to the case with no memory, even if the relative numerosity of the basins of attraction comes out to be modified in an ambiguous way.

When perturbations are inserted into this setting a deep change in prediction instead occurs. The whole population ends up choosing the global maximum, with no respect to the length of memory (the result holds for $k \geq 2$ with the notation used in the paper). By pairing mutations and memory the efficient population state is reached and their respective roles in the evolution towards efficiency turn out to be clarified: human rationality is limited by accessibility constraints and is only able to select local maxima; mutations bring novelties and allow to overcome human bounds, while memory creates the possibility to compare the old with the new (the wild type with the mutant in biological terms) and hence to select the best.

Finally, a couple of bounds to the rate of convergence are established. Some interesting research may concern a detailed analysis of the relation between the length of memory and the rate of convergence to the global maximum.

A comparison between the results got in this paper and the case of global accessibility may be useful. Under global accessibility, that is by using a matrix of accessibility where every entry is equal to 1, the global maximum is immediately reached. In the case of perturbed local accessibility unforgetful agents converge to the global maximum only after a long time, and they may not reach it at all if environmental changes occur in the meanwhile. The longer the time required for convergence the more demanding the requirement of constancy for the accessibility matrix and, above all, for the preference relation. Therefore, from the perspective of a public authority in the model there is room for a welfare enhancing intervention to the extent that convergence can be quickened.

## A Appendix

**Proof of lemma 1.** Since $B^t_{ba} = 1 \Leftrightarrow a = B^t(b)$ and $B^t_{ba} = 0 \Leftrightarrow a \neq B^t(b)$, then

$$x^t_a = \sum_{b \in \mathcal{C}} x^0_b B^t_{ba} = \sum_{\substack{b \in \mathcal{C}: \\ a = B^t(b)}} x^0_b \tag{10}$$

Since cycles are not admitted by transitivity of $\succ$ and the set of alternatives $\mathcal{C}$ is finite, then for any $b \in \mathcal{C}$ there must exist a positive integer $d_b$ such that $B^{d_b+1}(b) = B^{d_b}(b) \equiv a$. Alternative $a$ is the local maximum whose basin of attraction $b$ belongs to. Since $\mathcal{C}$ is finite, then there exists the maximum of $d_b$ over all the alternatives $b$ such that $x^0_b > 0$. Denote such maximum by $d(x^0)$.

For any alternative $a$, if $t \geq d(x^0)$ and $x^t_a > 0$, then $a$ is a local maximum since any $b$ such that $x^0_b > 0$ has already reached its local maximum. On the contrary, if $t < d(x^0)$ there exists an alternative $b$ which has not got its local maximum yet. This implies that an equilibrium cannot be reached before $d(x^0)$ periods. Finally, since for any local maximum $a$ and $t \geq d(x^0)$

$$x^t_a = \sum_{\substack{b \in \mathcal{C}: \\ a = B^t(b)}} x^0_b = \sum_{\substack{b \in \mathcal{C}: \\ a = B^{d(x^0)}(b)}} x^0_b = \sum_{\substack{b \in \tilde{S}(a): \\ x^0_b > 0}} x^0_b = \sum_{b \in \tilde{S}(a)} x^0_b \tag{11}$$

then an equilibrium is reached in exactly $d(x^0)$ periods. Define $\tilde{x} \equiv x^{d(x^0)}$ and the desired result is got. $\square$

**Proof of proposition 1.** Let $\hat{x}$ be the solution of $\hat{x} = \hat{x}B^\epsilon$. By taking into consideration that for any $a, c \in \mathcal{C}$, it is true that $B_{ca} = (1 - (n-1)\epsilon)$ if $B(c) = a$ and $B_{ca} = \epsilon$ if $B(c) \neq a$, any $\hat{x}_a$ can be expressed as follows:

$$\hat{x}_a = (1 - (n-1)\epsilon) \left( \sum_{c: B(c) = a} \hat{x}_c \right) + \epsilon \left( 1 - \sum_{c: B(c) = a} \hat{x}_c \right) \tag{12}$$

The proof of point 1 of the proposition is by induction. First notice that if an alternative is not the best accessible choice for any alternative, then it is selected by a fraction of population equal to $\epsilon$, as stated by (4). Then notice that (12) can be re-written as

$$\hat{x}_a = \epsilon + \sum_{c: B(c) = a} (1 - n\epsilon)\hat{x}_c \tag{13}$$

Now suppose that all the alternatives $c$ from which $a$ is the best accessible alternative, $a = B(c)$, are selected by fractions of population in accordance with (4) in proposition 1. Then, for any $c : B(c) = a$

$$
\begin{aligned}
(1-n\epsilon)\hat{x}_c = (1-n\epsilon)\epsilon\left(l(c) - \sum_{b\in\tilde{S}(c)}\left(1-(1-n\epsilon)^{d_b(c)}\right)\right) = \\
= \epsilon l(c) - n\epsilon^2 l(c) - \epsilon\sum_{b\in\tilde{S}(c)}\left((1-n\epsilon)-(1-n\epsilon)^{d_b(c)+1}\right) = \\
= \epsilon\left(l(c) - \sum_{b\in\tilde{S}(c)}\left(1-(1-n\epsilon)^{d_b(c)+1}\right)\right)
\end{aligned}
\tag{14}
$$

When $a$ is not a local maximum, $a \neq B(a)$, the following relations hold:

i) $1 + \sum_{c:B(c)=a} l(c) = l(a)$

ii) $d_b(c) + 1 = d_b(a)$ if $a = B(c)$ and $b \in \tilde{S}(c)$

iii) $\sum_{b\in\tilde{S}(a)}\left(1-(1-n\epsilon)^{d_b(a)}\right) - \sum_{c:B(c)=a}\sum_{b\in\tilde{S}(c)}\left(1-(1-n\epsilon)^{d_b(a)}\right) =$
$= \left(1-(1-n\epsilon)^{d_a(a)}\right) = 0$

Therefore, the equilibrium fraction of population selecting whatsoever non locally maximal alternative $a$ is so determined:

$$
\begin{aligned}
\hat{x}_a = \epsilon + \epsilon\sum_{c:B(c)=a}\left(l(c) - \sum_{b\in\tilde{S}(c)}\left(1-(1-n\epsilon)^{d_b(c)+1}\right)\right) = \\
= \epsilon\left(1 + \sum_{c:B(c)=a} l(c)\right) - \epsilon\sum_{c:B(c)=a}\sum_{b\in\tilde{S}(c)}\left(1-(1-n\epsilon)^{d_b(c)+1}\right) = \\
= \epsilon l(a) - \epsilon\sum_{b\in\tilde{S}(a)}\left(1-(1-n\epsilon)^{d_b(a)}\right)
\end{aligned}
\tag{15}
$$

Now consider the case where $a$ is a local maximum, $a = B(a)$. The expression in (12) can be re-written as

$$
\hat{x}_a = \frac{\epsilon + \sum_{\substack{c:B(c)=a,\\ c\neq a}} (1-n\epsilon)\hat{x}_c}{n\epsilon}
\tag{16}
$$

The expression in (14) remains the same. Similarly to the case of non locally maximal alternatives, when $a$ is a local maximum the following relations hold:

i) $1 + \sum_{\substack{c:B(c)=a,\\ c\neq a}} l(c) = l(a)$

ii) $d_b(c) + 1 = d_b(a)$ if $c \neq a$, $a = B(c)$ and $b \in \tilde{S}(c)$

iii) $\sum_{b\in\tilde{S}(a)} \left(1-(1-n\epsilon)^{d_b(a)}\right) - \sum_{\substack{c:B(c)=a,\\ c\neq a}} \sum_{b\in\tilde{S}(c)} \left(1-(1-n\epsilon)^{d_b(a)}\right) =$
$= \left(1-(1-n\epsilon)^{d_a(a)}\right) = 0$

Therefore, the equilibrium fraction of population selecting whatsoever locally maximal alternative $a$ is so determined:

$$\hat{x}_a = \frac{1}{n}\left(l(a) - \sum_{b\in\tilde{S}(a)} \left(1-(1-n\epsilon)^{d_b(a)}\right)\right) \tag{17}$$

As regards the results in point 2, they are got by taking the limit as $\epsilon \to 0$ of the above-calculated expressions. Such limits are easy to be computed since the expressions in (15) and (17) are polynomials.

Finally, I deal with the bound to the rate of convergence stated in point 3. Consider two population states, $\tilde{x}$ and $\bar{x}$ and let $2h$ be their triangular distance, $\sum_{a\in\mathcal{C}} |\tilde{x}_a - \bar{x}_a| = 2h$. I am interested in establishing a bound to the distance of their images according to matrix $B^\epsilon$, $\sum_{a\in\mathcal{C}} |(\tilde{x}B^\epsilon)_a - (\bar{x}B^\epsilon)_a| = \sum_{a\in\mathcal{C}} |((\tilde{x}-\bar{x})B^\epsilon)_a| = \sum_{a\in\mathcal{C}} |\sum_{b\in\mathcal{C}}(\tilde{x}_b - \bar{x}_b)B^\epsilon_{ba}|$. Let $x^+$ be a vector such that $x^+_a = \tilde{x}_a - \bar{x}_a$ when $\tilde{x}_a - \bar{x}_a > 0$ and $x^+_a = 0$ otherwise, and let $x^-$ be a vector such that $x^-_a = \tilde{x}_a - \bar{x}_a$ when $\tilde{x}_a - \bar{x}_a < 0$ and $x^+_a = 0$ otherwise. Notice that $\sum_{a\in\mathcal{C}} x^+_a = h$ and $\sum_{a\in\mathcal{C}} x^-_a = -h$. Moreover, each entry in matrix $B^\epsilon$ is not lower than $\epsilon$. Therefore, for every $a \in \mathcal{C}$, $(x^+B^\epsilon)_a \geq h\epsilon$ and $(x^-B^\epsilon)_a \leq -h\epsilon$. This means that at least $h\epsilon$ of the positive excess of $\tilde{x}$ over $\bar{x}$ will merge with at least $h\epsilon$ of the negative excess of $\tilde{x}$ over $\bar{x}$, so determining an overall reduction of at least $2h\epsilon$ for every alternative. Since there are $n$ alternatives and $x^+_a + x^-_a = \tilde{x}_a - \bar{x}_a$, the triangular distance between the images of $\tilde{x}_a$ and $\bar{x}_a$ according to matrix $B^\epsilon$ is at most $2h(1-n\epsilon)$. The distance after $t$ repetitions of $B^\epsilon$ is hence $2h(1-n\epsilon)^t$. If one of the two initial population states is the equilibrium one, then the result in (5) is obtained. $\square$

**Proof of proposition 2.** I prove point 1 by relying on the *radius-coradius* theorems of Ellison (2000). The global maximum of $B^k$ is an ergodic set of the unperturbed dynamics. Its radius $R(a)$, that is the minimum cost in terms of mutations for leaving its basin of strong attraction, is easily established to be equal to $k$. Its coradius $CR(a)$, that is the maximum over all states of the minimum cost in terms of mutations for reaching the global maximum from a given state, is trivially equal to 1. By theorem 1 of Ellison

(2000), since $R(a) > CR(a)$ for $k \geq 2$, only the global maximum can have a positive probability in the stochastically stable distribution, so proving what desired.

Now, I prove the inequality in (8). Consider matrix $(B^{k,\epsilon})^k$ coming out from the composition of matrix $B^{k,\epsilon}$ for $k$ times. The resulting matrix is such that each of its elements is not lower than $\epsilon^k$, since at any period with at least probability $\epsilon$ anything may be chosen and after $k$ periods any sequence may be formed starting from any other sequence. The number of different sequences of alternatives is $n^k$, and (8) is simply got in the same way of 5, where $\epsilon$ is substituted by $\epsilon^k$, $n$ is substituted by $n^k$ since the latter is the number of different sequences of alternatives, and the power to which $(1 - n^k\epsilon^k)$ is raised is equal to the number of complete repetitions of matrix $(B^{k,\epsilon})^k$, that is the integer part of the ratio $t/k$.

Finally, I deal with (9). Take two population distributions over sequences of choices, $\tilde{y}$ and $\bar{y}$, and let $2h$ be their triangular distance, $\sum_{a_1...a_k \in \mathcal{C}^k} |\tilde{y}_{a_1...a_k} - \bar{y}_{a_1...a_k}| = 2h$. Let $y^+$ be a vector such that $y^+_{a_1...a_k} = \tilde{y}_{a_1...a_k} - \bar{y}_{a_1...a_k}$ when $\tilde{y}_{a_1...a_k} - \bar{y}_{a_1...a_k} > 0$ and $y^+_{a_1...a_k} = 0$ otherwise, and let $y^-$ be a vector such that $y^-_{a_1...a_k} = \tilde{y}_{a_1...a_k} - \bar{y}_{a_1...a_k}$ when $\tilde{y}_{a_1...a_k} - \bar{y}_{a_1...a_k} < 0$ and $y^-_{a_1...a_k} = 0$ otherwise. Notice that $\sum_{a \in \mathcal{C}} y^+_{a_1...a_k} = h$ and $\sum_{a \in \mathcal{C}} y^-_{a_1...a_k} = -h$. Moreover, matrix $B^{k,\epsilon}$ is such that with at least probability $\epsilon$ whatever choice can be the next. Hence, at least $h\epsilon$ in $y^+B^{k,\epsilon}$ and $-h\epsilon$ in $y^-B^{k,\epsilon}$ are choosing the global maximum as last choice. From then on the vector of probabilities describing the next choice for both of the sequences will be the same, with probability $1-(n-1)\epsilon$ to choose the global maximum again and probability $\epsilon$ to choose any other alternative. Hence, recalling that $y^+ + y^- = \tilde{y} - \bar{y}$, after $k-1$ more periods at least $2h\epsilon$ of the difference between the initial population states will have disappeared. After $t$ periods, with $t \geq k$, at least $2h(\sum_{\alpha=1}^{t-k+1} \binom{t-k+1}{\alpha}(1-\epsilon)^{t-k+1-\alpha}\epsilon^\alpha)$ of the difference between the initial population states will have merged into the same sequences of choices, since $\sum_{\alpha=1}^{t-k+1} \binom{t-k+1}{\alpha}(1-\epsilon)^{t-k+1-\alpha}\epsilon^\alpha$ is an inferior bound to the probability of selecting at least once the global maximum in the first $t-k+1$ periods. Therefore, noticing that $\hat{y}^t = \hat{y}$ for any $t$,

$$\sum_{a_1...a_k \in \mathcal{C}^k} |y^t_{a_1...a_k} - \hat{y}_{a_1...a_k}| \leq$$
$$\leq \left(1 - \sum_{\alpha=1}^{t-k+1} \binom{t-k+1}{\alpha} \epsilon^\alpha (1-\epsilon)^{t-k+1-\alpha}\right) \sum_{a_1...a_k \in \mathcal{C}^k} |y^0_{a_1...a_k} - \hat{y}_{a_1...a_k}| =$$
$$= (1-\epsilon)^{t-k+1} \sum_{a_1...a_k \in \mathcal{C}^k} |y^0_{a_1...a_k} - \hat{y}_{a_1...a_k}| \tag{18}$$

□

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